On (a, d)- antimagic special trees, unicyclic graphs and complete bipartite graphs

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Abstract

A connected graph G(V, E) is said to be **(a, d)- antimagic** if there exist positive integers a and d and a bijection $f: E \rightarrow \{1, 2, \ldots, |E|\}$ such that the induced mapping $g_f: V \rightarrow N$, defined by $g_f(v) = \Sigma\{f(u,v) \mid (u, v) \in E(G)\}$ is injective and $g_f(V) = \{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. In this paper, we mainly investigate (a, d)- antimagic labeling of some special trees, complete bipartite graphs $K_{m,n}$ and categorize (a, d)- antimagic unicyclic graphs.

1. INTRODUCTION

In [7] G. Ringel (1994) introduced the concept of an antimagic graph. Each edge labeling f of a graph G = (V, E) from 1 through |E| induces a vertex labeling g_f where $g_f(v)$ is the sum of the labels of all edges that are incident upon vertex v. Labeling f is called **antimagic** if and only if (iff) the values $g_f(v)$ are pairwisely distinct for all vertices v of G. Graph G is called **antimagic** iff it has an antimagic labeling. Let Γ denote the set of all finite, connected undirected graphs G = (V, E) without loops and multiple edges. The main problem in the theory of antimagic graphs is the determination of all antimagic graphs in Γ. This problem still remains open. Ringel [7] conjectured that every graph $G \in \Gamma$ of order ≥ 3 is antimagic.

R. Bodendiek and G. Walther (1996) introduced the concept of an (a, d)- antimagic graph as a special antimagic graph where $a, d \in N$.

Definition 1.1 [2] Let $G(V, E) \in \Gamma$ be a graph of order $|V| \geq 3$ and $a, d \in N$. A bijective mapping $f : E \rightarrow \{1, 2, . . ., |E|\}$ with the induced mapping $g_f : V \rightarrow N$, defined by $g_f(v) := \sum_{e \in I(v)} f(e)$, $v \in V$, where $I(v) := \{e \in E \mid e \text{ is incident to } v\}$ for $v \in V$ is said to be **(a, d) antimagic** labeling iff $g_f(v)$ form an arithmetic progression with initial value a and step width d.

G is called **(a, d) antimagic** iff G admits an (a, d) antimagic labeling. Obviously every (a, d) antimagic graph is also antimagic. For example, C_4 is antimagic as shown in Figure 1. But C_4 does not admit an (a, d) antimagic labeling for any pair $a, d \in N$.

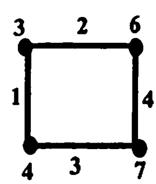

Figure 1. Antimagic labeling of C_4.

The **vertex weight** $w_f(v)$ (sometimes denoted as w(v)) of a vertex v in V(G) under an edge labeling f is the sum of values f(e) assigned to all edges incident to v. Let W denote the set of all vertex weights of the graph G.

R. Bodendiek and G. Walther [2, 3] proved the finiteness of two very interesting subsets of the set of all (a, d)- antimagic parachutes. M. Baca and I. Hollander [1] characterized all (a, d)- antimagic graphs of prisms $D_n = C_n \times P_2$ when n is even. They showed that when n is odd, the prism D_n are ((5n+5)/2, 2)- antimagic. They also conjectured that prisms with odd cycles of length n, $(n \geq 7)$, are ((n+7)/2, 4)- antimagic.

In [4] Bodendiek and Walther proved the following results that are useful to the ensuing sections in this paper.

Theorem 1.2 [4] If $G(V, E) \in \Gamma$, $|V| = n \geq 3$, $|E| = m \geq 2$, is an (a, d)- antimagic graph, then a, d $\in$ N satisfy the following conditions:

(a) a, d $\in$ N are positive solutions of the linear Diophantine equation

$$2an + n(n-1)d = 2m(m+1) \qquad (1)$$

(b) If δ denotes the minimum degree in G, then

$$a \geq 1 + 2 + \ldots + \delta = \delta(\delta + 1)/2. \qquad (2)$$ ∎

Theorem 1.3 [4] If a tree T = (V, E) of order $|V| = 2k + 1 \geq 3$ is (a, d)- antimagic, then it is necessarily the case that d = 1 and a = k. ∎

Theorem 1.4 [4] (a) If a tree T = (V, E) of order $|V| = 2k + 1 \geq 5$ contains a vertex v adjacent to at least three end vertices in T, then T is not (a, d)- antimagic.

