CONTINUED FRACTIONS, STATISTICS, AND GENERALIZED PATTERNS

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ABSTRACT

Recently, Babson and Steingrímsson (see [BS]) introduced generalized permutations patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation.

Following [BCS], let $e_k\pi$ (respectively; $f_k\pi$) be the number of the occurrences of the generalized pattern 12-3-...-k (respectively; 21-3-...-k) in π. In the present note, we study the distribution of the statistics $e_k\pi$ and $f_k\pi$ in a permutation avoiding the classical pattern 1-3-2.

We also present some applications of our results which relate the enumeration of permutations avoiding the classical pattern 1-3-2 according to the statistics e_k and f_k to Narayana numbers and Catalan numbers.

1. INTRODUCTION

Permutation patterns: Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern* (or *classical pattern*). We say α *avoids* τ, or is τ*-avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids any $\tau \in T$; the corresponding subset of permutations of S_n which avoid T is denoted $S_n(T)$. We denote by $\mathcal{S}$(1-3-2) by set of all 1-3-2-avoiding permutations of all sizes including the empty permutation.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SS]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]), and for $\tau_1, \tau_2 \in S_4$ (see [B, K] and references therein). Several recent papers [CW, MV1, Kr, MV2, MV3] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2. Another natural question is to study permutations avoiding τ_1 and containing τ_2 exactly t times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $t = 1$ was investigated in [R], and for certain $\tau_1 \in S_3$, $\tau_2 \in S_k$ in [RWZ, MV1, Kr, MV2]. The tools involved in these papers include continued fractions, Chebyshev polynomials, and Dyck paths.

Generalized permutation patterns: In [BS] Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation.

We write a classical pattern with dashes between any two adjacent letters of the pattern, say 1324, as 1-3-2-4. If we write, say 24-1-3, then we mean that if this pattern occurs in permutation π, then the letters in the permutation π that correspond to 2 and 4 are adjacent (see [C]). For example, the permutation $\pi = 35421$ has only two occurrences of the pattern 23-1, namely the subsequences 352 and 351, whereas π has four occurrences of the pattern 2-3-1, namely the subsequences 352, 351, 342 and 341.

Claesson [C] gave a complete answer for the number of permutations avoiding any single 3-letters generalized pattern with exactly one adjacent pair of letters. Later, Claesson and Mansour [CM] presented a complete solution for the number of permutations avoiding any double 3-letters generalized patterns with exactly one adjacent pair of letters. Besides, Kitaev [Ki] investigated simultaneous avoidance of two or more 3-letters generalized patterns without internal dashes.

On the other hand, Robertson, Wilf and Zeilberger [RWZ] found a simple continued fraction that records the joint distribution of the patterns 1-2 and 1-2-3 on 1-3-2-avoiding permutations. Recently, generalization of this theorem were given by Mansour and Vainshtein [MV1], by Krattenthaler [Kr], by Jani and Rieper [JR], and by Brändén, Claesson and Steingrímsson [BCS].

In the present note, we generalize [M, Th. 2.1] and [M, Th. 2.9] and give an analogue for the continued fraction theorem from [BCS] for generalized patterns. We prove it by using the arguments from [BCS] with a simple changes. In the last section we present applications of our results.

2. Main results

For all $k \geq 1$, we denote by $a_k(\pi)$ the number of the occurrences of the pattern 1-2-3-…-k in π. In [BCS, Th. 1] the following result is proved.

Theorem 2.1. (P. Brändén, A. Claesson, and E. Steingrímsson, [BCS, Th. 1]) *The generating function* $\sum_{\pi \in S(1\text{-}3\text{-}2)} \prod_{k\geq 1} x_k^{a_k(\pi)}$ *is given by the following continued fraction:*

$$\cfrac{1}{1-\cfrac{x_1^{\binom{0}{0}}}{1-\cfrac{x_1^{\binom{1}{0}}x_2^{\binom{1}{1}}}{1-\cfrac{x_1^{\binom{2}{0}}x_2^{\binom{2}{1}}x_3^{\binom{2}{2}}}{\ddots}}}}$$

in which the $(n+1)$*st numerator is* $\prod_{k=0}^{n} x_{k+1}^{\binom{n}{k}}$.

