PARTIAL PARALLELISMS WITH SHARPLY TWO-TRANSITIVE SKEW SPREADS

NORMAN L. JOHNSON   ROLANDO POMAREDA

Abstract

Partial parallelisms that admit a collineation group that fixes one spread Σ, fixes a line of it and acts sharply two-transitive on the remaining lines of Σ are completely classified.

1 Introduction.

A 'parallelism' in $PG(3,q)$ is a set of $1+q+q^2$ spreads of $PG(3,q)$ which forms a covering of the line set.

Various of the known infinite classes of parallelisms can be constructed using groups.

For example, the two infinite classes of Penttila and Williams [16] admit a collineation group transitive on the spreads of the parallelism. In this situation, the spreads are all Desarguesian.

At least some of the various infinite classes of Johnson [6], [7] and [8] can be obtained from the following construction process:

Let Σ, Σ' be two spreads of $PG(3,q)$ containing a given regulus R and let G be a collineation group of Σ which has the following properties:

(i) G fixes a line ℓ of R and is sharply 2-transitive on the lines of $\Sigma-\{\ell\}$,

(ii) G acts regularly on the set of lines not in Σ which are disjoint from ℓ,

(iii) G_R acts on Σ' and is regular on $\Sigma'-R$.

Furthermore, when such a collineation group exists, a parallelism may be obtained as follows: Let Σ'^* denote the spread obtained by the derivation of R. Then

$$\Sigma \cup_{g\in G} \Sigma'^* g$$

is a parallelism.

The Johnson parallelisms [6] may be obtained for a particular group G where both Σ and Σ' are Desarguesian. Furthermore, in [12], the authors completely enumerate the set of parallelisms which may be constructed in this manner when both Σ and Σ' are Desarguesian spreads and where G is a central collineation group of Σ with fixed axis ℓ.

The first author recently completely determined the parallelisms in $PG(3,q)$ which admit two-transitive groups (see [9]). In this situation, it turns out that the only possible parallelisms are the two parallelisms in $PG(3,2)$, which are equivalent to the Lorimer-Rahilly and Johnson-Walker translation planes of order 16.

Furthermore, Biliotti, Jha and Johnson [1] determine the possible collineation groups which can act transitively on the spreads of a parallelism.

Note that in the above construction process, the group G fixes a spread Σ and acts transitively on the remaining spreads of the parallelism. In this article, we determine the parallelisms which admit such groups. To be more precise, we provide some definitions and terminology.

Definition 1 *A 'partial parallelism' in $PG(3,q)$ is a set of spreads which are mutually disjoint on lines.*

A spread Σ 'skew' to a partial parallelism $\mathcal{P}$ is a spread such that $\{\Sigma\}\cup\mathcal{P}$ is a partial parallelism.

A collineation group of a partial parallelism is a subgroup of $\Gamma L(4,q)$ which permutes the spreads of the partial parallelism.

A 'skew collineation group' of a partial parallelism $\mathcal{P}$ is a collineation group of $\mathcal{P}$ which is also a collineation group of a skew spread Σ to $\mathcal{P}$.

Our main result determines the partial parallelisms which admit a skew collineation group that fixes a line of a skew spread Σ and acts sharply 2-transitive on the remaining lines of Σ and is given as follows.

Theorem 2 *Let $\mathcal{P}$ be a partial parallelism in $PG(3,q)$, for $q>2$ and $q\neq 8$ which admits a skew spread Σ. If there exists a skew collineation group of $\mathcal{P}$ and of Σ that fixes a component ℓ and acts sharply 2-transitively on the components of $\Sigma-\{\ell\}$.*

Then

(i) Σ is Desarguesian.

(ii) $\{\Sigma\}\cup\mathcal{P}$ is a parallelism in $PG(3,q)$.

(iii) For each regulus R of Σ containing ℓ, there is a unique spread Σ' of $\mathcal{P}$ containing the opposite regulus R^* of R. In addition, G_R fixes Σ' and is regular on the components $\Sigma'-R^*$.

(iv) G acts transitively on the spreads of $\mathcal{P}$.

(v) The spreads of $\mathcal{P}$ are spreads which may be derived from conical flock spreads that admit a collineation group that fixes a base regulus and acts transitively on the remaining $q-1$ base reguli.

