On inequivalent Hadamard matrices of order 36

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Abstract

The problem of classification of Hadamard matrices gets to be an NP hard problem as the order of the Hadamard matrices increase. In this paper we use a new criterion which inspired us to develop an efficient algorithm to investigate the lower bound of inequivalent Hadamard matrices of order 36. Using four $(1,-1)$ circulant matrices of order 9 in the Goethals - Seidel array we obtain many new Hadamard matrices of order 36 and we show that there are at least 1036 inequivalent Hadamard matrices for this order.

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1 Introduction

A Hadamard matrix of order n is an $n \times n$ $(1,-1)$-matrix satisfying $HH^T = nI_n$. A Hadamard matrix is normalized if all entries in its first row and column are equal to 1. Two Hadamard matrices are equivalent if one can be transformed into the other by a series of row or column permutations and negations. It is well known that if n is the order of a Hadamard matrix then n is necessarily 1, 2 or a multiple of 4.

The discussion of Hadamard equivalence is quite difficult, principally because of the lack of a good canonical form. The exact results which have been discovered are as follows : Hadamard matrices of orders less than 16 are unique up to equivalence. There are precisely five equivalence classes at order 16, and three equivalence classes at order 20, see [7, 8]. There are precisely 60 equivalence classes at order 24, see [9, 11]. There are precisely 487 equivalence classes at order 28, see [12, 13]. The classification of Hadamard matrices of orders $n \geq 32$ is still remains an open and difficult problem since an algorithmic approach of an exhaustive search is an NP hard problem. In particular for $n = 32$, Lin, Wallis and Lie [15] found 66104 inequivalent Hadamard matrices of order 32. Extensive results appear in [16] and [17]. Thus the lower bound for inequivalent Hadamard matrices of order 32 is 66104.

In this paper we show that there are at least 1036 inequivalent Hadamard matrices of order 36. In fact this number is obtained as follows: From Seberry's home page http://www.uow. edu.au/~jennie that there are 192 inequivalent Hadamard matrices of order 36. These are supplied by E. Spence (180 matrices) see [18], Z. Janko, (1 matrix of Bush-type) see [10] and V. D. Tonchev (11 matrices) see [19]. Using an efficient algorithm and the Magma software we found that 172 of their transposes are inequivalent to these.

In this paper we also improve this lower bound to 1036 by constructing 672 new Hadamard matrices as described in section 4.

Lam, Lam and Tonchev [14] showed that the lower bound for inequivalent Hadamard matrices of order 40 is 3.66×10^{11}.

Before we give a brief description of our algorithm we need the following notations and definitions. Let $A_j = \{a_{j1}, a_{j2}, ..., a_{jn}\}$, $j = 1, \ldots, \ell$, of length n be a set of ℓ sequences, denoted by A. The *non-periodic autocorrelation function* $N_A(s)$ of the above sequences is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{1}$$

If $A_j(z) = a_{j1} + a_{j2}z + \ldots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}). \tag{2}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function* $P_A(s)$ is defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{3}$$

For the results of this paper generally PAF is sufficient. However NPAF sequences imply PAF sequences exist.
The following theorem which uses four circulant matrices is very useful in our construction for Hadamard matrices.

Theorem 1 *[3, Theorem 4.49] or [6]. Suppose there exist four circulant matrices A, B, C, D of order n satisfying*

$$AA^T + BB^T + CC^T + DD^T = nI_n$$

Let R be the back diagonal matrix. Then

$$GS = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix}$$

is a Hadamard matrix of order $4n$.

Corollary 1 *If there are four sequences A, B, C, D of length n with entries from* $\{\pm 1\}$ *with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form a Hadamard matrix of order* $4n$. □

In this paper we use a simple algorithm to find four $(1, -1)$ sequences (A,B,C,D) of length 9, which have zero PAF ($P_A(s) + P_B(s) + P_C(s) + P_D(s) = 0$, $\forall s = 1, 2, 3, 4$) and are given in Hexadecimal (Hex) form in Table 1. From these sequences we can construct the appropriate circulant matrices A, B, C, D of order 9, which are used in theorem 1, for the construction of new inequivalent Hadamard matrices of order 36. The inequivalence of the Hadamard matrices was checked by an algorithm which is presented in section 2, and with the help of Magma software.

