CONSTRUCTIONS OF GRAPHS WITHOUT NOWHERE-ZERO FLOWS FROM BOOLEAN FORMULAS

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ABSTRACT. We show that a negation of tautology corresponds to a family of graphs without nowhere-zero group- and integer-valued flows.

1. INTRODUCTION

A graph admits a *nowhere-zero k-flow* if its edges can be oriented and assigned numbers $\pm 1, \ldots, \pm(k-1)$ so that for every vertex, the sum of the values on incoming edges equals the sum on the outcoming ones. Graphs which do not admit nowhere-zero k-flows are called *k-snarks* (see [K2]). It is well-known that a graph with a bridge (1-edge-cut) is a k-snark for any $k \geq 2$ (see, e. g., [J, K2, Z]). We refer to [K2] for more details about k-snarks.

Nontrivial cubic 4-snarks are called *snarks* (see, e. g., [K2]). By nontrivial we mean cyclically 4-edge-connected (deleting fewer than k edges does not result in a graph having at least two cyclic components) and with girth (the length of the shortest cycle) at least 5. Holyer [H] constructed a snark for any negation of tautology given in a conjunctive normal form. In [K1] is shown that a similar construction can be applied for bridgeless 5-snarks if there exists at least one bridgeless 5-snark. In this paper we generalize ideas from [H, K1] and show that a negation of a tautology corresponds to a family of k-snarks.

2. PRELIMINARIES

The graphs considered in this paper are all finite and unoriented. Multiple edges and loops are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. By a *multi-terminal*

1991 *Mathematics Subject Classification.* 05C15.
Key words and phrases. Nowhere-zero k-flow, k-snark, boolean formula.

network, briefly a *network*, we mean a pair (G, U) where G is a graph and $U = (u_1, \ldots, u_n)$ is an ordered set of pairwise distinct vertices of G. If no confusion can occur, we denote by U also the set $\{u_1, \ldots, u_n\}$. The vertices from U and $V(G) \setminus U$ are called the *outer* and *inner* vertices of the network (G, U), respectively. We allow $n = 0$, i. e., $U = \emptyset$.

We associate with each edge of G two distinct arcs, distinct for distinct edges (see also [K2]). If one of the arcs corresponding to an edge is denoted by x, the other is denoted by x^{-1}. If the ends of an edge e are the vertices u and v, one of the arcs corresponding to e is said to be *directed from u to v* (and the other from v to u). In particular, a loop corresponds to two distinct arcs both directed from a vertex to itself. Let $D(G)$ denote the set of arcs on G. Then $|D(G)| = 2|E(G)|$. If $v \in V(G)$, then $\omega_G(v)$ denotes the set of arcs of G directed from v to $V(G) \setminus \{v\}$.

If G is a graph and A is an additive Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that

$$\partial\varphi(v) = \sum_{x \in \omega_G(v)} \varphi(x) \qquad (v \in V(G))$$

is called the *boundary* of φ. An A-chain φ in G is called *nowhere-zero* if $\varphi(x) \neq 0$ for every $x \in D(G)$. If H is a subgraph of G, then $\varphi_{|H}$ denotes the restriction of φ to H. If (G, U) is a network, then an A-chain φ in G is called an *A-flow* in (G, U) if $\partial\varphi(v) = 0$ for every inner vertex v of (G, U). The following statement is proved in [K1, K2].

Lemma 1. *If φ is an A-flow in a network (G, U), then $\sum_{u \in U} \partial\varphi(u) = 0$.*

If k is an integer ≥ 2, then by a *(nowhere-zero) k-flow* φ in a network (G, U) we mean a (nowhere-zero) $\mathbb{Z}$-flow in (G, U) such that $|\varphi(x)| < k$ for every $x \in D(G)$ and $|\partial\varphi(u)| < k$ for every $u \in U$.

With every A-flow in a network (G, U), $U = (u_1, \ldots, u_n)$, is associated a *characteristic vector* $\chi(\varphi) = (z_1, \ldots, z_n)$ so that $z_i = 0$ if $\partial\varphi(u_i) = 0$ and $z_i = 1$ otherwise. The *A-characteristic set* $\chi_A(G, U)$ (*k-characteristic set* $\chi_k(G, U)$) of the network (G, U) is the set of all characteristic vectors $\chi(\varphi)$ where φ is a nowhere-zero A-flow (nowhere-zero k-flow) in (G, U).

By a (*nowhere-zero*) *A-flow* and *k-flow* in a graph G we mean a (nowhere-zero) A-flow and k-flow in the network $(G, \emptyset)$, respectively. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in Jaeger [J] and Zhang [Z]. The following theorems are proved in [K2, Section 2] and generalize the classical results of Tutte [T1, T2].