(vi) If G is linear (i.e. in $GL(4,q)$) then the spreads of $\mathcal{P}$ are Hall spreads.

(vii) If $q = p^r$ where $(r, q^2 - 1) = 1$ then the spreads of $\mathcal{P}$ are Hall spreads.

2 Skew Collineation Groups.

We assume the hypothesis of the theorem stated in the introduction and give the proof as a series of lemmas.

Lemma 3 *Σ is Desarguesian.*

Proof. First of all note that G is a subgroup of $\Gamma L(4, q)$ of order exactly $q^2(q^2 - 1)$. Let $q = p^r$. Consider $G \mid \ell \leq \Gamma L(2, q)$. Suppose there are no elations in G with axis ℓ. Then q^2 must divide qr which implies that q divides r so that that $(p, r) = (2, 1)$. Since we have assumed that $q > 2$, we must have an elation in G. But, G is 2-transitive on the components of $\Sigma - \{\ell\}$ which implies that the set of elations with axis ℓ is transitive on $\Sigma - \{\ell\}$. It follows that the spread is a semifield spread.

Thus, by Cordero-Figueroa [3], Σ is a generalized twisted field spread as $q \neq 8$.

Thus, we now have a generalized twisted field spread within $PG(3, q)$, which implies, in the notation of Biliotti, Jha and Johnson [2] that when

$$x * m = xm - cx^{p^a} m^{p^b}$$

defines the generalized twisted pre-semifield then $a = r$ or $2r$ (in which case the plane is Desarguesian) so $a = r$. In this case, the results of Biliotti, Jha and Johnson show that $(2r, a) = (2r, a, b)$ and since $a = r$, it follows that $(2r, r, b) = r$ so that $b = r$ or $2r$. In either case, we have a Desarguesian spread. ∎

Lemma 4 *Let E denote the elation group obtained in the proof of the previous lemma. Then each E orbit of spreads of $\mathcal{P}$ has length q.*

Proof. Consider a spread Σ' of $\mathcal{P}$. Then $E_{\Sigma'}$ has order dividing q. To see this, we note that the axis of E appears in the affine plane corresponding to Σ' as a Baer subplane fixed pointwise by $E_{\Sigma'}$. By the structure of Baer groups of Foulser [4], the assertion follows. Assume that the length of the E orbit of Σ', $O(\Sigma')$, is strictly larger than q. We note that E is normal in G so that the E-orbits of a particular length of spreads of $\mathcal{P}$ are permuted by G. Since the number of spreads on $\mathcal{P}$ is $\leq q(q + 1)$, there are say k E-orbits of length $O(\Sigma')$ so that $1 \leq k < q + 1$.

Therefore, the stabilizer of Σ has order which is divisible by

$$|E_{\Sigma'}|\,(q^2 - 1)/(k, q^2 - 1).$$

If $G = EH$, for H a subgroup which fixes two components of Σ, then clearly $H_{\Sigma'}$ normalizes $E_{\Sigma'}$. Since EH is sharply 2-transitive on the components of $\Sigma - \{\ell\}$, then each non-identity element of H fixes exactly two components of Σ. Thus, $H_{\Sigma'}$ normalizes $E_{\Sigma'}$ and no non-identity element of $H_{\Sigma'}$ can centralize a non-identity element of $E_{\Sigma'}$ as non-trivial elations fix exactly one component. This implies that $|H_{\Sigma'}| < q - 1$ since $|E_{\Sigma'}| < q$. However, $k < q + 1$ so that

$$\begin{aligned} q - 1 &> (q^2 - 1)/(k, q^2 - 1) \geq (q^2 - 1)/k \\ &> (q^2 - 1)/(q + 1) = q - 1. \end{aligned}$$

Hence, we have a contradiction and thus the proof. ■

Lemma 5 *There is a Dickson nearfield $(D, +, *)$ of order h^n and kernel $GF(h)$ where $h^n = q^2$ such that for $G = EH$ then H may be chosen to have the following form:*

$$\langle \sigma_m : (x, y) \longmapsto (x^{\sigma^m} m^i, (y * m = y^{\sigma^m} m) m^i); m \in GF(q^2) - \{0\} \rangle$$

for i a fixed integer between 0 and $q^2 - 1$.