2 The algorithm

The following algorithm was first given in [4]. In the same paper the authors prove that this algorithm can be used as necessary and sufficient criterion to check equivalence of Hadamard matrices.

The *Hamming distance distribution* ($W(x)$) and the *symmetric Hamming distance distribution* ($SW(x)$), of a projection in k columns, is defined to be

$$W_k(x) = a_0 + a_1x^1 + \ldots + a_kx^k \text{ and}$$

$$SW_k(x) = \begin{cases} \sum_{i=0}^{(k-1)/2} (a_i + a_{k-i})x^i, & \text{when k is odd} \\ \sum_{i=0}^{(k-2)/2} (a_i + a_{k-i})x^i + a_{\frac{k}{2}}x^{\frac{k}{2}}, & \text{when k is even} \end{cases}$$

respectively, where a_m is the number describing how many pairs of rows of the projection have distance m.

Example 1 Consider the projections for $k = 3$ and $n = 8$. A Hadamard matrix of order 8 is

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \end{matrix}$$

Since $k = 3$ the projections are all possible 3-sets of columns. We will just illustrate with the sets of columns 2, 3, 4 and 2, 3, 5.

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{matrix}$$

We now consider the distance between all pairs of rows of these 8×3 matrices. The first set has distance 3 (4 times), 2 (12 times) and 1 (12 times) so its Hamming distance distribution and its symmetric Hamming distance distribution is

$$W_3(x) = 0 + 12x + 12x^2 + 4x^3, \quad SW_3(x) = 4 + 24x$$

respectively, while the second set has 0 (4 times) and 2 (24 times) so its Hamming distance distribution and its symmetric Hamming distance distribution is

$$W_3(x) = 4 + 24x^2, \quad SW_3(x) = 4 + 24x$$

respectively. □

The Hamming distance distribution $W_k(x)$ is invariant only to permutations of columns or rows, or negations of columns while the symmetric Hamming distance distribution $SW_k(x)$ is invariant to permutations and negations of both rows and columns.

Lemma 1 *Two equivalent projections have the same symmetric Hamming distance distribution.*

Lemma 2 *All projections of two Hadamard matrices H_1, H_2 of order n in $k = 1, 2$ columns are the same (actually these give only one inequivalent projection) even thought the Hadamard matrices are inequivalent.*

Lemma 3 *Let H be a Hadamard matrix of order n. Any two rows of the Hadamard matrix have Hamming distance distribution and symmetric Hamming distance distribution $W_n(x) = SW_n(x) = x^{n/2}$.*

Definition 1 Let H be a Hadamard matrix of order n and P_k a set of k columns of H. We define the *complementary projection* of P_k to be the set of the columns of H which are not contained in P_k. Obviously the complementary projection of P_k consist of $n - k$ columns.

Remark 1 Let H_1, H_2 be two Hadamard matrices of order n. Suppose $r = \{r_1, r_2, \ldots, r_k\}$ and $p = \{p_1, p_2, \ldots, p_k\}$ be two rows of a projection of H_1 and $q = \{q_1, q_2, \ldots, q_k\}$ and $s = \{s_1, s_2, \ldots, s_k\}$ be two rows of a projection of H_2. Then $SW(x)$ of rows r, p is equal to $SW(x)$ of rows q, s if and only if the symmetric Hamming distance distribution of the corresponding rows of their complementary projections is equal.

Example 2 The complementary projections of the projections given in example 1 are

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{matrix} \quad \textit{and} \quad \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{matrix}$$

with symmetric Hamming distance distribution $SW_{8-3}(x) = SW_5(x) = 4 + 24x$.

From Lemmas 1, 2 and 3 it is obvious that:

Corollary 2 *All projections of two Hadamard matrices H_1, H_2 of order n in $k = 1, 2$ and $k = n$ columns have the same symmetric Hamming distance distribution.*

Using Remark 1 and the above lemmas we can conclude:

Corollary 3 *Let H_1, H_2 be two Hadamard matrices of order n. We need only to check the symmetric Hamming distance distribution of projections for $k = 3, 4, \ldots, n/2$ because if these have the same symmetric Hamming distance distribution, then the corresponding complementary projections will have the same symmetric Hamming distance distribution as well.*

The Symmetric Hamming distance distribution algorithm:

(i) Set $k = 3$.