(b) Every tree T = (V, E) of order $|V| = 2k + 1 \geq 5$, containing at least k + 2 end vertices, is not (a, d)- antimagic. ∎

Using the above results, Bodendiek and Walther [4] proved that cycles of even order, paths of even order, stars $K_{1,n}$, $n \geq 3$, complete binary trees are not (a, d)- antimagic.

In this paper, we mainly investigate the (a, d)- antimagic labeling of trees, complete bipartite graphs and unicyclic graphs. Section 2 deals with (a, d)- antimagic labeling of some special classes of caterpillars and spiders. Section 3 encompasses the study of (a, d)- antimagic labeling of complete bipartite graphs $K_{m,n}$. A necessary condition is obtained for the complete bipartite graph $K_{m,n}$ to be (a, d)- antimagic. Using this condition it is proved that $K_{m,n}$ is not (a, d)- antimagic when m + n is prime and $K_{n,n}$ is not (a, d)- antimagic when n is odd. Several properties of (a, d)- antimagic labeling of $K_{n,n+2}$ have been derived.

In Section 4 we investigate the categorization of (a, d)- antimagic labeling of unicyclic graphs. In particular, (a, d)- antimagic labeling of unicyclic graphs such as $C_n \odot mK_1$ and C_n @ P_m are discussed.

Throughout this paper a graph always means a connected graph, p denotes the number of vertices and q the number of edges of the graph unless otherwise mentioned. For graph theoretic definitions and terminology, we refer to Harary [6] and Gallian [5].

Main Results

2. On (a, d)- antimagic labeling of trees.

Bodendiek and Walther [3] observed that trees of even order are not (a, d)- antimagic. We focus our attention on (a, d)- antimagic labeling of trees T of odd order. We consider a **caterpillar** as a tree which has a path $P_n = a_1a_2 \ldots a_n$ of order n and is obtained by attaching x_i (possibly zero) end vertices at the vertex a_i of P_n, $i = 1, 2, \ldots, n$ by end edges. It is denoted as $T = S(x_1, x_2, x_3, \ldots, x_n)$. Clearly the order of T is $n + \sum x_i$. Now the following theorem is an immediate consequence of Theorem 1.4(b).

Theorem 2.1 The caterpillar $T = S(x_1, x_2, \ldots, x_n)$ of odd order is (a, d)- antimagic only if $\sum x_i \leq n + 1$. ∎

Corollary 2.2 Let $T = S(x_1, x_2, \ldots, x_n)$ be a caterpillar of odd order. If $x_i \geq 2$ for $i = 1, 2, \ldots, n$, then T is not (a, d)- antimagic.

Proof: Since $x_i \geq 2$ for each i, we observe that $\sum x_i \geq 2n > n + 1$. Hence the result follows from Theorem 2.1. ∎

Corollary 2.3 If T is an (a, d)- antimagic caterpillar, then $\deg(v) \leq 4$ for any vertex $v \in V(T)$. ∎

Corollary 2.4 If T is an (a, d)- antimagic caterpillar of odd order p and having a vertex v of degree 4, then $p \geq 9$.

Proof: Let v be an internal vertex of the path of degree 4. If we assign the labels a and a+1 for the end edges emanating from v and the labels 1 and 2 for the edges of the path incident at v, then we must have $1 + 2 + a + (a+1) \leq a + p - 1$. This implies that $p \geq 9$ since $a = (p-1)/2$ using Theorem 1.3. ∎

Example: The caterpillar S(0, 0, 2, 0, 0, 0, 0) is (4, 1)- antimagic as shown in Figure 2.

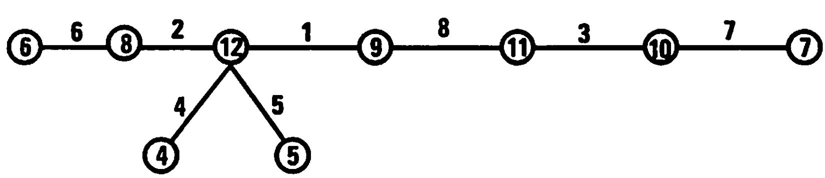

Figure 2. (4, 1) - antimagic labeling of S(0, 0, 2, 0, 0, 0, 0)

However one can easily verify that the caterpillar S(0, 0, 0, 2, 0, 0, 0) shown in Figure 3 is not (a, d)- antimagic though p = 9 (we omit a formal proof).

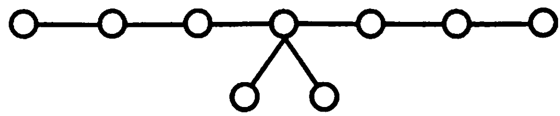

Figure 3 S(0, 0, 0, 2, 0, 0, 0) is not (a, d) - antimagic

Theorem 2.5 Let T = $S(x_1, x_2, x_3, \ldots, x_n)$ be of odd order where $x_i = 2$ for $i = 1, 2, \ldots, k\,(< n)$ and $x_i = 1$ for $i = k+1, k+2, \ldots, n$. If T is (a, d)- antimagic, then k = 1.

