New Classes of Graceful Graphs *

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Abstract

We prove the gracefulness of two classes of graphs.

Let G be a graph with q edges. G is numbered if each vertex v is assigned a non-negative integer $\phi(v)$ and each edge uv is assigned the value $|\phi(u)-\phi(v)|$. The numbering is called graceful if, further, the vertices are labelled with distinct integers from $\{0,1,2,\ldots,q\}$ and the edges with integers from 1 to q. A graph which admits a graceful numbering is said to be graceful. For the literature on graceful graphs see [1, 2] and the relevant reference given in them.

Definition 1. Let G_p be a graph of order p with m edges and containing a path of length $p-1$, and let S_k be a star with k edges. Let $G_p \times S_{2^{n-1}-m}$, $m \leq 2^{n-1}-2$, be the graph obtained by identifying a vertex of G_p which is an end vertex of a path of

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length $p-1$ with any vertex of $S_{2^{n-1}-m}$ other than the centre of $S_{2^{n-1}-m}$.

Definition 2. Let G be a graceful graph with m edges. Then G has a graceful numbering ϕ such that $\phi(w) = m$ for some vertex w of G. Let $G \bullet S_{n-m}$ be the graph obtained by identifying the vertex w of G with any vertex of S_{n-m} other than the centre.

Recently Kathiresan [3] has proved that $K_n \times S_{2^{n-1}-\binom{n}{2}}$ are graceful for all values of n. In this article we prove the following general results.

Theorem 1. For all values of n, p, m with $p-1 \le m \le 2^{n-1}-2$, and for all graphs G_p of order p with m edges and containing a path of length $p-1$, the graphs $H(p,m,n) = G_p \times S_{2^{n-1}-m}$ are graceful.

Proof: Clearly $H(p,m,n)$ has $p + 2^{n-1} - m$ vertices and 2^{n-1} edges. Let u_i, $i = 0, 1, \ldots, p-1$ be the vertices of G_p such that $u_0, u_1, \ldots, u_{p-1}$ is a path of length $p-1$ in G_p and let v_i, $i = 0, 1, \ldots, 2^{n-1} - m$ be the vertices of $S_{2^{n-1}-m}$ where v_0 is the centre of the star and $u_{p-1} = v_{2^{n-1}-m}$. Denote by E_1 and E the edge set of G_p and $H(p,m,n)$ respectively.

Define ϕ on the vertices of $H(p,m,n)$ by the following rule.

$$\phi(u_i) = 2^i, i = 0, 1, \ldots, p-1,$$
$$\phi(v_0) = 0$$

and assign the $2^{n-1}-m-1$ numbers from the set $\{1, 2, \ldots, 2^{n-1}\}$ $-[\{2^j - 2^i | u_j u_i \in E_1, 0 \le i \le p-2, i+1 \le j \le p-1\} \cup \{2^{p-1}\}]$ to the vertices $v_1, v_2, \ldots, v_{2^{n-1}-m-1}$ of $S_{2^{n-1}-m}$ in any way so that each vertex receives exactly one number.

Note that $\{2^0, 2^1, \ldots, 2^{p-2}\} \subseteq \{2^j - 2^i | u_j u_i \in E_1, 0 \le i \le p-2, i+1 \le j \le p-1\}$. It can be verified readily that ϕ is a one-to-one map from the vertex set of $H(p,m,n)$ into $\{0, 1, \ldots, 2^{n-1}\}$.