Theorem 1. *Let (G, U) be a network and A be an Abelian group of order $k \geq 2$. Then (G, U) has a nowhere-zero k-flow iff (G, U) has a nowhere-zero A-flow. Furthermore, $\chi_k(G, U) = \chi_A(G, U)$.*

Theorem 2. *If a network (G, U) admits a nowhere-zero k-flow, then it admits a nowhere-zero $(k+1)$-flow. Furthermore, $\chi_k(G, U) \subseteq \chi_{k+1}(G, U)$.*

Thus the study of nowhere-zero k-flows is, in certain sense, equivalent to the study of nowhere-zero A-flows where A is an Abelian group of order k. But flows with values from finite groups are easier to handle than integral flows. With respect to this fact, we define a *(nowhere-zero) k-flow* and *k-chain* in a network (G, U) to be every (nowhere-zero) A-flow and A-chain in (G, U), respectively, where A is an Abelian group of order k. Similarly we shall use notation $\chi_k(G, U)$ instead of $\chi_A(G, U)$ (which is correct by Theorem 2).

3. Construction

A network is called *k-proper* (*k-improper*) if every vector from $\chi_k(G, U)$ has all coordinates equal 1 (0). In [K2, Proposition 2.2.] is proved the following.

Lemma 2. *If a graph G is a k-snark, then $(G, (u_1, u_2))$ is k-proper for every two different vertices u_1 and u_2 of G. Furthermore, if u_1 and u_2 are joined by an edge e, then $(G - e, (u_1, u_2))$ is k-improper.*

It is known that the Petersen graph P is a snark (see Fig. 1). Thus, by Lemma 2, $(P, (u_1, u_2))$ is a 4-proper network.

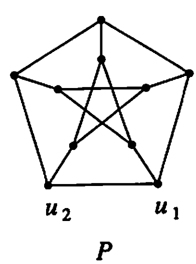

FIG. 1

A network (G, U) is called *k-even* (*k-odd*) if every vector from $\chi_k(G, U)$ has an even (odd) number of nonzero coordinates. The following statement is proved in [K2, Propositions 6.1 and 7.1].

Lemma 3. *Let G be a k-snark and H be the graph arising from G after deleting edges (u, u_1), (u, u_2), (u, u_3) where u, u_1, u_2, u_3 are pairwise different vertices of G. Then $(H, (u_1, \ldots, u_n))$ is k-even.*

Let v be a vertex of P and v_1, v_2, v_3 be the vertices of valency two in $P - v$ (see Fig. 2). By Lemma 3, $(P - v, (v_1, v_2, v_3))$ is a 4-even network. If $u_1, \ldots, u_n$ are the vertices of valency one in $K_{1,n}$ and $n \geq 2$ is even (odd), then $(K_{1,n}, (u_1, \ldots, u_n))$ is k-even (k-odd).

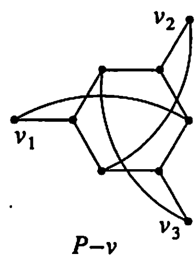

FIG. 2

A network $(G, (u_1, u_2, u_3))$ is called *k-inverting* if each of its k-characteristic vectors is either $(1, 0, 1)$, or $(0, 1, 1)$.

Proposition 1. *Let r be an odd (even) positive integer, $(G, (u_1, \ldots, u_{r+2}))$ be a k-even (k-odd) network and $(G', (u'_1, \ldots, u'_{r+1}))$ be a k-proper network. Identify the sets of vertices $\{u_1, u'_1\}, \ldots, \{u_r, u'_r\}$ to new vertices $v_1, \ldots, v_r$, respectively. Then the resulting network $(G'', (u_{r+1}, u_{r+2}, u'_{r+1}))$ is k-inverting.*

Proof. Follows directly from the definitions. □

A network $(G, (u_1, u_2, u_3))$ is called *k-oriented* if each of its k-characteristic vectors is either $(0, 0, 0)$, or $(1, 1, 0)$, or $(1, 1, 1)$.

Proposition 2. *Let $(G, (u_1, u_2, u_3))$ and $(G', (u'_1, u'_2, u'_3))$ be two distinct k-inverting networks and H the graph obtained after identifying the sets of vertices $\{u_2, u'_2\}$ and $\{u_3, u'_3\}$ to new vertices v_2 and v_3, respectively. Then the network $(H, (u_1, u'_1, v_3))$ is k-oriented.*

Proof. Let φ be a nowhere-zero k-flow in $(H, (u_1, u'_1, v_3))$. If $\partial\varphi(u_1) = \partial\varphi_{|G}(u_1) \neq 0$, then $\partial\varphi_{|G}(u_2) = -\partial\varphi_{|G'}(u'_2) = 0$ and $\partial\varphi_{|G'}(u'_1) = \partial\varphi(u'_1) \neq 0$ (if we write $\partial\varphi_{|G}$, we always mean $\partial(\varphi_{|G})$).