Proof: Here $\sum x_i = 2k + (n - k) = n + k$. If T is (a, d)- antimagic, then $n + k \leq n + 1$ by Theorem 2.1. This implies that $k \leq 1$. Now k = 0 makes |V(T)| even and hence it follows that k = 1. ∎

Theorem 2.6 T = $S(2, x_2, x_3, \ldots, x_n)$, where $x_2 = x_3 = \ldots = x_n = 1$, is (n, 1)- antimagic.

Proof: The given caterpillar is an odd tree with p = 2n+1, q = 2n and $\sum x_i = n+1$. Let $a_1, a_2, \ldots, a_n$ be the path vertices, u_1 and u_2 be the two end vertices incident to a_1 and v_i be the end vertex incident to a_i, $i = 2, \ldots, n$. Now Figure 4 depicts (n, 1)- antimagic labeling for T which completes the proof. ∎

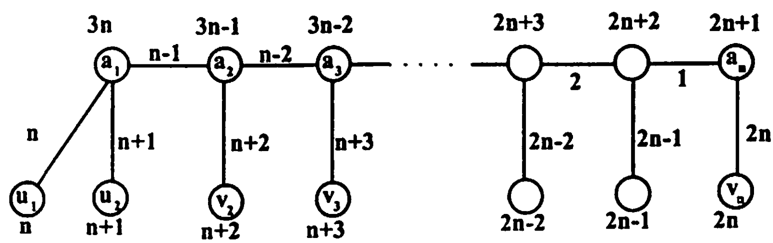

Figure 4. (n, 1)- antimagic labeling of S(2, 1, 1, . . ., 1).

A spider $SP(P_n, 2)$ [5] is a caterpillar $S(x_1, x_2, \ldots, x_n)$ where $x_n = 2$ and $x_i = 0$, for i = 1, 2, . . ., n − 1.

Theorem 2.7 The spider $SP(P_{2k+1}, 2)$, where $k \geq 1$, is (k+1, 1)- antimagic.

Proof: Let $a_1, a_2, \ldots, a_{2k+1}$ be the path vertices and u_1 and u_2 be the two end vertices attached at the vertex a_{2k+1}. Figure 5 concludes the proof by giving a (k+1, 1)- antimagic labeling of the spider $SP(P_{2k+1}, 2)$. ∎

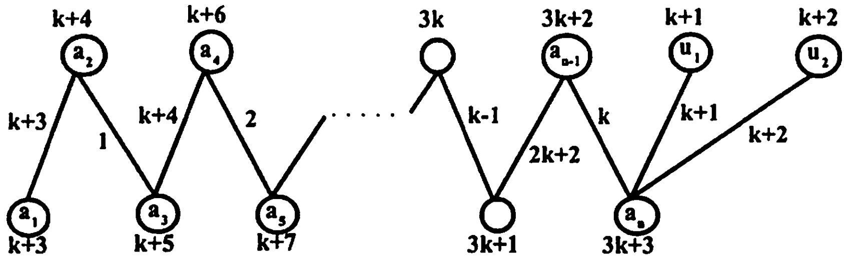

Figure 5. (k+1, 1)- antimagic labeling of SP(P_{2k+1}, 2).

A **regular spider** SP(k : n) is a tree obtained by identifying one end vertex of k number of paths, each of length n [5]. Obviously SP(k : n) has kn + 1 vertices.

Theorem 2.8 If a regular spider SP(k : n) of odd order is (a, d)- antimagic, then $k \leq 3n-1$.

Proof: Let SP(k : n) be (a, d)- antimagic. Then a = kn/2 and d = 1. It has one vertex of degree k and hence if we assign the integers 1, 2, . . ., k for the edges incident to that vertex, then we must have $1+2+\ldots+k \leq a+(p-1)d$. That is, $k \leq 3n-1$ since $p = kn+1$. Hence the result is proved. ∎

Theorem 2.9 SP(n : 2) is (a, d)- antimagic if and only if $n \leq 4$.

Proof: Replacing k and n with n and 2 respectively in Theorem 2.8, we can show that SP(n : 2) is (a, d)- antimagic only if $n \leq 5$.

Claim: SP(5 : 2) is not (a, d)- antimagic.