The labels of the edges of G_p are $2^j - 2^i$ where $u_j u_i \in E_1$ with $0 \le i \le p-2$ and $i+1 \le j \le p-1$ and the labels of edges of the star are $\{1, 2, \ldots, 2^{n-1}\} - \{2^j - 2^i | u_j u_i \in E_1, 0 \le i \le p-2, i+1 \le$

$j \leq p-1\}$. Hence $\{|\phi(u)-\phi(v)||uv \in E\} = \{1, 2, \ldots, 2^{n-1}\}$. It follows that $H(p, m, n)$ is graceful. □

We extend the class of graceful graphs in Theorem 1 further. Let k be a positive integer. For any i with $1 \leq i \leq k$, take G as a disjoint union of $G_{p_1}, G_{p_2}, \ldots, G_{p_k}$ where G_{p_i} is a graph of order p_i with m_i edges and containing a path of length $p_i - 1$ for $1 \leq i \leq k$. $\sum_{i=1}^{k} m_i = m$, $\sum_{i=1}^{k} p_i = p$, $k \leq 2^{n-1} - m$. Let $G(p_1, \ldots, p_k) \times S_{2^{n-1}-m}$ be the graph obtained from G and $S_{2^{n-1}-m}$ by identifying a vertex of G_{p_i} which is an end vertex of a path of length $p_i - 1$ with a distinct vertex of $S_{2^{n-1}-m}$ other than the centre for each i with $1 \leq i \leq k$. Clearly $G(p_1) \times S_{2^{n-1}-m_1}$ is just $G_{p_1} \times S_{2^{n-1}-m_1}$.

Let $u_{0,i}, u_{1,i}, \ldots, u_{p_i-1,i}$ be a path of length $p_i - 1$ in G_{p_i} and let $v_i, i = 0, 1, \ldots, 2^{n-1} - m$ be the vertices of $S_{2^{n-1}-m}$ where v_0 the the centre of the star and $u_{p_i,i} = v_{2^{n-1}-m-i+1}$. For $1 \leq i \leq k$, let E_i be the edge set of G_{p_i} and $E'_i = \{2^t - 2^s | u_t u_s \in E_i, \sum_{r=1}^{i-1} p_r \leq s \leq \sum_{r=1}^{i} p_r - 2, s+1 \leq t \leq \sum_{r=1}^{i} p_r - 1\} \cup \{2^{\sum_{r=1}^{i} p_r - 1}\}$ with $\sum_{r=1}^{0} p_r = 0$. Define

$$\phi(u_{j,i}) = 2^{\sum_{r=1}^{i-1} p_r + j}, 1 \leq i \leq k, 0 \leq j \leq p_i - 1,$$
$$\phi(v_0) = 0$$

and assign the $2^{n-1} - m - k$ numbers from the set $\{1, 2, \ldots, 2^{n-1}\} - E'_1 \cup \cdots \cup E'_k$ to the vertices $v_1, v_2, \ldots, v_{2^{n-1}-m-k}$ of $S_{2^{n-1}-m}$ in any way so that each vertex receives exactly one number. By similar arguments as in Theorem 1, the graphs $G(p_1 \ldots, p_k) \times S_{2^{n-1}-m}$ are graceful.

Theorem 2. For all values n, m with $m \leq n - 2$, and for all graceful graphs G with m edges and a graceful numbering ϕ such that $\phi(w) = m$, the graphs $G \bullet S_{n-m}$ are graceful.

Proof: Note that $G \bullet S_{n-m}$ has n edges. Let $v_i, i = 0, 1, \ldots, n - m$ be the vertices of S_{n-m} where v_0 is the centre and $v_{n-m} = w$. Define $\bar{\phi}$ on the vertices of $G \bullet S_{n-m}$ as follows:

$$\bar{\phi}(u) = \phi(u) + 1, u \in V(G),$$
$$\bar{\phi}(v_0) = 0$$

and assign the $n-m-1$ numbers from the set $\{m+2,\ldots,n\}$ to the vertices $v_1, v_2, \ldots, v_{n-m-1}$ in any way so that each vertex receives exactly one number. Clearly $\bar{\phi}$ is a one-to-one map from the vertex set of $G \bullet S_{n-m}$ into $\{0,1,\ldots,n\}$ and $\{|\bar{\phi}(u) - \bar{\phi}(v)| | uv \in E(G \bullet S_{n-m})\} = \{1,2,\ldots,n\}$. Hence $G \bullet S_{n-m}$ is graceful. □

References

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