For all $k \geq 3$, we denote by $e_k(\pi)$ (respectively; $f_k(\pi)$) the number of occurrences of the generalized pattern 12-3-…-k (respectively; 21-3-…-k) in π. Furthermore, let $e_2(\pi)$, $f_2(\pi)$, and $e_1(\pi) = f_1(\pi)$ denote the number of occurrences of the pattern 12, 21 and 1, respectively, and let $e_0(\pi) = f_0(\pi) = 1$ for all π.

We now present an analogue of Theorem 2.1 where the statistic a_k is replaced by e_k for all k.

Theorem 2.2. *The generating function* $\sum_{\pi \in S(1\text{-}3\text{-}2)} \prod_{k\geq 1} x_k^{e_k(\pi)}$ *is given by the following continued fraction:*

$$\cfrac{1}{1-x_1+x_1x_2^{\binom{0}{0}}-\cfrac{x_1x_2^{\binom{0}{0}}}{1-x_1+x_1x_2^{\binom{1}{0}}x_3^{\binom{1}{1}}-\cfrac{x_1x_2^{\binom{1}{0}}x_3^{\binom{1}{1}}}{1-x_1+x_1x_2^{\binom{2}{0}}x_3^{\binom{2}{1}}x_4^{\binom{2}{2}}-\cfrac{x_1x_2^{\binom{2}{0}}x_3^{\binom{2}{1}}x_4^{\binom{2}{2}}}{\ddots}}}}$$

in which the $(n+1)$*st numerator is* $x_1 \cdot \prod_{k=0}^{n} x_{k+2}^{\binom{n}{k}}$.

Proof. Let π be a nonempty permutation avoiding 1-3-2 such that $\pi = (\pi', n, \pi'')$ and $j = \pi^{-1}(n)$, and every element in π' is greater than every element in π''. Thus $\pi', \pi'' \in \mathcal{S}(1\text{-}3\text{-}2)$, so

$$e_k(\pi) = \begin{cases} e_k(\pi') + e_{k-1}(\pi') + e_k(\pi''), & \text{for all } k \geq 3; \\ e_2(\pi') + e_2(\pi'') + 1 - \delta_{\pi',\varnothing}, & k = 2; \\ e_1(\pi') + e_1(\pi'') + 1, & k = 1 \end{cases},$$

where $\delta_{\pi',\emptyset}$ is the Kronecker delta.

It follows that the generating function

$$C(x_1, x_2, \dots) = \sum_{\pi \in \mathcal{S}(1\text{-}3\text{-}2)} \prod_{k \geq 1} x_k^{e_k(\pi)}$$

satisfies (see [M, Th. 2.1])

$$C(x_1, x_2, \dots) = 1 + x_1 C(x_1, x_2, \dots) + \\ + x_1 x_2 C(x_1, x_2, \dots)(C(x_1, x_2 x_3, x_3 x_4, \dots) - 1),$$

which is equivalent to

$$C(x_1, x_2, x_3, x_4, \dots) = \frac{1}{1 - x_1 + x_1 x_2 - x_1 x_2 C(x_1, x_2 x_3, x_3 x_4, \dots)},$$

and the theorem follows now by induction. □

Similarly, it is easy to see that

$$f_k(\pi) = \begin{cases} f_k(\pi') + f_{k-1}(\pi') + f_k(\pi''), & \text{for all } k \geq 3; \\ f_2(\pi') + f_2(\pi'') + 1 - \delta_{\pi'',\varnothing}, & k = 2; \\ f_1(\pi') + f_1(\pi'') + 1, & k = 1 \end{cases},$$

where $\delta_{\pi'',\emptyset}$ is the Kronecker delta. Consequently, by using [M, Th. 2.9] and the argument of the proof of Theorem 2.1 we get the following result.