Suppose to the contrary, SP(5 : 2) is (a, d)- antimagic. Let u be the vertex of degree 5. In light of the above arguments, four edges that are incident at u receive labels 1, 2, 3, 4. The fifth edge can only be labeled with 5 to get w(u) = 15, which is the maximum vertex- weight required. Labeling the end edges with 6, 7, . ., 10 in any order, no vertex would get a vertex weight 5, the minimum vertex- weight. Hence the claim is proved.

Now Figure 6 depicts (a, d)- antimagic labeling of SP(n : 2) when $n \leq 4$. ∎

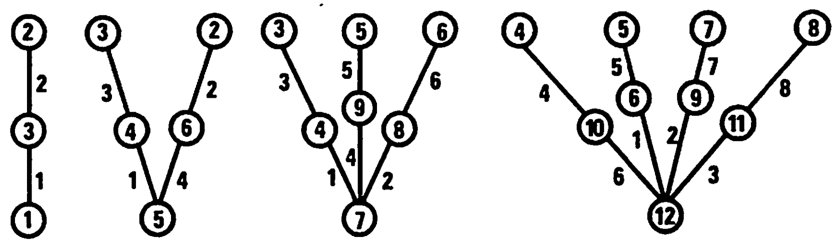

Figure 6

The converse of the Theorem 2.8 is not true. For example, SP(5 : 2) is not (a, d)- antimagic by the above claim in Theorem 2.9, though k = 5 and n = 2 satisfy $k \le 3n - 1$.

3. On (a, d)- antimagic labeling of $K_{m,n}$

In this section, we mainly focus on the (a, d)- antimagic labeling of the complete bipartite graphs $K_{n,n+2}$ and $K_{n,n}$.

Consider an (a, d)- antimagic labeling of $K_{m,n}$, $m \le n$, if it exists. Then the vertex weights form an arithmetic progression W = {a, a+d, a+2d, . . ., a+(m+n−1)d}. Let L denote the set of last m terms of the arithmetic progression W and L_d denote the sum of the terms of L. Then $L_d = (a+nd) + (a+(n+1)d) + \ldots + (a+(m+n-1)d)$. That is, $L_d = ma + d(mn + ½(m^2 - m))$. Let S denote the sum $1 + 2 + 3 + \ldots + q$. We know that $2S = a + (a+d) + \ldots + (a+(m+n-1)d)$. Then $S = ½(m+n)(a + ½(m+n-1)d)$. Let D_d denote the difference between L_d and S. That is, $D_d = L_d - S$. Then we have the following relation

$D_d = ½\,ma - ½\,na + d(mn + ½(m^2 - m) - ¼(m^2 + 2mn + n^2 - m - n))$.

That is, $D_d = ¼(m-n)(2a + d(m+n-1)) + ½\,d\,m\,n$. (3)

Lemma 3.1 If f is any (a, d)- antimagic labeling of $K_{m,n}$, $m \le n$, then $D_d \ge 0$.

Proof: Let W = {a, a+d, a+2d, . . ., a+(m+n−1)d} be the set of vertex weights corresponding to f. Suppose $D_d < 0$. Then $L_d < S$. That is, sum of last m terms (in fact the largest m terms of the arithmetic series) is less than S. This implies that the sum of any m terms of W does not add up to the required sum S for the m vertices of a partite set of $K_{m,n}$, a contradiction since f is an (a, d)- antimagic labeling of $K_{m,n}$. Hence $D_d \ge 0$. ∎

Now we obtain a necessary condition for $K_{m,n}$ to have (a, d)- antimagic labeling.

Theorem 3.2 If $K_{m,n}$, $m \le n$ has an (a, d)- antimagic labeling, then d divides D_d.

Proof: It is obvious when $D_d = 0$. Suppose $D_d > 0$. For any (a, d)- antimagic labeling of $K_{m,n}$, the sum of the vertex weights in each partite set must be S. Then few (or all) terms of L have to be exchanged with equal number of terms from the first n terms of W to adjust the excess D_d. Since each term of W differs from the other by a multiple of d and D_d is adjusted by shifting the terms of W, it follows that D_d must be a multiple of d. Hence the proof is complete. ∎

Theorem 3.3 If n is odd, then $K_{n,n}$ is not (a, d)- antimagic.