Proof. See Johnson and Pomareda [13]. ■

Lemma 6 *(i) $\{\Sigma\} \cup \mathcal{P}$ is a parallelism in $PG(3, q)$ and G acts transitively on the $q(q + 1)$ spreads of $\mathcal{P}$.*

(ii) The spreads of $\mathcal{P}$ are derived conical flock spreads.

(iii) Furthermore, there is a set of $q + 1$ elation subgroups E_i each of order q, for $i = 1, 2, ..., q + 1$, such that $E - \langle 1 \rangle = \cup_{i=1}^{q+1} (E_i - \langle 1 \rangle)$.

Proof. Each spread of $\mathcal{P}$ is fixed by an elation group E^- of order exactly q. This group acts on the fixed spread as a Baer group of order q. This implies by Johnson [5] and Payne and Thas [15] that the net of degree $q + 1$ defined by the Baer group is, in fact, a regulus net and the plane obtained by derivation of this regulus net is a conical flock plane.

Consider any orbit of E of spreads of $\mathcal{P}$. Any such orbit has length q and there is a subgroup E^- of order q which fixes a given spread of this orbit. Since E is elementary Abelian, it follows directly that E^- fixes each spread of this orbit.

Now consider E^- acting on the spread of Σ. We know from Johnson [5] that E^- is 'regulus-inducing' in the sense that the axis of E^- and the E^--orbit of any 2-subspace disjoint from the axis defines a regulus. Since G acts 2-transitively on the components not equal to ℓ of Σ, it follows that G acts transitively on the set of $q(q + 1)$ reguli of Σ that contain ℓ. The subgroup E^- defines a set of q reguli of Σ containing ℓ. The stabilizer of a regulus R of Σ containing ℓ has order $q(q - 1)$. Thus, there is a set of $q + 1$

'regulus-inducing' groups $\{E_i; i = 1, 2, ..., q+1\} = \{E^{-}g; g \in H\}$, which are clearly disjoint.

Note that each E_i fixes a set of exactly $q(q+1)$ Baer subplanes of Σ (of the associated affine plane) and these are components of spreads of $\mathcal{P}$. Hence, each E_i fixes spreads of $\mathcal{P}$ and we have seen above that E_i fixes qz_i spreads. Since $\langle E_i, E_j \rangle$ for $i \neq j$ is E, it follows that E_i cannot fix a spread fixed by E_j. Hence, E_i fixes exactly q spreads and the group H acting transitively on the set $\{E_i; i = 1, 2, ..., q+1\}$ is also transitive on the set of spreads fixed by the subgroups E_i. This implies that EH is transitive on the spreads of $\mathcal{P}$ and the number of these spreads is $q(q+1)$. Thus, $\{\Sigma\} \cup \mathcal{P}$ is a parallelism. This completes the proof of the lemma. ■

Lemma 7 *For each regulus R of Σ incident with ℓ, there is a unique spread in $\mathcal{P}$ containing the opposite regulus R^*. This spread is a derived conical flock spread which is derived using the regulus R.*

Proof. We have seen that the spreads of $\mathcal{P}$ are derived conical flock spreads with associated derived reguli containing ℓ (the opposite regulus contains ℓ as a Baer subplane). What we don't yet know is whether these reguli are reguli of Σ.

In a Dickson nearfield of order $h^n = q^2$, there is a cyclic subgroup of order $(q^2-1)/n$. In the representation of the group H mentioned above, we consider the stabilizer of the standard regulus R of Σ corresponding to $GF(q) \cup (\infty)$ where we take ℓ as $x = 0$ and Σ coordinatized by $GF(q^2)$. Now H_R has the following form:

$$\langle \sigma_m : (x,y) \longmapsto (x^{\sigma_m} m^i, (y * m = y^{\sigma_m} m) m^i); m \in GF(q) - \{0\} \rangle$$

for i a fixed integer between 0 and $q^2 - 1$.

