New Conditions for k-ordered Hamiltonian Graphs

Guantao Chen*
Georgia State University
Atlanta GA 30303

Ronald J. Gould
Emory University
Atlanta GA 30322

Florian Pfender
Emory University
Atlanta GA 30322
fpfende@mathcs.emory.edu

Abstract

We show that in any graph G on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y, we can fix the order of k vertices on a given cycle and find a hamiltonian cycle encountering these vertices in the same order, as long as $k < n/12$ and G is $\lceil (k+1)/2 \rceil$-connected. Further we show that every $\lfloor 3k/2 \rfloor$-connected graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y is k-ordered hamiltonian, i.e. for every ordered set of k vertices we can find a hamiltonian cycle encountering these vertices in the given order. Both connectivity bounds are best possible.

1 Introduction

One of the most widely studied classes of graphs are hamiltonian graphs. In this paper we are interested in the following question: When can we guarantee a certain set S of vertices to appear on a hamiltonian cycle in a given order? In [6], Ng and Schultz first explored the following related concept introduced by Chartrand. A graph is called *k-ordered hamiltonian*, if for every vertex set S of size k there is a hamiltonian cycle encountering the vertices in S in a given order. Clearly, every hamiltonian graph is 3-ordered hamiltonian. Ng and Schultz [6] showed that k-ordered hamiltonian graphs must be $(k-1)$-connected. Further, they showed the following theorem.

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Theorem 1 *[6] Let G be a graph of order n and let k be an integer with $3 \leq k \leq n$. If $d(u) + d(v) \geq n + 2k - 6$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

This bound was later improved in [3] and [4] by Faudree et al. for small values of k.

Theorem 2 *[4] Let G be a graph of order n and let k be an integer with $3 \leq k \leq n/2$. If $d(u) + d(v) \geq n + (3k - 9)/2$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

Instead of increasing the bound on the degree sum from the Ore-bound for hamiltonicity as in these papers, we choose to ask for a higher connectivity with the resultant effect of being able to lower the degree sum condition. We will first prove the following theorem.

Theorem 3 *Let G be a graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y. Let $k < n/12$ be an integer, and let C be a cycle encountering a vertex sequence $S = \{x_1, \ldots, x_k\}$ in the given order. If G is $\lceil (k+1)/2 \rceil$-connected, then G has a hamiltonian cycle encountering S in the given order.*

Corollary 4 *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Let $k < n/12$ be an integer, and let C be a cycle encountering a vertex sequence $S = \{x_1, \ldots, x_k\}$ in the given order. If G is $\lceil (k+1)/2 \rceil$-connected, then G has a hamiltonian cycle encountering S in the given order.*

The connectivity bound is best possible, as illustrated by the following graph G_1. Let L, K, R be complete graphs with $|R| = \lceil (2n - k)/4 \rceil$, $|K| = \lfloor k/2 \rfloor$, $|L| = n - |K| - |R|$. Let G_1 be the union of the three graphs, adding all possible edges containing vertices of K. Clearly, $\delta(G_1) > n/2$, and G_1 is $\lfloor k/2 \rfloor$-connected. Let $S = \{x_1, \ldots, x_k\}$ with $x_i \in K$ if i is even and $x_i \in R$ otherwise. The cycle $C = x_1 x_2 \ldots x_k x_1$ contains S in the right order, but no cycle containing S in the right order can contain any vertices of L.

A graph is called *k-ordered*, if for every vertex sequence S of size k there is a cycle encountering the vertices in S in the given order. Now observe that every k-ordered graph is $(k-1)$-connected. Thus, we get the following corollaries (these are very similar to theorems used in [6] and [4]).

Corollary 5 *Let G be a graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y. Let $k < n/12$ be an integer, and suppose that G is k-ordered. Then G is k-ordered hamiltonian.*

Corollary 6 *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Let $k < n/12$ be an integer, and suppose that G is k-ordered. Then G is k-ordered hamiltonian.*

We further prove the following theorem.

