Redundance of Trees

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ABSTRACT. The redundance $R(G)$ of a graph G is the minimum, over all dominating sets S, of $\sum_{v \in S} 1 + \mathrm{d}(v)$, where $\mathrm{d}(v)$ is the degree of vertex v. We establish a sharp upper bound on the redundance of trees and characterize all trees that achieve the bound.

Preliminaries. A dominating set S for a graph G is a subset of the vertices of G such that every vertex $v \in G$ is either in S or is adjacent to an element of S. The redundance of a graph G, $R(G)$, is the minimum, over all dominating sets S, of $I(S) = \sum_{v \in S} 1 + \mathrm{d}(v)$, where $\mathrm{d}(v)$ is the degree of vertex v; $I(S)$ is the *influence* of S.

Goddard et al. [1] showed that if T is a tree on at least two vertices then $R(T) \leq 3n/2 - 1$, where $n = |V(T)|$. For every $k \geq 1$ they produced a graph on $4k + 2$ vertices that attains the bound. Here we refine the upper bound so that it is sharp for all values of n and characterize the trees that attain the bound.

We note that it is an easy exercise to show for any bipartite G that $R(G) \leq |V(G)|/2 + |E(G)|$; applying this to a tree with n vertices and $n - 1$ edges leads to the bound $3n/2 - 1$.

The Fundamental Observation. Suppose that a vertex v in G is adjacent to $p_v \geq 1$ pendant vertices. Any dominating set S for G must contain either v or every one of the pendant vertices; the corresponding contribution to the influence of S is either $1 + \mathrm{d}(v)$ or $2p_v$. If every nonpendant vertex is adjacent to at least one pendant vertex, then $R(G) = \sum_v \min(1 + \mathrm{d}(v), 2p_v)$, where the sum is over all nonpendant v. A bit of experimentation makes it appear that this sum is largest when $1 + \mathrm{d}(v)$ and $2p_v$ are equal, or nearly so, for every nonpendant v. They are equal when $p_v = (1 + \mathrm{d}(v))/2$, or equivalently, when v is adjacent to to p_v pendant vertices and $n_v = p_v - 1$ nonpendant vertices.

Suppose the tree T has $m > 1$ nonpendant vertices, and that $p_v = n_v + 1$ for all nonpendant v. Then T has $n = 4m - 2$ vertices and redundance $6m - 4 = 3n/2 - 1$, showing that the bound is sharp when $n \equiv 2 \pmod 4$. (This is also true for the special case $n = 2$.) Such trees are easy to construct, by starting with a tree on m vertices and adding pendant edges to each vertex.

It turns out that this bound, $3n/2 - 1$, is not sharp for other values of n, but it is very close. In all cases trees that do achieve the appropriate upper bound have pendant vertices adjacent to every nonpendant vertex, with $2p_v$ equal, or nearly equal, to $1 + \mathrm{d}(v)$ for every nonpendant v.

When $p_v = n_v + 1$ for all nonpendant vertices v, we may form a dominating set S that achieves the redundance $3n/2 - 1$ by placing either v or every pendant vertex adjacent to v into S, for every nonpendant v. This means that when $p_v = n_v + 1$ for all nonpendant v, given a vertex w, we may find two dominating sets that realize the redundance, one that contains w and one that does not. More generally, if p_v is non-zero for all nonpendant v, let $k_v = p_v - n_v - 1$. If $k_v > 0$, a dominating set that realizes the redundance will contain v, while if $k_v < 0$, such a dominating set will contain all of the pendant vertices adjacent to v. If we replace v by the pendant vertices, or replace the pendant vertices by v, respectively, the resulting dominating set has influence $|k_v|$ greater than the redundance. Thus, given a vertex w, we may find two dominating sets, one that contains w and one that does not, but in one of the two cases we pay a 'penalty' of $|k_v|$.

The Characterization. We are now ready for the precise statement of the upper bounds and the description of the extremal graphs.