Proof: By putting m = n in equation (3), we get $D_d = \frac{1}{2} n^2 d$, which is not divisible by d when n is odd. Hence the result follows from Theorem 3.2. ∎

If the graph $K_{m,n}$ has an (a, d)- antimagic labeling, then $p = m+n$ and $q = mn$ must satisfy the equation (1) in Theorem 1.2 (a). Without loss of generality let $n = m+c$ where $c \in N \cup \{0\}$ is an integer. Then we get $2(m^2 + mc)(m^2 + mc+1) = (2m + c)(2a + (2m + c - 1)d)$. Therefore, $2(m^2+mc)(m^2+mc+1) \equiv 0 \pmod{2m+c}$. That is, $2(m^2+ mc)(m^2+ mc+1) = k(2m+c)$ for some integer k. This implies that $2m^4 - 2m^2 + 2m(2m+c) = (k - 2m^2c)(2m+c)$. This becomes that $2m^4 - 2m^2 = (k - 2m - 2m^2c)(2m+c)$. Thus $(k - 2m - 2m^2c) = (2m^4 - 2m^2) / (2m+c)$. Since the right hand side turns out to be an integer, by usual polynomial division, we get $c^2(c^2 - 4) \equiv 0 \pmod{2m+c}$, which is satisfied by c = 0, 2 among other values.

Theorem 3.4 If m + n is prime, then the complete bipartite graph $K_{m,n}$, where $n > m > 1$, is not (a, d)- antimagic.

Proof: Let $n = m + c$, where $c \in N$. Then $m + n = 2m+c$ and using the above remarks $2m+c$ divides $c^2(c^2-4)$. If $(2m+c)$ is prime, then $(2m+c)$ divides c or $(c-2)(c+2)$. If $(2m+c)$ divides c, then m = 0 which is not admissible. Suppose $(2m+c)$ divides $(c-2)(c+2)$. Since $(2m+c)$ cannot divide $(c - 2)$, it follows that $(2m+c)$ must divide $(c+2)$. This implies that m = 1, again not admissible. Hence the result is proved. ∎

We note that the case m = 1 has already been considered in [4].

Theorem 3.5 If $K_{n,n+2}$ is (a, d)- antimagic, then d is even and $(n+1) \le d < (n+1)^2/2$.

Proof: Let (X, Y) be the bipartition of $K_{n,n+2}$ where $|X| = n$ and $|Y| = n+2$. Here $p = 2(n+1)$ and $q = n(n+2)$ which when substituted in equation (1) of Theorem 1.2(a), we get $n(n+1)(n+2) = 2a + (2n+1)d$. (4)

Therefore, d is even for any n. Since the minimum degree of any vertex in $K_{n,n+2}$ is n, putting $a \geq \frac{1}{2}n(n+1)$ in equation(4), we get $d < (n+1)^2/2$. (5)

By Lemma 3.1, we have $L_d \geq S$, where $S = \frac{1}{2}(n(n+2)(n(n+2)+1)) = \frac{1}{2}n(n+1)^2(n+2)$ and $L_d = n(n+1)^2(n+2) - ((n+2)a + \frac{1}{2}d(n+1)(n+2))$. Then $n(n+1)^2 \geq 2a+(n+1)d$. (6)

Subtracting (4) from (6), we get $n+1 \leq d$. Hence the result follows from (5). ∎

Now the following corollaries are the immediate consequences of the above theorem.

Corollary 3.6 For any (a, d)- antimagic labeling of $K_{n,n+2}$, $D_d = 0$ if and only if $d = n+1$. Further, in this case n is odd.

Proof: For any (a, d)- antimagic labeling of $K_{n,n+2}$, $D_d = \frac{1}{2}((n^2-1)d - 2a)$ by equation (3). Now $D_d = 0$ if and only if $(n^2-1)d - 2a = 0$. Putting the value of 2a from equation (4), we get $(n^2-1)d + (2n+1)d - n(n+1)(n+2) = 0$. This in turn, is true if and only if $d = n+1$. In this case n is odd since d is even, by Theorem 3.5. ∎

Corollary 3.7 If n is even and $K_{n,n+2}$ has an (a, d)- antimagic labeling, then $D_d > 0$. ∎

Proof: If $D_d = 0$, then $d = n+1$ by Corollary 3.6. Since n is even, d is odd contradicting Theorem 3.5. Hence $D_d > 0$. ∎

In fact, for any (a, d)- antimagic $K_{n,n+2}$, when n is even, $d = n+2$ is the minimum value of d.

Corollary 3.8 $K_{2,4}$ is not (a, d)- antimagic.

Proof: Suppose $K_{2,4}$ is (a, d)- antimagic. Putting $n = 2$ in Theorem 3.5, we get $3 \leq d \leq 4$ which implies $d = 4$. Putting $d = 4$ in equation (4), we get $a = 2$, a contradiction since $a \geq 3$, by Theorem 1.2(b). Hence the result is proved. ∎

Theorem 3.9 Let $K_{n,n+2}$ be (a, d)- antimagic graph.