Now we have the kernel homology group $Kern$ of order $q-1$ acting both on Σ and the parallelism. Hence, by multiplication of the kernel homologies $(x,y) \longmapsto (xm^{-i}, ym^{-i})$, we obtain the corresponding group in $H_R Kern$:

$$\langle (x,y) \longmapsto (x^{\sigma_m}, (y * m = y^{\sigma_m} m)); m \in GF(q) - \{0\} \rangle.$$

We note that $\sigma_m = 1$ if and only if $m \in C_{(q^2-1)/n}$ where the notation denotes the cyclic subgroup of $GF(q^2) - \{0\}$ of order $(q^2-1)/n$.

We assert that, for some $m \in GF(q) - \langle 1 \rangle$, $m \in C_{(q^2-1)/n}$. If this were not the case, then $((q^2-1)/n, q-1) = 1$ which implies that $(q^2-1, n) = (q-1)(2, q-1)$. Letting $h = p^t$, this says that

$$p^{t(q-1)(2,q-1)} \text{ divides } q^2.$$

However, $q - 1 \geq r$ for all $q = p^r$. Hence, either $q - 1 = r$ and $q = 2$ or $q - 1 > r$ so that $t = 1$ and $q - 1 = 2r$, implying that $q = 3$. But, then $3^{2 \cdot 2} = 3^2$, a contradiction.

So, there is an element acting on the spread of the form:

$$\tau : (x,y) \longmapsto (x, ym) \text{ for } m \in GF(q) - \{0\}.$$

This element normalizes a group E^- which fixes exactly q spreads of $\mathcal{P}$. Now τ acts on a spread Σ' fixed by E^- where E^- is a Baer group of order q acting on Σ'. Hence, the associated regulus net has $q+1$ Baer subplanes incident with the zero vector and τ must fix two of these Baer subplanes (assuming that the order of τ is a prime). Consider the second of these Baer subplanes $\mathcal{M}$ within the structure of the affine plane π_Σ corresponding to Σ. If $\mathcal{M}$ is not a component of π_Σ, it cannot be fixed by τ as τ is a central collineation group of Σ and $\mathcal{M}$ is disjoint from the axis or coaxis of τ. Hence, $\mathcal{M}$ is a component of Σ. But, E^- acts transitively and regular on the remaining q Baer subplanes of the associated regulus net. Thus, the opposite regulus to this regulus in Σ' is a regulus on Σ containing ℓ. ■

Lemma 8 *For each spread Σ' containing an opposite regulus R^*, where R is a regulus of Σ containing ℓ, the group G_R acts regularly on $\Sigma' - R^*$.*

More generally, G_R is semi-regular on the set of Baer subplanes of Σ which are disjoint from ℓ.

Proof. We have seen that we may assume that there is a group

$$\langle (x,y) \longmapsto (x^{\sigma_m}, (y * m = y^{\sigma_m} m)); m \in GF(q) - \{0\} \rangle$$

acting on the spread containing the opposite regulus to the standard regulus.

We see that the basic assertion follows from the statement about regularity.

The Baer subplanes of Σ disjoint from $x = 0$ (the axis of E), have the general form $y = x^q a + xb$ for all $a \neq 0, b \in GF(q^2)$.

The image of an element of the group above corresponding to m is:

$$y = x^q a^{\sigma_m} m + x b^{\sigma_m} m = x^q (a * m) + x(b * m)$$

where $*$ denotes Dickson nearfield multiplication. This Baer subplane is fixed if and only if

$$a * m = a \text{ and } b * m = b.$$

Since

$$a * m = a = a * 1$$

in a Dickson nearfield, we have that $m = 1$ which implies that the group element is the identity. This proves the lemma. ■

Now the theorem is proven except to show that when the original group G is taken within $GL(4,q)$, the spreads of $\mathcal{P}$ are Hall spreads. If the group

G is in $GL(4,q)$, it follows that the corresponding Dickson nearfield has kernel $GF(q)$ or $GF(q^2)$. In the latter case, the Dickson nearfield is a field and we obtain the group:

$$\langle (x,y) \longmapsto (x, ym); m \in GF(q) - \{0\} \rangle$$

acting on the spread containing the opposite to the standard regulus. In the former case, q is forced to be odd and we may represent the Dickson nearfield where $x * m = x^q m$ if m is a non-square and $x * m = xm$ if m is square in $GF(q^2)$. Since $m \in GF(q)$ is always square, it follows that $\sigma_m = 1$. Now this group leaves invariant each Desarguesian spread containing R and the group G_R then acts regularly on the components not in R of each such Desarguesian spread. Given a component ℓ' of $\Sigma - R^*$, there is a unique Desarguesian spread ρ containing ℓ' and R. Hence, the group G_R acts regularly on $\Sigma' - R^*$ and on $\rho - R$ and both contain ℓ'. Thus, the derived spread $\Sigma'^* = \rho$ so that Σ' is Hall. If $(r, q^2 - 1) = 1$, then $h^n = q^2 = p^{2r} = p^{tn}$ where n divides $q^2 - 1$. But, $(n, 2r) = n$ so that $n = 1$ or 2. We then have the previous situation.

This completes the proof of the theorem stated in the introduction.

3 The Structure of the Spreads.

In this section, we give the complete representation of the spreads of a partial parallelism with a skew collineation group G that acts two-transitive on the lines different from a given line of a spread Σ.

By the previous section, we may choose the skew spread as a Desarguesian spread in $PG(3,q)$. We choose the regulus R in vector form and given as follows:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in GF(q),$$

where the underlying vector space V_4 has vectors (x_1, x_2, y_1, y_2) for all x_i, $y_i \in GF(q)$ and $i = 1, 2$.

We represent the Desarguesian spread Σ by the following:

$$x = 0, y = x \begin{bmatrix} u + \rho t & \gamma t \\ t & u \end{bmatrix} \forall u, t \in GF(q)$$

where $x^2 + \rho x - \gamma$ is irreducible over $GF(q)$. The axis of the elation group E is $x = 0$ and E has the following form:

$$E = \left\langle \begin{bmatrix} 1 & 0 & u + \rho t & \gamma t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall u, t \in GF(q) \right\rangle.$$

Furthermore, the group H developed in the previous section is as follows:

$$H = \langle (x,y) \longmapsto (x^{\sigma_m} m^i, y^{\sigma_m} m^{i+1}); \forall m \in GF(q^2) \rangle .$$

In this representation, the mapping denoted by σ_m is an automorphism of $GF(q^2)$. In order to explicate the spreads, we need to determine the form of H written over $GF(q)$ and to determine the form of the spread Σ', where the notation is taken as development in the previous section.

Let $\{1, e\}$ be a basis for $GF(q^2)$ over $GF(q)$ and let

$$(x_1 e + x_2)^{\sigma_m} = (x_1^{\sigma_m}(e\alpha_m + \beta_m) + x_2^{\sigma_m},$$

for all $x_i \in GF(q)$ for $i = 1, 2$ for some $\alpha_m, \beta_m \in GF(q)$.

Hence, we obtain, writing $(x_1, x_2) = x_1 e + x_2$:

$$\begin{aligned} H &= \langle (x_1, x_2, y_1, y_2) \longmapsto (x_1^{\sigma_m}, x_2^{\sigma_m}, y_1^{\sigma_m}, y_2^{\sigma_m}) AM \rangle \\ \text{such that } A &= \begin{bmatrix} \alpha_m & \beta_m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_m & \beta_m \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } M = \begin{bmatrix} m^i & 0 \\ 0 & m^{i+1} \end{bmatrix} \\ \text{where } m &= \begin{bmatrix} u_m + \rho t_m & \gamma t_m \\ t_m & u_m \end{bmatrix}, \text{ for a unique pair } (u_m, t_m) \neq (0,0) \\ & \text{of elements of } GF(q) \times GF(q). \end{aligned}$$

We now determine (α_m, β_m) relative to ρ and γ.

Since the collineation

$$g : (x_1, x_2, y_1, y_2) \longmapsto (x_1^{\sigma_m}, x_2^{\sigma_m}, y_1^{\sigma_m}, y_2^{\sigma_m})$$

is a collineation of Σ, it follows immediately that

$$y = x \begin{bmatrix} u + \rho t & \gamma t \\ t & u \end{bmatrix}$$

is mapped by g onto:

$$y = x \begin{bmatrix} \alpha_m & \beta_m \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u^{\sigma_m} + \rho^{\sigma_m} t^{\sigma_m} & \gamma^{\sigma_m} t^{\sigma_m} \\ t^{\sigma_m} & u^{\sigma_m} \end{bmatrix} \begin{bmatrix} \alpha_m & \beta_m \\ 0 & 1 \end{bmatrix}.$$