THEOREM 1 The following are sharp bounds on $R(T)$, where T is a tree on $n > 1$ vertices:

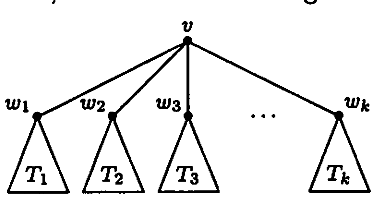
$$R(T) \leq \begin{cases} \frac{3n}{2} - 2 & \text{if } n \equiv 0 \pmod 4; \\ \lfloor \frac{3n}{2} - 1 \rfloor & \text{if } n \equiv 1,3 \pmod 4; \\ \frac{3n}{2} - 1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

T is a tree with redundance equal to the upper bound if and only if every nonpendant vertex v has $k_v = 0$ except:

1. If $n \equiv 0 \pmod 4$, either one nonpendant vertex v has $k_v = \pm 2$, or two have $k_v = 1$, or two have $k_v = -1$.
2. If $n \equiv 1 \pmod 4$, one nonpendant vertex v has $k_v = -1$.
2. If $n \equiv 3 \pmod 4$, one nonpendant vertex v has $k_v = 1$.

Proof. It is simple to check that trees with the claimed properties have redundance as claimed. Moreover, it is easy to construct trees with the desired form, by starting with any tree and adding pendant edges at each vertex so that each of the original vertices v becomes a nonpendant vertex with the desired value of k_v.

The remainder of the proof is by induction; $n = 2$ is trivial. Suppose first that T on n vertices has a nonpendant vertex v that is adjacent to no pendant vertex; we will show that $R(T)$ is strictly less than $3n/2 - 2$. Vertex v is adjacent to vertices w_i, $i = 1, \ldots, k$, $k \geq 2$, and each w_i is in a tree T_i on $n_i \geq 2$ vertices, as indicated in the figure.



Let j be the number of trees T_i that are extremal with $n_i \equiv 2 \pmod 4$; without loss of generality, these trees are $T_1, \ldots, T_j$. Choose sets S_i in T_i that realize $R(T_i)$. We consider a number of cases.

1. If $w_i \in S_i$ for some $i > j$, replace sets S_i, $1 \le i \le j$, by new sets S_i that do not include w_i and realize $R(T_i)$ (this is possible, as we remarked above). Then $S = \bigcup_{i=1}^{k} S_i$ is a dominating set for T, and

$$\begin{aligned} I(S) &\le \sum_{i=1}^{j}\left(\frac{3}{2}n_i - 1\right) + \sum_{i=j+1}^{k}\left(\frac{3}{2}n_i - \frac{3}{2}\right) + k - j \\ &= \frac{3}{2}n - \frac{1}{2}j - \frac{1}{2}k - \frac{3}{2} \le \frac{3}{2}n - \frac{5}{2}. \end{aligned}$$

2. If $w_i \notin S_i$ for all $i > j \ge 1$, replace S_1 by one that includes w_1 and realizes $R(T_1)$, and replace sets S_i, $2 \le i \le j$, by ones that do not include w_i and realize $R(T_i)$. Then $S = \bigcup_{i=1}^{k} S_i$ is a dominating set for T, and

$$\begin{aligned} I(S) &\le 1 + \sum_{i=1}^{j}\left(\frac{3}{2}n_i - 1\right) + \sum_{i=j+1}^{k}\left(\frac{3}{2}n_i - \frac{3}{2}\right) \\ &= \frac{3}{2}n + \frac{1}{2}j - \frac{3}{2}k - \frac{1}{2} \le \frac{3}{2}n - \frac{5}{2}. \end{aligned}$$

3. Suppose $j = 0$ and $w_i \notin S_i$ for all i and for every i, either n_i is 0 (mod 4) or T_i is not extremal. Then $S = \{v\} \cup \bigcup_{i=1}^{k} S_i$ is a dominating set for T, and

$$I(S) \le \sum_{i=1}^{k}\left(\frac{3}{2}n_i - 2\right) + k + 1 = \frac{3}{2}n - k - \frac{1}{2} \le \frac{3}{2}n - \frac{5}{2}.$$

4. If $j = 0$ and $w_i \notin S_i$ for all i and, without loss of generality, n_1 is either 1 or 3 (mod 4) and T_1 is extremal, replace S_1 by a new dominating set S_1 that contains w_1 with $I(S_1) \le 3n_1/2 - 3/2 + 1$. This is possible by the remark above, since by the induction hypothesis, $R(T_1) = 3n_1/2 - 3/2$ and $|k_w| \le 1$ for all nonpendant w in T_1. Then $S = \bigcup_{i=1}^{k} S_i$ is a dominating set for T, and

$$I(S) \le 1 + 1 + \sum_{i=1}^{k}\left(\frac{3}{2}n_i - \frac{3}{2}\right) = \frac{3}{2}n + \frac{1}{2} - \frac{3}{2}k \le \frac{3}{2}n - \frac{5}{2}.$$

Now suppose that every nonpendant vertex in T is adjacent to a pendant vertex. If $k_v > 0$ and $k_w < 0$, then by moving a pendant edge from v to w we get a tree on n vertices with redundance strictly larger than $R(T)$, so $R(T)$ is not maximum.