(i) If n is odd, then a is even and d divides a.

(ii) If n is even, then d divides 2a.

Proof: Let A, B be the bipartition of the vertex set of $K_{n,n+2}$, where $|A| = n$. Then $\sum\{w_f(v) \mid v \in A\} = \sum\{w_f(v) \mid v \in B\} = S = n(n+1)^2(n+2)/2$ (see proof of Theorem 3.5). Since $n(n+1)^2(n+2)/2$ is even for any n, we have $\sum\{w_f(v) \mid v \in A\}$ is even. This implies that $(a+r_1d) + (a+r_2d) + \ldots + (a+r_nd)$ is even, where $a+r_id$ are the weights of the vertices in A. Then $na + (r_1 + r_2 + \ldots + r_n)d$ is even, which implies that na is even. Since d is even and n is odd, we get a is even. Also $D_d = \frac{1}{2}((n^2-1)d - 2a)$ by equation (3). The fact that d must divide D_d implies that d divides a when n is odd and d divides 2a when n is even. ∎

Theorem 3.10 $K_{3,5}$ is (a, d)- antimagic if and only if (a, d) = (16, 4).

Proof: For $K_{3,5}$, equation (4) in Theorem 3.5 becomes $2a+7d = 60$ and the possible solutions for (a, d) are (9, 6) and (16, 4) subject to the conditions on d stated in Theorem 3.5. But a = 9 is not admissible by Theorem 3.9(i). Now we display (16, 4)- antimagic labeling of $K_{3,5}$ in Figure 7. ∎

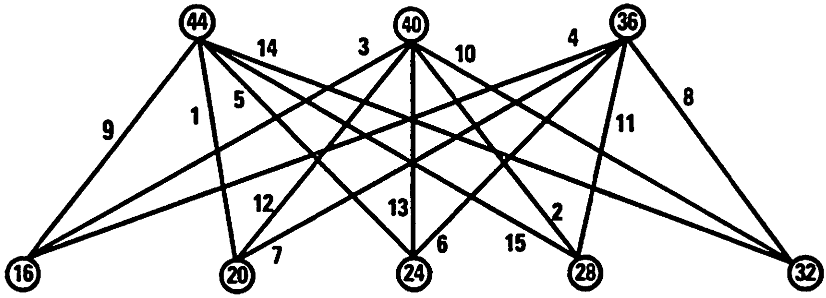

Figure 7. (16, 4) - antimagic labeling of $K_{3,5}$

Theorem 3.11 If $K_{n,n}$ is (a, d)- antimagic, then n and d are even and $0 < d < n^2/2$.

Proof: Suppose $K_{n,n}$ is (a, d)- antimagic. Then n is even by Theorem 3.3. Putting $p = 2n$ and $q = n^2$ in Theorem 1.2, we get $n(n^2+1) = 2a+(2n-1)d$ (7). This implies that d is even. Since $\delta(K_{n,n}) = n$, we have $a \geq \frac{1}{2}n(n+1)$ and (7) becomes $n(n^2+1) \geq n(n+1) + (2n-1)d$. That is, $d \leq n(n^2-1)/(2n-1) < n^2(n-1)/(2n-1) = n^2/2$. Hence the proof is complete. ∎

Example In Figure 8, (27, 2)- antimagic labeling of $K_{4,4}$ is illustrated.

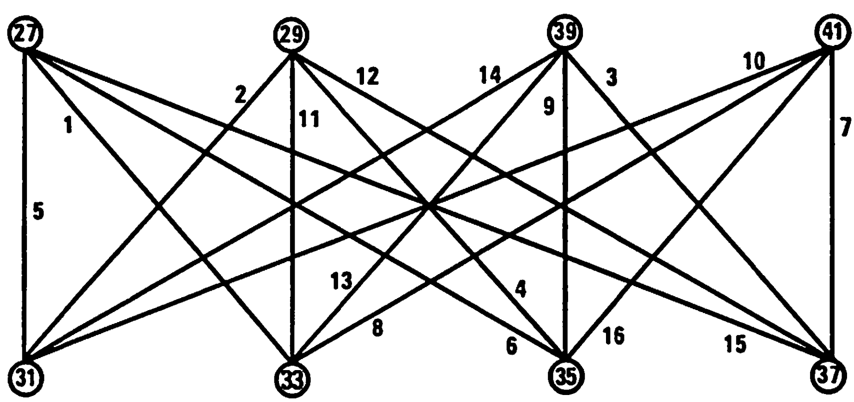

Figure 8 (27, 2) - antimagic labeling of $K_{4,4}$

Remark 3.12 In Table 1, all the possible (a, d)- antimagic labelings are enlisted for the graph $K_{n,n+2}$, $n \leq 10$. We do not know whether all the labelings enlisted in Table 1 exist. However we do know that no other (a, d)- antimagic labeling exists for $K_{n,n+2}$ other than those given in the Table 1.