Theorem 7 *Let G be a graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y. Let $k \leq n/176$ be an integer. If G is $\lfloor 3k/2 \rfloor$-connected, then G is k-ordered hamiltonian.*

The connectivity bound is best possible, as illustrated by the following graph G_2. Let L_2, K_2, R_2 be complete graphs with $|R_2| = \lfloor k/2 \rfloor$, $|K_2| = 2\lfloor k/2 \rfloor - 1$, $|L_2| = n - |K_2| - |R_2|$. Let G_2' be the union of the three graphs, adding all possible edges containing vertices of K_2. Let $x_i \in L_2$ if i is odd, and let $x_i \in R_2$ otherwise. Add all edges $x_i x_j$ whenever $|i-j| \notin \{0, 1, k-1\}$, and the resulting graph is G_2. The degree sum condition is satisfied and G_3 is $(\lfloor 3k/2 \rfloor - 1)$-connected. But there is no cycle containing the x_i in the right order, since such a cycle would contain $2\lfloor k/2 \rfloor$ paths through K_2.

For the analogous theorem with a bound on the minimum degree we get a slight improvement on the connectivity bound for odd k.

Theorem 8 *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Let $k \leq n/176$ be an integer. If G is $3\lfloor k/2 \rfloor$-connected, then G is k-ordered hamiltonian.*

Again, the connectivity bound is best possible, as illustrated by the following graph G_3. Let L_3, K_3, R_3 be complete graphs with $|R_3| = \lceil (n-k)/2 \rceil$, $|K_3| = 2\lfloor k/2 \rfloor - 1$, $|L_3| = n - |K_3| - |R_3|$. Let G_3' be the union of the three graphs, adding all possible edges containing vertices of K_3. Let $x_i \in L_3$ if i is odd, and let $x_i \in R_3$ otherwise. Add all edges $x_i x_j$ whenever $|i - j| \notin \{0, 1, k-1\}$, and the resulting graph is G_3. The degree condition is satisfied, and G_3 is $(3\lfloor k/2 \rfloor - 1)$-connected. But there is no cycle containing the x_i in the right order, since such a cycle would contain $2\lfloor k/2 \rfloor$ paths through K_3.

2 Proof of Theorem 3

Assume that C is a maximal cycle encountering S in the given order. If C is hamiltonian, we are done. So, assume $|C| < n$, and let H be a component of $G - C$, say $|H| = r$. The sequence S splits C into k segments $[x_1 C x_2], \ldots, [x_k C x_1]$.

Claim 1 *There is at most one adjacency of H in each segment $[x_i C x_{i+1}]$.*

Suppose the contrary. Let x, y be two adjacencies of H inside $[x_i C x_{i+1}]$ with no other adjacencies of H in (xCy). Let $v \in H \cap N(x)$. Let $|(xCy)| = s$. Since v is not insertible in C we get

$$d(v) \leq r - 1 + \frac{n - r - s + 1}{2}.$$

Insert the vertices of (xCy) one by one into $[yCx]$. If all of them can be inserted, we can extend C through v, so there is a vertex w that can not be inserted. We get

$$d(w) \leq s - 1 + \frac{n - r - s + 1}{2},$$

so

$$d(v) + d(w) \leq n - 1,$$

a contradiction. This proves the claim. □

By claim 1, C has at most k adjacencies to H. Let $v \in H$, and $w \in C$ be a vertex not adjacent to H. Then

$$n \leq d(v) + d(w) \leq (r - 1 + k) + (n - r - 1) = n + k - 2.$$

Thus, w is adjacent to all but at most $k - 2$ vertices of $G - H$. Further, v is adjacent to all but at most $k - 2$ vertices in H. We claim that H is hamiltonian connected as follows: Either H is complete and we are done, or two vertices $v, u \in H$ are not adjacent. Then $|H| \geq \frac{d(v)+d(u)}{2} - k \geq \frac{n}{2} - k$, using Claim 1 and the degree sum condition. Now $\delta_H(H) \geq |H| - k + 2 > |H|/2 + 1$, which implies hamiltonian connectedness.

Claim 2 *$G - C$ has at most one component.*

Suppose the contrary, let H' be another component with $|H'| = r'$. Let $v \in H$, $v' \in H'$. Since G is $\lceil (k+1)/2 \rceil$-connected, H can be adjacent to at most $\lfloor (k-1)/2 \rfloor$ vertices from S, else there is a contradiction with Claim 1. The same is true for H'. Thus, for some i, $x_i \notin N(H) \cup N(H')$. But now,

$$\begin{aligned} 3n \leq 2(d(x_i) + d(v) + d(v')) \leq \\ 2((n - r - r' - 1) + (r - 1 + k) + (r' - 1 + k)) = \\ 2n + 4k - 6, \end{aligned}$$

a contradiction that proves the claim. □

Since G is $\lceil (k+1)/2 \rceil$-connected, there is a segment $[x_j C x_{j+2})$ with two adjacencies y, z of H. By claim 1, we may assume that $y \in [x_j C x_{j+1})$, and $z \in (x_{j+1} C x_{j+2})$. If $|H| \geq k$ we can even guarantee that $|(N(y) \cup N(z)) \cap H| \geq 2$.