The matrix of this previous component is:

$$\begin{bmatrix} u^{\sigma_m} + \beta_m t^{\sigma_m} + \rho^{\sigma_m} t^{\sigma_m} - 2t^{\sigma_m}\beta_m & \alpha_m^{-1}(\gamma^{\sigma_m} t^{\sigma_m} + \rho^{\sigma_m} t^{\sigma_m} - \beta_m)\beta_m \\ t^{\sigma_m}\alpha_m & u^{\sigma_m} + t^{\sigma_m}\beta_m \end{bmatrix}.$$

Since, up to isomorphism, we may choose any Desarguesian spread Σ to initiate our parallelism construction, we may choose $\rho = 0$ and γ a non-square when q is odd. When q is even then since the polynomial $x^2+\rho x+\gamma$ is non-singular, we may assume that $\gamma = 1$ by dividing by $\gamma = \epsilon^2$ to obtain the corresponding irreducible polynomial $x^2 + (\rho/\epsilon)x + 1$.

Hence, when $\rho = 0$ then $\beta_m = 0$.

Thus, we obtain the following requirements:

$$\begin{aligned}\rho(t^{\sigma_m}\alpha_m) &= \rho^{\sigma_m}t^{\sigma_m} \text{ and}\\ \gamma(t^{\sigma_m}\alpha_m) &= \alpha_m^{-1}(\gamma^{\sigma_m}t^{\sigma_m} + \rho^{\sigma_m}t^{\sigma_m} - \beta_m t^{\sigma_m})\beta_m.\end{aligned}$$

Now assume that q is odd. Then, $\rho = 0 = \beta_m$, and

$$\begin{aligned}\gamma(t^{\sigma_m}\alpha_m) &= \alpha_m^{-1}(\gamma^{\sigma_m}t^{\sigma_m}) \text{ so that}\\ \alpha_m^2 &= \gamma^{\sigma_m - 1}.\end{aligned}$$

When q is even then $\gamma = 1$ and

$$\begin{aligned}\alpha_m &= \rho^{\sigma_m - 1} \text{ and}\\ \alpha_m &= \alpha_m^{-1}(1 + \rho^{\sigma_m} + \beta_m)\beta_m.\end{aligned}$$

The parallelism is determined once the spread different from Σ and containing R is determined. (Note that the parallelism is obtained by the derivation of R of the spread in question.)

Hence, we consider a conical flock spread Σ'^* of the following form:

$$x = 0, y = x\begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix} \forall u, t \in GF(q)$$

where f and g are functions on $GF(q)$. (The reader is directly to the survey article by Johnson and Payne [11] or the article by Gevaert and Johnson [10] to see that the representation of the spread is as maintained.)

We need only consider the action of the group:

$$\langle (x, y) \longmapsto (x^{\sigma_m}, y^{\sigma_m}m) \text{ for all } m \in GF(q)\rangle$$

on the spread Σ'^* :

This group must fix the standard regulus and act regularly on the remaining $q - 1$ reguli R_t for $t \neq 0$ determined by the element t in the $(2,1)$-entry of the matrix spread set. Hence, we take $t = 1$, $g(1) = g$, $f(1) = f$ and apply the above group elements.

So, the element

$$y = x\begin{bmatrix} \rho & \gamma \\ 1 & 0 \end{bmatrix},$$

where $\rho = 0$ if q is odd, and $\gamma = 1$ if q is even, maps (after an appropriate addition by an element $\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$ to make the (2, 2)-entry 0) to

$$\begin{aligned} y &= x \begin{bmatrix} \alpha_m & \beta_m \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} g^{\sigma_m} & f^{\sigma_m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_m m & \beta_m m \\ 0 & m \end{bmatrix} \text{ which gives:} \\ y &= x \begin{bmatrix} g^{\sigma_m} m & \alpha_m^{-1}(g^{\sigma_m}\beta_m m + f^{\sigma_m} m) \\ \alpha_m m & 0 \end{bmatrix}. \end{aligned}$$