Theorem 2.3. *The generating function* $\sum_{\pi \in \mathcal{S}(1\text{-}3\text{-}2)} \prod_{k \geq 1} x_k^{f_k(\pi)}$ *is given by the following continued fraction:*

$$1 - \cfrac{x_1}{x_1 x_2^{\binom{0}{0}} - \cfrac{1}{1 - \cfrac{x_1}{x_1 x_2^{\binom{1}{0}} x_3^{\binom{1}{1}} - \cfrac{1}{1 - \cfrac{x_1}{x_1 x_2^{\binom{2}{0}} x_3^{\binom{2}{1}} x_4^{\binom{2}{2}} - \cfrac{1}{\ddots}}}}}}.$$

Following [BCS], we define $\mathcal{A}$ to be the ring of all infinite matrices with a finite number of non zero entries in each row, that is,

$$\mathcal{A} = \{A : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \mid \text{for all } n \ (A(n,k) = 0 \text{ for almost every } k)\},$$

with multiplication defined by $(AB)(n,k) = \sum_{i\geq 1} A(n,i)B(i,k)$. With each $A \in \mathcal{A}$ we now associate a family of statistics $\{< \mathbf{q}, A_k >\}_{k\geq 1}$ defined on $\mathcal{S}(1\text{-}3\text{-}2)$, where $\mathbf{q} = (q_1, q_2, q_3, \dots)$ and

$$< \mathbf{q}, A_k >= \sum_{i\geq 1} A(i,k) q_i.$$

Following [BCS], let us define mathematical objects with respect to A as follows. Let $\mathbf{q} = (q_1, q_2, \dots)$, where the q_i are indeterminates; for each $A \in \mathcal{A}$ and $\pi \in \mathcal{S}(1\text{-}3\text{-}2)$ we define three objects as follows:

The *weight* $\eta(\pi, A; \mathbf{q})$, the *weight* $\mu(\pi, A; \mathbf{q})$, and the *weight* $\nu(\pi, A; \mathbf{q})$ of π with respect to A, by

$$\eta(\pi, A; \mathbf{q}) = \prod_{k\geq 1} q_k^{<\mathbf{a}, A_k>\pi}, \qquad \mu(\pi, A; \mathbf{q}) = \prod_{k\geq 1} q_k^{<\mathbf{e}, A_k>\pi},$$

$$\nu(\pi, A; \mathbf{q}) = \prod_{k\geq 1} q_k^{<\mathbf{f}, A_k>\pi},$$

respectively, where $\mathbf{a} = (a_1, a_2, \dots)$, $\mathbf{e} = (e_1, e_2, \dots)$ and $\mathbf{f} = (f_1, f_2, \dots)$ are the statistics which we defined earlier.

The *generating function with respect to* A of the statistics $\{< \mathbf{a}, A_k > \}_{k\geq 1}$, statistics $\{< \mathbf{e}, A_k >\}_{k\geq 1}$, and statistics $\{< \mathbf{f}, A_k >\}_{k\geq 1}$ is respectively defined by

$$F_A(\mathbf{q}) = \sum_{\pi\in\mathcal{S}(1\text{-}3\text{-}2)} \eta(\pi, A; \mathbf{q}), \qquad G_A(\mathbf{q}) = \sum_{\pi\in\mathcal{S}(1\text{-}3\text{-}2)} \mu(\pi, A; \mathbf{q}),$$

$$H_A(\mathbf{q}) = \sum_{\pi\in\mathcal{S}(1\text{-}3\text{-}2)} \nu(\pi, A; \mathbf{q}).$$

The *continued fractions with respect to* A are respectively defined by

$$C_A(\mathbf{q}) = \cfrac{1}{1 - \cfrac{\prod_{k\geq 0} q_{k+1}^{A(1,k)}}{1 - \cfrac{\prod_{k\geq 0} q_{k+1}^{A(2,k)}}{\ddots}}},$$

$$D_A(\mathbf{q}) = \cfrac{1}{1 - q_1 + q_1 \prod_{k\geq 1} q_{k+1}^{A(1,k)} - \cfrac{q_1 \prod_{k\geq 1} q_{k+1}^{A(1,k)}}{1 - q_1 + q_1 \prod_{k\geq 1} q_{k+1}^{A(2,k)} - \cfrac{q_1 \prod_{k\geq 1} q_{k+1}^{A(2,k)}}{\ddots}}},$$

and by

$$E_A(\mathbf{q}) = 1 - \cfrac{q_1}{q_1 \prod_{k\geq 1} q_{k+1}^{A(1,k)} - \cfrac{1}{1 - \cfrac{q_1}{q_1 \prod_{k\geq 1} q_{k+1}^{A(2,k)} - \cfrac{1}{\ddots}}}}.$$

The main result in [BCS] can be formulated as follows.