Claim 3 $|C| \geq n/2$.

Suppose $|C| < n/2$. Then $|H| \geq n/2$, and y, z could be picked such that $uy, vz \in E(G)$ for two vertices $u, v \in H$. Find a hamiltonian path P in H

from u to v. Observe that $N(x_{j+1}) \cup N(x_{j+2}) \subseteq C$. If $x_{j+1}x_{j+2} \in E(G)$, then the cycle $uPvzC^{-}x_{j+1}x_{j+2}Cx_{j}u$ is longer than C, a contradiction. Thus, $x_{j+1}x_{j+2} \notin E(G)$. But now

$$|C| \geq \frac{d(x_{j+1}) + d(x_{j+2})}{2} + 2 > \frac{n}{2},$$

the contradiction proving the claim. □

For the final contradiction we differentiate two cases.

Case 1 *There exists a vertex* $w \in (yCx_{j+1}) \cup (zCx_{j+2})$.

Let $N = N(x_{j+1}) \cap N(x_{j+2}) \cap N(w)$. Since none of the vertices x_{j+1}, x_{j+2}, w is adjacent to H, each is adjacent to all but at most $k-2$ vertices of the cycle. Thus, $|N| \geq |C| - 3k + 6$.

Claim 4 *For some* i, $|N \cap [x_iCx_{i+1}]| \geq 4$.

Suppose not, then

$$n/2 \leq |C| \leq 3k + |C| - |N| \leq 6k - 6,$$

a contradiction for $n \geq 12k$. □

Let i be as in the last claim, and let $v_1, v_2, v_3, v_4 \in N \cap [x_iCx_{i+1}]$ be the first four of these vertices in that order.
If $v_4 \in (yCx_{j+1}]$, define a new cycle as follows:$C' = zC^{-}v_4x_{j+2}CyuPvz$ (see Figure 1).
If $v_4 \in (zCx_{j+2}]$, let $C' = zC^{-}x_{j+2}v_4CyuPvz$.
Otherwise observe that by claim 1, there is at most one adjacency x of H in $[v_1Cv_4]$.
For $i \neq j+1$, define the new cycle C' as follows:
If $x \in [v_1Cv_2]$, let $C' = zC^{-}x_{j+1}v_3x_{j+2}Cv_2wv_4CyuPvz$ (see Figure 2).
If $x \in [v_3Cv_4]$, let $C' = zC^{-}x_{j+1}v_2x_{j+2}Cv_1wv_3CyuPvz$.
Otherwise, let $C' = zC^{-}x_{j+1}v_2Cv_3x_{j+2}Cv_1wv_4CyuPvz$.
For $i = j+1$, a very similar construction works:
let $C' = zC^{-}v_4wv_1C^{-}x_{j+1}v_2Cv_3x_{j+2}CyuPvz$.
In any case, no vertex in $C - C'$ is adjacent to H, so all of them have high degree to C and thus high degree to $C \cap C'$. Therefore, we can insert them one by one into C' creating a longer cycle, a contradiction.

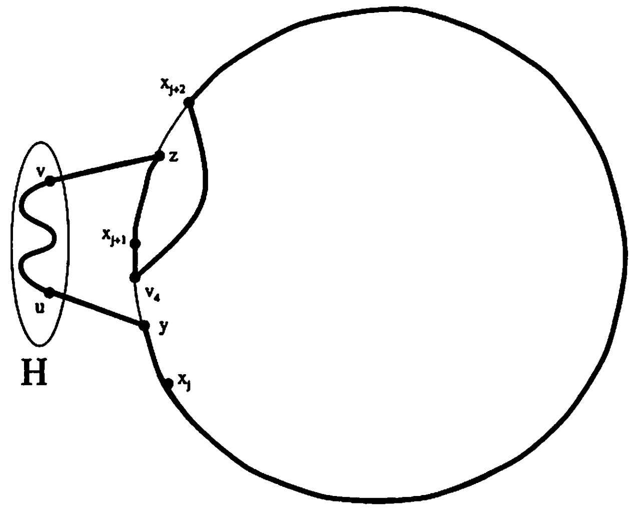

Figure 1: a possible C'