If $\partial\varphi(u_1) = \partial\varphi_{|G}(u_1) = 0$, then $\partial\varphi_{|G}(u_2) = -\partial\varphi_{|G'}(u'_2) \neq 0$ and $\partial\varphi_{|G'}(u'_1) = \partial\varphi(u'_1) = 0$. Furthermore, by Lemma 1, we have $\partial\varphi_{|G}(u_3) = -\partial\varphi_{|G}(u_2) = \partial\varphi_{|G'}(u'_2) = -\partial\varphi_{|G'}(u'_3)$, whence $\partial\varphi(v_3) = 0$.

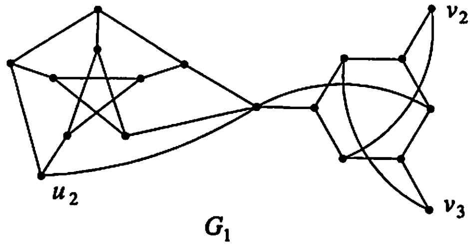

FIG. 3

Thus $\chi_k(H, (u_1, u_1', v_3)) \subseteq \{(0,0,0), (1,1,0), (1,1,1)\}$. $\square$

Consider the 4-proper network $(P, (u_1, u_2))$ from Fig. 1 and the 4-even network $(P - v, (v_1, v_2, v_3))$ from Fig. 2. Identify the vertices v_1 and u_1 to a new inner vertex. The resulting network $(G_1, (v_2, v_3, u_2))$ is 4-inverting by Proposition 2 (see Fig. 3). From two copies of this network we get a 4-oriented network $(G_2, (v_2, v_2', w_3))$ (see Fig. 4).

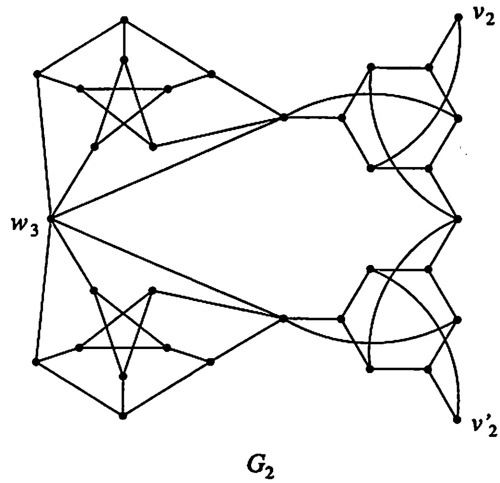

FIG. 4

A network (G, U) is called *k-stable* if every vector from $\chi_k(G, U)$ has all coordinates equal.

Proposition 3. *Let* $(G_1, (u_{1,1}, u_{1,2}, u_{1,3}))$, … , $(G_n, (u_{n,1}, u_{n,2}, u_{n,3}))$ *be k-oriented networks and G the graph obtained from* $G_1, \ldots, G_n$ *after identifying the sets of vertices* $\{u_{1,3}, u_{2,2}\}$, $\{u_{2,3}, u_{3,2}\}$, … , $\{u_{n-1,3}, u_{n,2}\}$, $\{u_{n,3},$

$u_{1,2}\}$ *to new vertices* $v_1, v_2, \ldots, v_{n-1}, v_n$*, respectively. Then the network* $(G, (u_{1,1}, u_{2,1}, \ldots, u_{n,1}))$ *is k-stable.*

Proof. Let φ be a nowhere-zero k-flow in the network $(G, (u_{1,1}, u_{2,1}, \ldots, u_{n,1}))$ and assume that there exists an index i so that $\partial\varphi(u_{i,1}) = 0$. Without loss of generality we can assume that $i = 1$. Then $\partial\varphi(u_{1,1}) = \partial\varphi_{|G_1}(u_{1,1}) = \partial\varphi_{|G_1}(u_{1,2}) = \partial\varphi_{|G_1}(u_{1,3}) = 0$. Therefore $\partial\varphi_{|G_2}(u_{2,2}) = -\partial\varphi_{|G_1}(u_{1,3}) = 0$ and $\partial\varphi_{|G_2}(u_{2,2}) = \partial\varphi_{|G_2}(u_{2,3}) = \partial\varphi_{|G_2}(u_{2,1}) = \partial\varphi(u_{2,1}) = 0$. Similarly, using induction, we can show that $\partial\varphi(u_{j,1}) = 0$ for every $j = 1, \ldots, n$. This implies the statement. □

A network $(G, (u_1, \ldots, u_n))$ is called *k-satisfaction testing* if u_n is k-proper.