n	n+2	Possible (a, d) - antimagic labelings	Remarks
1	3	Nil	Proved in [4]
2	4	Nil	Proved (Corollary 3.8)
3	5	(16, 4)	Proved (Theorem 3.10)
4	6	(33, 6), (15, 10)	?
5	7	(72, 6), (50, 10), (28, 14)	?
6	8	(116, 8), (90, 12), (77, 14)	?
7	9	(192, 8), (72, 24)	?
8	10	(275, 10), (258, 12), (207, 18), (190, 20), (156, 24), (105, 30), (54, 36)	?
9	11	(400, 10), (324, 18), (286, 22), (210, 30)	?
10	12	(534, 12), (450, 20), (429, 22), (345, 30), (198, 44)	?

Table 1. Possible (a, d)- antimagic labeling of $K_{n,n+2}$ for $n \leq 10$.

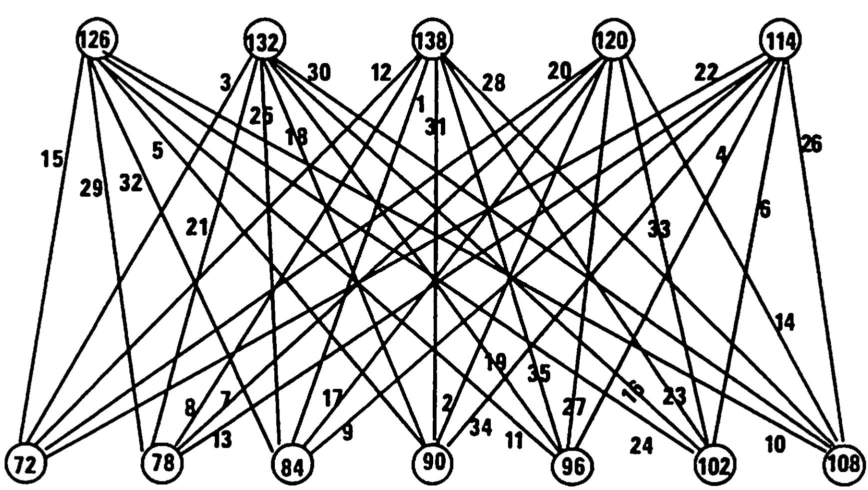

Figure 9. (72, 6) - antimagic labeling of $K_{5,7}$

It is felt that one can use trial and error method for the given values of n to achieve the (a, d)- antimagic labeling listed in Table 1. It would be of great interest to formulate a general rule to achieve the same. In Theorem 3.10, we have proved that $K_{3,5}$ is (16, 4)- antimagic. In Figure 9, we have illustrated a (72, 6)- antimagic labeling of the graph $K_{5,7}$. Hence we propose the following conjecture.

Conjecture 1 For any odd $n \geq 3$, the graph $K_{n,n+2}$ is $(\frac{1}{2}(n+1)(n^2-1), (n+1))$- antimagic.

4. On (a, d)- antimagic labeling of Unicyclic graphs.

It is observed by Bodendiek et al. [4] that even cycles are not (a, d)- antimagic. We give a categorization of the (a, d)- antimagic labeling of unicyclic graphs in the following theorem.

Theorem 4.1 Let G be an (a, d)- antimagic unicyclic graph of order n.

(i) If n is even, then (a, d) = (2, 2).

(ii) If n is odd, then (a, d) = (2, 2) or ((n+3)/2, 1).

Proof: Since p = q = n, equation (1) in Theorem 1.2 implies that $2(n+1) = 2a + (n-1)d$.

(i) If n is even, then d must be even. Since even cycles are not (a, d)- antimagic, $\delta(G) = 1$ which implies that $(n-1)d \leq 2n$ and hence d = 2 and a = 2.

(ii) If n is odd and G is not a cycle, then d = 1 or 2 which correspond to a = (n+3)/2 or 2 respectively. If G is an odd cycle, then $a \geq 3$ since $\delta(G) = 2$. Then $d \leq 2 - 2/(n-1) < 2$, which implies that d = 1 and a = (n+3)/2. ∎

The graph $C_n \odot mK_1$ is a unicyclic graph with p = q = n(m+1) obtained from the cycle C_n by attaching m pendent edges at each vertex of the cycle C_n.