Now if q is odd then $\beta_m = 0$ so we obtain:

$$y = x \begin{bmatrix} g^{\sigma_m} m & \alpha_m^{-1} f^{\sigma_m} m) \\ \alpha_m m & 0 \end{bmatrix},$$

where $\alpha_m^2 = \gamma^{\sigma_m - 1}$. We note from the previous section that the indicated group G_R is regular on $\Sigma'' - R$, so it follows that the function $m \longmapsto \alpha_m m$ is bijective on $GF(q) - \{0\}$. Let $\alpha_m m = t(m) = t_m$ so that the spread is

$$\begin{aligned} y &= x \begin{bmatrix} u + \alpha_m^{-1} g^{\sigma_m} t_m & \alpha_m^{-2} f^{\sigma_m} t_m) \\ t_m & u \end{bmatrix} \forall u, t_m \in GF(q), \\ \text{where } \alpha_m^2 &= \gamma^{\sigma_m - 1}. \end{aligned}$$

When q is even $\gamma = 1$ so we obtain:

$$y = x \begin{bmatrix} u + \alpha_m^{-1} g^{\sigma_m} t_m & \alpha_m^{-2}(g^{\sigma_m}\beta_m t_m + f^{\sigma_m} t_m) \\ t_m & u \end{bmatrix} \forall u, t \in GF(q),$$

where

$$\begin{aligned} \alpha_m &= \rho^{\sigma_m - 1} \text{ and} \\ \alpha_m &= \alpha_m^{-1}(1 + \rho^{\sigma_m} + \beta_m)\beta_m. \end{aligned}$$

(Note that, in this case, $\alpha_m m = \alpha_n n$ if and only if $\rho^{\sigma_m - 1} m = \rho^{\sigma_n - 1} n$ if and only if $\rho^{\sigma_m} m = \rho^{\sigma_n} n$. Equivalently, $\rho * m = \rho * n$ if and only if $m = n$ as we obtain a Dickson nearfield multiplication $*$.)

We summarize as follows:

Theorem 9 *Let $\mathcal{P}$ be a partial parallelism in $PG(3, q)$ for $q > 2$ and $q \neq 8$ which admits a skew spread Σ such that there admits a skew collineation group of $\mathcal{P}$ and of Σ that fixes a component ℓ and acts sharply 2-transitively on the components of $\Sigma - \{\ell\}$.*

Then

(i) Σ is Desarguesian.

(ii) $\{\Sigma\} \cup \mathcal{P}$ *is a parallelism in* $PG(3,q)$.

(iii) For each regulus R *of* Σ *containing* ℓ, *there is a unique spread* Σ' *of* $\mathcal{P}$ *containing the opposite regulus* R^* *of* R *and* G_R *fixes* Σ' *and is regular on the components* $\Sigma' - R^*$.

(iv) G *acts transitively on the spreads of* $\mathcal{P}$.

(v) The spreads of $\mathcal{P}$ *are spreads which may be derived from conical flock spreads that admit a collineation group that fixes a base regulus and acts transitively on the remaining* $q-1$ *base regulus.*

The spreads of Σ *and the derived versions of the spreads of* $\mathcal{P}$ *may be represented as images under* $G = EH$ *of* Σ'^* *where* Σ *is given by*

$$
\begin{aligned}
x &= 0, y = x\begin{bmatrix} u+\rho t & \gamma t \\ t & u \end{bmatrix} \forall u,t \in GF(q), \\
\text{where } \rho &= 0 \text{ if } q \text{ is odd, and } \gamma = 1 \text{ if } q \text{ is even.}
\end{aligned}
$$

$G = EH$ *is given as follows:*

$$
E = \left\langle \begin{bmatrix} 1 & 0 & u+\rho t & \gamma t \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \forall u,t \in GF(q) \right\rangle \text{ and}
$$

$$
\begin{aligned}
H &= \langle (x_1,x_2,y_1,y_2) \longmapsto (x_1^{\sigma_m}, x_2^{\sigma_m}, y_1^{\sigma_m}, y_2^{\sigma_m})AM \rangle \\
\text{such that } A &= \begin{bmatrix} \alpha_m & \beta_m & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_m & \beta_m \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and} \\
M &= \begin{bmatrix} m^i & 0 \\ 0 & m^{i+1} \end{bmatrix} \\
\text{where } m &= \begin{bmatrix} u_m + \rho t_m & \gamma t_m \\ t_m & u_m \end{bmatrix} \\
& \quad \text{for a unique pair } (u_m, t_m) \neq (0,0).
\end{aligned}
$$