Theorem 2.4. (P. Brändén, A. Claesson, and E. Steingrímsson, [BCS, Th. 2]) *For $A \in \mathcal{A}$, $F_A(\mathbf{q}) = C_{BA}(\mathbf{q})$ where $B = \left[\binom{i}{j}\right]$, and conversely $C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q})$.*

Given the above definitions, we get an analog of Theorem 2.4 for the statistics e_k and f_k as follows.

Theorem 2.5. *Let $A \in \mathcal{A}$. Then*

$$G_A(\mathbf{q}) = D_{BA}(\mathbf{q}), \qquad H_A(\mathbf{q}) = E_{BA}(\mathbf{q}),$$
$$D_A(\mathbf{q}) = G_{B^{-1}A}(\mathbf{q}), \qquad E_A(\mathbf{q}) = H_{B^{-1}A}(\mathbf{q}),$$

where $B = \left[\binom{i}{j}\right]$, $\binom{n}{k} = 0$ for all $k < 0$ or $n < k$.

Proof. By using the definitions and the same arguments as in the proof of Theorem 2 in [BCS], we obtain that

$$\begin{aligned} \mu(\pi, A; \mathbf{q}) &= \textstyle\prod_{k\geq 1} q_k^{<e,A_k>_\pi} \\ &= \textstyle\prod_{k\geq 1} \prod_{j\geq 1} q_k^{A(j,k)e_j(\pi)} \\ &= \textstyle\prod_{j\geq 1} \left(\prod_{k\geq 1} q_k^{A(j,k)}\right)^{e_j(\pi)}. \end{aligned}$$

Let $x_{j+1} = \prod_{k\geq 1} q_k^{A(j,k)}$. Theorem 2.2 yields

$$\prod_{j\geq 1} x_{j+1}^{\binom{n-1}{j-1}} = \prod_{j\geq 1} \left(\prod_{k\geq 1} q_k^{A(j,k)}\right)^{\binom{n-1}{j-1}} = \prod_{k\geq 1} q_k^{\langle(\binom{n-1}{0},\binom{n-1}{1},\binom{n-1}{2},\dots),A_k\rangle}.$$

Consequently, again, by definitions we have $G_A(\mathbf{q}) = D_{BA}(\mathbf{q})$. Observing that $B^{-1} = \left[(-1)^{i-j}\binom{i}{j}\right] \in \mathcal{A}$, we also obtain $D_A(\mathbf{q}) = G_{B^{-1}A}(\mathbf{q})$.

Similarly, we have $E_{BA}(\mathbf{q}) = H_A(\mathbf{q})$ and $E_A(\mathbf{q}) = H_{B^{-1}A}(\mathbf{q})$. □

Remark 2.6. *The general approach which is described in* [BCS] *for the statistics $a_k(\pi)$ also works with others statistics. Its works in particular for the statistics $e_k(\pi)$ and $f_k(\pi)$. So, a natural question to be asked is the following: Is there any description for all the statistics for which this approach will work? We re not able to offer an answer to this question.*

3. Applications

In the current section, we present examples of applications of the results in the above section.

3.1. Narayana numbers. Let $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ be the Narayana numbers. We denote the corresponding generating function for the Narayana numbers we denote by $N(x,t)$. Then

$$N(x,t) := \sum_{n,k\geq 0} N(n,k)x^k t^n = 1 + xtN^2(x,t) - xtN(x,t) + tN(x,t).$$

This allows us to express $N(x,t)$ as a continued fraction:

$$N(x,t) = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{tx}{1 - \cfrac{t}{1 - \cfrac{tx}{\ddots}}}}}.$$

Proposition 3.1. *The number $N(n,k)$ equals the number of permutations $\pi \in S_n(1\text{-}3\text{-}2)$ with $e_2(\pi) = k$.*

Proof. Let $A(n,k) = \delta_{(n,k),(1,1)} + \delta_{(n,k),(2,2)}$ where δ is Kronecker delta, so by applying Theorem 2.5 we get that

$$N'(x,t) := \sum_{\pi \in S(1\text{-}3\text{-}2)} x^{e_2(\pi)} t^{|\pi|} = \frac{1}{1 - t - xt + xtN'(x,t)} =$$

$$= \cfrac{1}{1 - t - xt + \cfrac{xt}{1 - t - xt + \cfrac{xt}{\ddots}}},$$

so $N'(x,t)$ satisfies the same functional equation as $N(x,t)$, hence $N'(x,t) = N(x,t)$. □

Again, the same argument work also to statistics on f_k as follows.