Theorem 4.2 The graph $C_n \odot mK_1$ is (a, d)- antimagic if and only if m = 1.

Proof: When m = 1, $C_n \odot K_1$ is called crown that is proved (2, 2)- antimagic in [8]. Now to prove the necessary part, suppose $C_n \odot mK_1$ has (a, d)- antimagic labeling. We have to consider the following two cases:

CASE 1: n(m+1) is even.

Using Theorem 4.1(i), we have a = 2 = d. Then all the end edges should be labeled with distinct even positive integers. There are mn end edges and there are exactly $n(m+1)/2$ even integers between 1 and $n(m+1)$. Then $mn \le n(m+1)/2$ which implies $m = 1$.

CASE **2** : $n(m+1)$ is odd.

By Theorem 4.1(ii), $d = 1$ or 2. When $d = 2$ then $a = 2$ and hence as in CASE 1, $mn \le (n(m+1) - 1)/2$ which implies that $m \le (n-1)/n$. Hence $m < 1$ which is not admissible. When $d = 1$, $a = (n(m+1) + 3)/2$. Then all the integers from 1 to $((n(m+1) + 3)/2) - 1$ should be used to label only the edges of the cycle C_n. Then $((n(m+1)+3)/2) - 1 \le n$, implying that $m < 1$, again not admissible. Hence the proof is complete. ∎

The graph $G = C_n @ P_m$, called '**dragon**' consists of a cycle C_n together with a path P_m, one end vertex u_1 of P_m is joined with a node v_1 of C_n. That is, $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, \ldots, v_n\}$ of vertices of cycle C_n and $V_2 = \{u_1, u_2, \ldots, u_m\}$ of the path P_m. $E(G) = E(C_n) \cup E(P_m) \cup \{v_1u_1\}$. $C_n @ P_m$ contains m+n vertices and equal number of edges.

Theorem 4.3 If $m = n$ or $m = n-1$, the graph $G = C_n @ P_m$, $n \ge 3$, is (2, 2)- antimagic.

Proof: We define an edge labeling $f : E(G) \to \{1, 2, \ldots, n+m\}$ as follows:

$f(u_iu_{i+1}) = 2(m - i), i = 1, 2, \ldots, m-1.$

$f(v_1u_1) = 2m$

$f(v_iv_{i+1}) = 2i - 1,\ i = 1, 2, \ldots, n$. (Here $v_{n+1} = v_1$). Hence,

$w_f(V_1) = \{2(m+n), 4, 8, \ldots, 4(n-1)\}$ and $w_f(V_2) = \{4m-2, 4m-6, \ldots, 6, 2\}$.

Therefore, when $m = n$, $w_f(V) = \{4n, 4, 8, \ldots, 4(n-1)\} \cup \{4n-2, 4n-6, \ldots, 6, 2\}$.

When $m = n-1$, $w_f(V) = \{4n-2, 4, 8, \ldots, 4(n-1)\} \cup \{4n-6, 4n-10, \ldots, 6, 2\}$.

In both cases f is (2, 2)- antimagic labeling for $C_n @ P_m$. ∎

Example (2, 2)- antimagic labeling of $C_5@ P_5$ and $C_6@ P_5$ are shown in Figure 10.

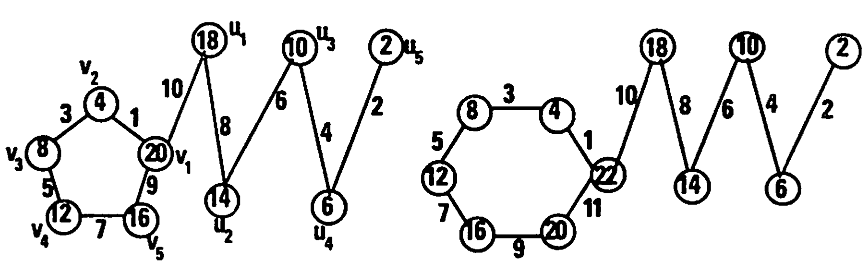

Figure 10. (2, 2) - antimagic labeling of $C_5@ P_5$ and $C_6@ P_5$

Theorem 4.4 Let $G = C_n @ P_m$ be of even order. G is (a, d)- antimagic if and only if $m = n$ or $n-1$.

Proof: If $m = n$ or $n-1$, then G is (2, 2)- antimagic, by Theorem 4.3. Conversely, suppose G is (a, d)- antimagic. By Theorem 4.1(i), G is only (2, 2)- antimagic. In this case, we can use even integers only to label the path edges and odd integers only to label the cycle edges. This is possible only when $m = n$ or $n-1$. Hence the result is proved. ∎

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