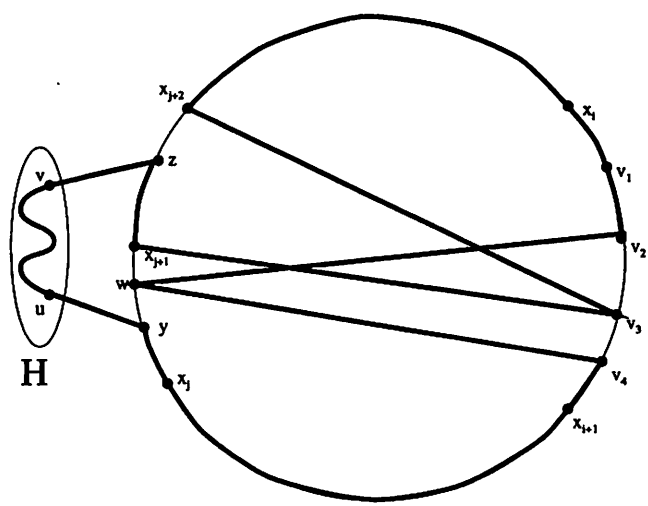

Figure 2: a possible C'

Case 2 *Suppose* $(yCx_{j+1}) \cup (zCx_{j+2}) = \emptyset$.

Let $N' = N(x_{j+1}) \cap N(x_{j+2})$. Then $|N'| \geq |C| - 2k + 4$.

Claim 5 *For some* l, $|N' \cap [x_l C x_{l+1}]| \geq 5$.

Suppose not. Then

$$n/2 \leq |C| \leq 4k + |C| - |N'| \leq 6k - 4,$$

a contradiction for $n \geq 12k$. □

Let l be as in the last claim, and let $z_1, z_2, z_3, z_4, z_5 \in N' \cap [x_l C x_{l+1}]$ be the first five of these vertices in that order. At most one of them is adjacent to H, say z_2. Now a very similar argument as in the last case gives the desired contradiction, just replace x_{j+1} by z_1, x_{j+2} by z_5, and w by z_4. One possible cycle would then be (for $l < i < j$): $C' = zC^- x_{j+1} z_2 C z_3 x_{j+2} C z_1 v_2 C v_3 z_5 C v_1 z_4 v_4 C y u P v z$ (see Figure 3). □

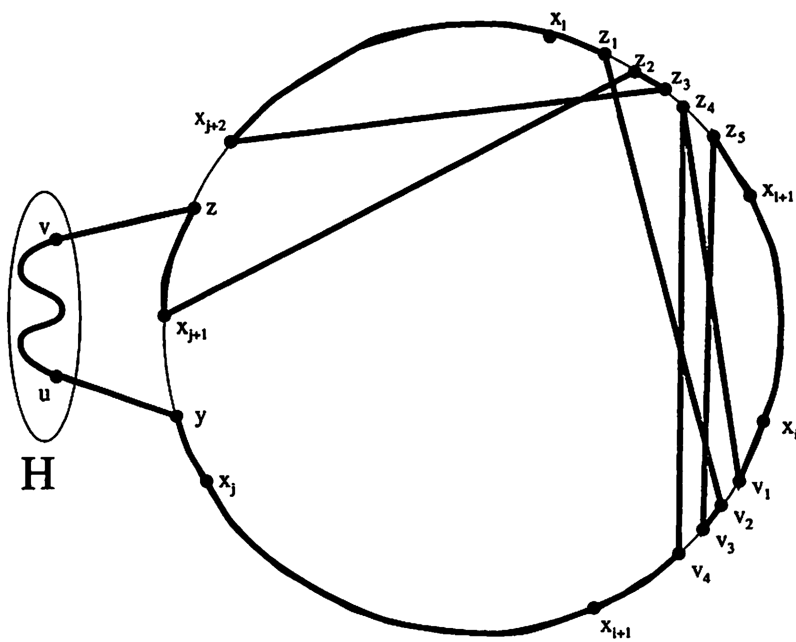

Figure 3: a possible C'

3 Proof of Theorems 7 and 8

By Corollary 5, all we need to show is that G is k-ordered. For this purpose, we will use a slightly stronger concept.