Proposition 3.2. (see Simion [S]) *The number $N(n,k)$ equals the number of permutations $\pi \in S_n(1\text{-}3\text{-}2)$ with $f_2(\pi) = k$.*

Proof. Let $A(n,k) = \delta_{(n,k),(1,1)} + \delta_{(n,k),(2,2)}$ where δ is Kronecker delta, so by applying Theorem 2.5 we get that

$$N''(x,t) := \sum_{\pi \in \mathcal{S}(1\text{-}3\text{-}2)} x^{f_2(\pi)} t^{|\pi|} = 1 - \cfrac{t}{xt - \cfrac{1}{1 - tN''(x,t)}} = $$

$$= 1 - \cfrac{t}{xt - \cfrac{1}{1 - \cfrac{t}{xt - \cfrac{1}{1 - \cfrac{t}{\ddots}}}}},$$

so $N''(x,t)$ satisfies the same functional equation as $N(x,t)$, hence $N''(x,t) = N(x,t)$. □

3.2. **Increasing subsequences.** Following to [BCS], we define as follows. The subsequence $\pi_{i_1}\pi_{i_1+1}\pi_{i_2}\dots\pi_{i_k}$ $(k \geq 2)$ of π is called *2-increasing subsequence* if $\pi_{i_j} < \pi_{i_{j+1}}$, $i_j < i_{j+1}$ and $i_1 + 1 < i_2$. Hence, the total number of 2-increasing subsequences in a permutation is counted by $e_2 + e_3 + \dots$. An application of Theorem 2.2 gives the following continued fraction for the distribution of $e_2 + e_3 + \dots$:

$$\sum_{\pi \in \mathcal{S}(1\text{-}3\text{-}2)} x^{e_2(\pi)+e_3(\pi)+\dots} t^{|\pi|} =$$

$$= \cfrac{1}{1 - t(1-x) - \cfrac{xt}{1 - t(1-x^2) - \cfrac{x^2t}{1 - t(1-x^4) - \cfrac{x^4t}{\ddots}}}}.$$

The subsequence $\pi_{i_1}\pi_{i_1+1}\pi_{i_2}\dots\pi_{i_k}$ $(k \geq 2)$ of π is called *almost 2-increasing subsequence* if $\pi_{i_j} < \pi_{i_{j+1}}$, $i_j < i_{j+1}$ for $j = 2, 3, \dots, k-1$, $i_1 + 1 < i_2$ and $\pi_{i_1+1} < p_{i_1}$. Hence, the total number of almost 2-increasing subsequences in a permutation is counted by $f_2 + f_3 + \dots$. An application of Theorem 2.3 gives the following continued fraction for the distribution of $f_2 + f_3 + \ldots$:

$$\sum_{\pi \in \mathcal{S}(1\text{-}3\text{-}2)} x^{f_2(\pi)+f_3(\pi)+\cdots} t^{|\pi|} = 1 - \cfrac{t}{xt - \cfrac{1}{1 - \cfrac{t}{x^2t - \cfrac{1}{1 - \cfrac{t}{x^4t - \cfrac{1}{1 - \cfrac{t}{x^8t - \ddots}}}}}}}$$

3.3. **Catalan numbers.** The nth Catalan number is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$, and the corresponding generating function is given by $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. Theorem 2.5 yields for the statistic $s = 0$ ($s(\pi) = 0$ for all π) the following. The generating function $C(x)$ for the number of permutations avoiding 1-3-2 can be expressed, again, in terms of continued fractions:

$$C'(x) := \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}}}, \qquad C''(x) := 1 - \cfrac{x}{x - \cfrac{1}{1 - \cfrac{x}{x - \cfrac{1}{\ddots}}}}$$

In the above two cases $C'(x) = C''(x) = C(x)$.

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