Let m denote the number of nonpendant vertices in T, and let $k = \sum k_v$, taking the sum over all nonpendant vertices v. Note that $n = m + \sum_v (1 + n_v + k_v) = 4m + k - 2$. If $k \geq 0$ then $k_v \geq 0$ for all v, and likewise if $k \leq 0$ then $k_v \leq 0$ for all v.

Suppose first that $k \geq 0$, so $R(T) = \sum_v \min(1 + \mathrm{d}(v), 2p_v) = \sum_v 1 + \mathrm{d}(v)$. Then

$$R(T) = \sum_v 1 + \mathrm{d}(v) = \sum_v 1 + 2n_v + 1 + k_v = 6m + k - 4,$$

where the sums are over all nonpendant v.

If $k \equiv 0 \pmod 4$ then $n \equiv 2 \pmod 4$ and $6m + k - 4 \leq 3n/2 - 1$, with equality only if $k = 0$. Thus $R(T)$ has the maximum value $3n/2 - 1$ only if $k_v = 0$ for all nonpendant v.

If $k \equiv 1 \pmod 4$ then $n \equiv 3 \pmod 4$ and $6m + k - 4 \leq 3n/2 - 3/2$, with equality only if $k = 1$. Thus $R(T)$ has the maximum value $3n/2 - 3/2$ only if one $k_v = 1$ and the remainder are 0.

If $k \equiv 2 \pmod 4$ then $n \equiv 0 \pmod 4$ and $6m + k - 4 \leq 3n/2 - 2$, with equality only if $k = 2$. Thus $R(T)$ has the maximum value $3n/2 - 2$ only if either one $k_v = 2$ and all others are 0, or $k_v = k_w = 1$ for two vertices and the remainder are 0.

If $k \equiv 3 \pmod 4$ then $n \equiv 1 \pmod 4$ and $6m + k - 4 < 3n/2 - 3/2$, so $R(T)$ does not realize the maximum value.

Now suppose that $k \leq 0$, so $R(T) = \sum_v \min(1 + \mathrm{d}(v), 2p_v) = \sum_v 2p_v$. Now

$$R(T) = \sum_v 2p_v = \sum_v 1 + n_v + k_v = 6m + 2k - 4,$$

where the sums are over all nonpendant v.

If $k \equiv 0 \pmod 4$ then $n \equiv 2 \pmod 4$ and $6m + 2k - 4 \leq 3n/2 - 1$, with equality only if $k = 0$. Thus $R(T)$ has the maximum value $3n/2 - 1$ only if $k_v = 0$ for all nonpendant v.

If $k \equiv 1 \pmod 4$ then $n \equiv 3 \pmod 4$ and $6m + k - 4 < 3n/2 - 3/2$, so $R(T)$ does not realize the maximum value.

If $k \equiv 2 \pmod 4$ then $n \equiv 0 \pmod 4$ and $6m + k - 4 \leq 3n/2 - 2$, with equality only if $k = -2$. Thus $R(T)$ has the maximum value $3n/2 - 2$ only if either one $k_v = -2$ and all others are 0, or $k_v = k_w = -1$ for two vertices and the remainder are 0.

If $k \equiv 3 \pmod 4$ then $n \equiv 1 \pmod 4$ and $6m + k - 4 \leq 3n/2 - 3/2$, with equality only if $k = -1$. Thus $R(T)$ has the maximum value only if one $k_v = -1$ and the remainder are 0.

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References.

[1] W. Goddard, O. R. Oellermann, P. J. Slater, and H. C. Swart, *Bounds on the total redundance and efficiency of a graph*, Ars Combinatoria, 54 (2000), pp. 129–138.