We will say that a graph G on at least $2k$ vertices is *k-linked*, if for every vertex set $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ of $2k$ vertices, there are k disjoint x_iy_i-paths. The property remains the same if we allow repetition in T, and ask for k internally disjoint x_iy_i-paths. Thus, as an easy consequence, every k-linked graph is k-ordered.

An important theorem about k-linked graphs is the following theorem of Bollobás and Thomason:

Theorem 9 *[1] Every $22k$-connected graph is k-linked.*

The following lemmas will be used later.

Lemma 10 *If a $2k$-connected graph G has a k-linked subgraph H, then G is k-linked.*

Proof: Let $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ be a set of $2k$ vertices in $V(G)$. Since G is $2k$-connected, there are $2k$ disjoint paths from T to $V(H)$ (trivial paths for vertices in $T \cap H$). Now we can connect these paths in the desired way inside H, since H is k-linked. □

Lemma 11 *If G is a graph, $v \in V(G)$ with $d(v) \geq 2k-1$, and if $G - v$ is k-linked, then G is k-linked.*

Proof: Let $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ be a set of $2k$ vertices in $V(G)$. If $v \notin T$, we can find disjoint x_iy_i-paths inside $G - v$. Thus assume that $v \in T$, without loss of generality we may assume that $v = x_1$. If $y_1 \in N(v)$, we can find disjoint x_iy_i-paths for all $i \geq 2$ in $G - v - y_1$, since $G - v - y_1$ is $(k-1)$-linked. Adding the path vy_1 completes the desired set of paths in G. If $y_1 \notin N(v)$, then there exists a vertex $x_1' \in N(v) - T$, since $d(v) \geq 2k-1$. We can find disjoint x_iy_i-paths for $i \geq 2$ and a $x_1'y_1$-path in $G - v$, which we can then extend to an x_1y_1-path in G. □

Further, we will use a theorem of Mader about dense graphs:

Theorem 12 *[5] Every graph G with $|V(G)| = n \geq 2k-1$, and $|E(G)| \geq (2k-3)(n-k+1)+1$ has a k-connected subgraph.*

Corollary 13 *[5] Every graph G with $|V(G)| = n \geq 2k-1$, and $|E(G)| \geq 2kn$ has a k-connected subgraph.*

Proof of Theorem 7. Let G be a graph fulfilling the stated conditions. Let $S = \{x_1, \dots, x_k\}$ be a set of k vertices. To show that G is k-ordered we need to find a cycle C including the vertices of S in the given order. Corollary 5 will then provide Theorem 7. Let K be a minimal cutset of G. Let L and R be two components of $G - K$ with $|L| \leq |R|$.

Case 1 *Suppose* $|K| \geq 2k$.

The degree sum condition forces $|E(G)| \geq n^2/4 \geq 44kn$. By Corollary 13, G has a $22k$-connected subgraph H, which is k-linked by Theorem 9. By Lemma 10, G is k-linked and thus k-ordered.

Case 2 *Suppose* $3\lfloor k/2 \rfloor \leq |K| \leq 2k - 1$.

First note that L and R are the only components of $G - K$. Otherwise, let $x \in L$, $y \in R$, $z \in G - (K \cup L \cup R)$, then

$$\begin{aligned} 3n &\leq 2d(x) + 2d(y) + 2d(z) \\ &\leq 2|L| + 2|K| + 2|R| + 2|K| + 2(n - |L| - |R|) \\ &\leq 2n + 4|K| < 2n + 8k, \end{aligned}$$

a contradiction.

Claim 1 *R is k-linked, and L is k-linked or complete.*

Let $v \in L, w \in R$. Then

$$n \leq d(v) + d(w) \leq |L| - 1 + |K| + |R| - 1 + |K| \leq n + 2k - 3.$$

Thus w is connected to all but at most $2k - 3$ vertices in R. Therefore, R is $2k$-connected. Again,

$$|E(R)| \geq |R|(|R| - 2k + 2) \geq |R|(n/2 - 3k + 2) \geq 44k|R|.$$

Thus, R has a $22k$-connected and therefore k-linked subgraph, and so R is k-linked by Corollary 13, Theorem 9 and Lemma 10.