Proposition 4. *Let* $(G, (u_1, \ldots, u_{r+n}))$ *be a network and let* $(G', (u'_1, \ldots, u'_{r+1}))$ *a k-proper network,* $r \geq 1$*. Identify the sets of vertices* $\{u_1, u'_1\}, \ldots, \{u_r, u'_r\}$ *to new vertices* $v_1, \ldots, v_r$*, respectively. Then the resulting network* $(G'', (u_{r+1}, \ldots, u_{r+n}, u'_{r+1}))$ *is k-satisfaction testing.*

Proof. Follows directly from the definitions. □

Construction 1. Suppose that C is a Boolean formula given in a conjunctive normal form, i. e., C is a set of clauses $\{C_1, \ldots, C_n\}$ in variables $y_1, y_2, \ldots, y_m$ and each clause C_i consists of literals $l_{i,1}, l_{i,2}, \ldots, l_{i,n_i}$ where a literal $l_{i,j}$ is either a variable y_s or its negation $\overline{y}_s$. If we have a truth assignment to the variables of C, then a clause is satisfied if at least one of its literals has value "true". C is satisfiable if there exists a truth assignment which satisfies all the clauses in C. Let C be not satisfiable.

Take a k-satisfaction testing network $(G_i, (u_{i,1}, \ldots, u_{i,n_i}, u_{i,n_i+1}))$ for each clause C_i of C. The outer vertices $u_{i,1}, \ldots, u_{i,n_i}$ are called the *outputs* of G_i.

Let y_s be a variable in C which appears in just p clauses of C. Then take a k-stable network $(H_s, (v_{s,1}, \ldots, v_{s,q}))$, $q \geq p$. The outer vertices $v_{s,1}, \ldots, v_{s,p}$ are called the *outputs* of H_s.

Assume that the operation of negation of a variable appears t times in C. Then, for $r = 1, \ldots, t$, let $(F_r, (w_{r,1}, w_{r,2}, w_{r,3}))$ be a k-inverting network. The vertices $w_{r,1}$ and $w_{r,2}$ are called the *outputs* of F_r.

Now we construct a new network (G, U) from these networks. The construction runs as follows. Suppose the literal $l_{i,j}$ in clause C_i is y_s. Then identify one output from G_i with one output from H_s to a new inner vertex (of (G, U)). If $l_{i,j}$ is $\overline{y}_s$, then take one F_r and identify one output from G_i with one output of this F_r to a new inner vertex and identify one output from H_s with the second output of F_r to another new inner vertex.

Repeating this process for all clauses and its literals we get (G, U). Its outer vertices are the outer vertices of all networks used in the construction which have not been outputs. We claim that (G, U) is a k-snark. On the contrary suppose there exists a nowhere-zero k-flow φ in (G, U). Then take a truth assignment to the variables of C so that a variable y_s has value "true" if $\partial\varphi_{|H_s}(v_{s,1}) \neq 0$, and value "false" otherwise. By Lemma 1, for each clause C_i, there exists at least one output $u_{i,j}$ of G_i so that $\partial\varphi_{|G_i}(u_{i,j}) \neq 0$. But then, in our assignment, each clause is satisfied, that means C is satisfiable. Thus (G, U) is a k-snark.

In some special cases we can deduce also a converse, namely that C is not satisfiable if (G, U) is a k-snark (see [H, K1]). Construction 1 gives nontrivial k-snarks for $k = 3, 4$, and if the 5-flow conjecture is false, then also for $k = 5$ (see [K1]).

Formula $y_1 \wedge \overline{y}_1$ is a negation of tautology and has literals $C_1 = y_1$ and $C_2 = \overline{y}_1$. Taking one copy of the 4-inverting network $(G_1, (v_2, v_3, u_2))$ from Fig. 3, two copies of the 4-satisfaction testing network $(K_2, (w_1, w_2))$, one copy of the 4-stable network $(K_3, (u_1, u_2))$, and applying Construction 1 we get a 4-snark $(G_3, (w_2, w_2', u_2))$ indicated in Fig. 5. Taking two copies of this network and identifying pairs of corresponding outer vertices to new inner vertices we get a bridgeless 4-snark.

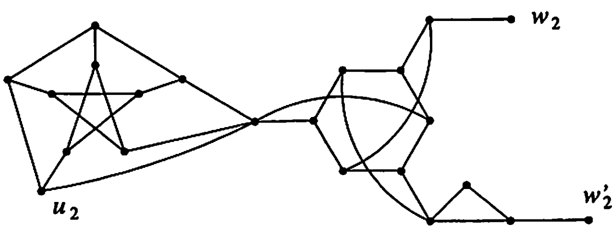

FIG. 5

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