Furthermore, the spread for Σ'^* *is given when* q *is odd by*

$$
\begin{aligned}
y &= x\begin{bmatrix} u + \alpha_m^{-t} g^{\sigma_m} t_m & \alpha_m^{-2} f^{\sigma_m} t_m) \\ t_m & u \end{bmatrix} \forall u, t_m \in GF(q), \\
\text{where } \alpha_m^2 &= \gamma^{\sigma_m - 1}.
\end{aligned}
$$

and when q *is even, we obtain:*

$$
y = x\begin{bmatrix} u + \alpha_m{}^{1} g^{\sigma_m} t_m & \alpha_m{}^{2}(g^{\sigma_m}\beta_m t_m + f^{\sigma_m} t_m) \\ t_m & u \end{bmatrix} \forall u,t \in GF(q),
$$

where

$$\begin{aligned} \alpha_m &= \rho^{\sigma_m - 1} \text{ and} \\ \alpha_m &= \alpha_m^{-1}(1 + \rho^{\sigma_m} + \beta_m)\beta_m. \end{aligned}$$

References

[1] M. Biliotti, V. Jha and Norman L. Johnson. Transitive Parallelisms, Resultate der Mathematik 37 (2000), 308–314.

[2] M. Biliotti, V. Jha and Norman L. Johnson, The collineation groups of generalized twisted field planes. Geom. Ded. 76 (1999), 97-126.

[3] M. Cordero and R. Figueroa, Transitive autotopism groups and the generalized twisted field planes, Mostly Finite Geometries, Lecture Notes in Pure and Applied Math. Vol. 190, edited by Johnson, N.L., Marcel Dekker, New York-Basel-Hong Kong, 1997, 191-196.

[4] D.A. Foulser, Foulser, D.A., Baer p-elements in translation planes, J. Algebra 31 (1974), 354-366.

[5] N.L. Johnson, Translation planes admitting Baer groups and partial flocks of quadric sets. Simon Stevin, vol. 63. no. 3(1989), 163-187.

[6] N.L. Johnson, Subplane Covered Nets, Monographs/Textbooks in Pure and Applied Mathematics, vol. 222, Marcel Dekker, New York 2000.

[7] N.L. Johnson, Some New Classes of Finite Parallelisms, Note di Mat. 20 (2000/2001) (appeared in 2002), 77-88.

[8] N.L. Johnson, Dual parallelisms, Note di Mat. 21 (2002), 137-150.

[9] N.L. Johnson, Two-transitive parallelisms, Designs, Codes, and Cryptography, 22 (2001), 179-189.

[10] H. Gevaert and Norman L. Johnson, Flocks of quadratic cones, generalized quadrangles, and translation planes. Geom. Ded. 27, no. 3(1988), 301-317.

[11] N.L. Johnson and S. E. Payne, Flocks of Laguerre Planes and Associated Geometries, Mostly Finite Geometries, Lecture Notes in Pure and Applied Math. Vol. 190, edited by Johnson, N.L., Marcel Dekker, New York-Basel-Hong Kong, 1997, 51-122.

[12] N.L. Johnson and R. Pomareda, Transitive Partial Parallelisms of Deficiency one, European J. Combinatorics, 232 (2002), 969-986.

[13] N.L. Johnson and R. Pomareda, Parallelism-Inducing Groups, Aequationes Math. 65 (2003), 133-157.

[14] H. Lüneburg, Translation Planes, Springer-Verlag, 1980, Berlin-New York.

[15] S.E. Payne and J.A. Thas, conical flocks, partial flocks, derivation and generalized quadrangles, Geom. Dedicata 38 (1991), 229-243.

[16] T. Penttila and B. Williams, Regular packings of $PG(3,q)$, European J. Combin., 19 (1998), 713–720.

Norman L. Johnson
Mathematics Dept.
University of Iowa
Iowa City, IA. 52242

Rolando Pomareda
Mathematics Dept.
University of Chile
Casilla 653
Santiago, Chile