If L is complete we are done. Otherwise, let $x, y \in L$ with $xy \notin E$, then

$$|L| \geq \frac{d(x) + d(y)}{2} - |K| \geq \frac{n}{2} - 2k + 1.$$

Every vertex in L is connected to all but at most $2k - 3$ vertices in L, therefore L is $2k$-connected. By a similar argument as before, L is k-linked, establishing the claim. □

Claim 2 *For every vertex $v \in K$, at least one of the following holds:*

1. $d_R(v) \geq 2k$,
2. $d_L(v) \geq 2k$,
3. $d_L(v) = |L|$.

Suppose the claim is false for some vertex $v \in K$. Let $x \in L - N(v)$, $y \in R - N(v)$. Then

$$\begin{aligned} 2n &\le d(x) + 2d(v) + d(y) \\ &< |L| + |K| + 2(|K| + 4k) + |R| + |K| \\ &\le n + 3|K| + 4k < n + 10k, \end{aligned}$$

a contradiction. □

The last claim yields a partition of K as follows:

$$\begin{aligned} K_R &= \{v \in K \mid d_R(v) \ge 2k\}, \\ K_{L1} &= \{v \in K \mid d_L(v) \ge 2k\} - K_R, \\ K_{L2} &= \{v \in K \mid d_L(v) = |L|\} - (K_R \cup K_{L1}). \end{aligned}$$

Note that either $K_{L1} = \emptyset$ or $K_{L2} = \emptyset$, and that the graph induced on K_{L2} is complete, since all vertices in K_{L2} have degree less than $4k$.

Now let $R' = \langle R \cup K_R \rangle, L' = \langle L \cup K_{L1} \cup K_{L2} \rangle$. By Claim 1, Claim 2 and Lemma 11, R' is k-linked and L' is k-linked or complete.

For the last part of the proof, let $S_L = L' \cap S$, $S_R = R' \cap S$. Create a new graph G' as follows: For every i with $x_i \in S_L$ and $x_{i-1}, x_{i+1} \in S_R$, add a vertex x_i' with $N(x_i') = N(x_i) \cup \{x_i\}$. It is easy to see that G' is $\lfloor 3k/2 \rfloor$-connected. Therefore, $G' - S_R$ is $(\lfloor 3k/2 \rfloor - |S_R|)$-connected. Using this fact, we can find independent paths in $G' - S_R$ from each of the vertices in $S_L \cup \bigcup x_i'$ into $R' - S_R$, since $|S_L \cup \bigcup x_i'| \le \min\{k, 2|S_L|\} \le 3k/2 - |S_R|$. Denote the set of last edges of these paths by M. Now contract the edges $x_i x_i'$ to get back to G.

The existence of the cycle C is now guaranteed, since we can pick appropriate vertices in $S_L \cup (M \cap L')$ and in $S_R \cup (M \cap R')$, and then use the fact that R' is k-linked and L' is k-linked or complete to find the necessary connections. This completes the proof of Theorem 7.

□

Proof of Theorem 8. Observe that the connectivity only played a role in the last part of the previous proof. Let G be a graph as in Theorem 8. If G is $\lfloor 3k/2 \rfloor$-connected, we are done by Theorem 7. Thus, we may assume that k is odd and G has a minimal cut set of size $3\lfloor k/2 \rfloor$. Further, we know that G splits in two parts L' and R', each of which is k-linked (observe that the degree condition forces $|L'| > 2k$) by the proof of Theorem 7.

Since k is odd, there are two consecutive vertices in S on the same side, we may assume x_1 and x_k is such a pair. Since G is $(3(k-1)/2)$-connected, there exists a matching $M = \{e_1, \dots, e_{3(k-1)/2}\}$ of edges between R' and

L'. We can renumber the edges of M such that $e_i \cap S \subseteq \{x_i\}$ for all $i \leq k-2$, and $e_{k-1} \cap S \subseteq \{x_{k-1}, x_k\}$. Let $x_{k+1} = x_1$. To construct the cycle C, we need to find $x_i x_{i+1}$-paths for all $i \leq k$. If $x_i \in L'$ and $x_{i+1} \in R'$, or if $x_i \in R'$ and $x_{i+1} \in L'$, we want to find a path from x_i to e_i through $L'(R')$ and a path from e_i to x_{i+1} through $R'(L')$. Note that this case can only occur if $i \leq k-1$. If $x_i, x_{i+1} \in L'(R')$, we want to find a $x_i x_{i+1}$-path in $L'(R')$. The simultanuous existence of all these paths is guaranteed since R' and L' are k-linked. This completes the proof of Theorem 8.

□

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