(ii) Find all projections for each Hadamard matrix of a given order n and k columns by taking all possible k columns of the entire $n \times n$ Hadamard matrix. These are $\binom{n}{k}$ projections in total.

(iii) In the projections found in step (ii) calculate the symmetric Hamming distance distributions for any two rows of the projection. These are $\binom{n}{2}$ symmetric Hamming distance distributions and save different symmetric Hamming distance distributions and how many times each of them appear.

(iv) Check if the set of all different symmetric Hamming distance distributions of the first Hadamard matrix is the same with the set of all different symmetric Hamming distance distribution of the second Hadamard matrix.

(v) If the answer in step (iv) is false, then stop and say that these two Hadamard matrices are inequivalent, otherwise increase k by 1.

(vi) If now $k < n/2$ then go to step (ii) and continue, otherwise stop and say that this algorithm can not decide for the equivalence of these Hadamard matrices.

3 Inequivalent Hadamard matrices of order 36

Some Hadamard matrices of order 36 are constructed, among others, in [2, 5, 6, 18, 19].

In this section we discuss the equivalence in Hadamard matrices of order 36. We know from Seberry's home page http://www.uow.edu.au/~jennie that there are 192 inequivalent Hadamard matrices of order 36. These are supplied by E. Spence (180 matrices) see [18], Z. Janko, (1 matrix of Bush-type) see [10] and V. D. Tonchev (11 matrices) see [19]. We shall refer to these matrices as $H_1, \ldots, H_{192}$.

When we apply our criterion to these 192 matrices we obtain the following results:

- For $k = 2, 3$, all 192 Hadamard matrices of order 36 have the same symmetric Hamming distance distributions and thus we obtain only one of the 192 inequivalent Hadamard matrices of this order. The result for $k = 2$ is a computational verification of lemma 2.
- For $k = 4$, only 173 Hadamard matrices give different symmetric Hamming distance distributions and thus we obtain only the 173 of the 192 inequivalent matrices.
- For $k = 5$, we have found 190 different symmetric Hamming distance distributions and thus we obtain 190 of the 192 inequivalent Hadamard matrices of order 36.
- Finally for $k = 6$, we have found 192 different symmetric Hamming distance distributions and thus we obtain all 192 inequivalent Hadamard matrices of order 36.

From the above computational results it seems that with $k = 6$ our algorithm will give us sufficient partial results on the investigation for inequivalent Hadamard matrices of order 36.

However, this algorithm (with $k = 6$), cannot decide for the equivalence of the transposes of these 192 Hadamard matrices. The algorithm needs to move to $k > 6$ to decide if these are inequivalent. Thus the computational time increases. But it is more convenient and more efficient to use the Magma software to solve this problem. We write a simple program in Magma and we apply the function "IsHadamardEquivalent" of this software to decide for the equivalence of the remaining unsolved cases. The book by Cannon and Playoust [1] was very useful in our study.

The matrices numbered 1 and 174 given by E. Spence (see [18]) are symmetric and so are equal to their transposes. Thus we have in total 190 Hadamard matrices to check for equivalence. After the application of the above program we obtain that the transposes of matrices $H_1, \ldots, H_{192}$ numbered H_i^T, $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,$ $22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45,$ $46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67,$ $68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89,$

90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 176, 178, 187, 188 are inequivalent with the 192 known Hadamard matrices, while the other 18 of the 190 transposes are equivalent. Thus we have shown that there are $192 + 172 = 364$ inequivalent Hadamard matrices. We shall refer to these matrices as $G_1, \ldots, G_{364}$.

Moreover, using a simple algorithm we have constructed several Goethals-Seidel Hadamard matrices of order 36 and when we checked these for equivalence we obtain that 663 of these are inequivalent to the 364 matrices constructed before. We shall refer to these matrices as $F_1, \ldots, F_{663}$. These 663 matrices can be constructed using the information given in Table 1. We will denote the new inequivalent Hadamard matrices we have found by F_i, $i = 1, 1, \ldots, 663$, and these are given in table 1. Nine of their transposes are inequivalent to all $364 + 663 = 1027$ Hadamard matrices. These matrices are F_i^T, $i = 18, 26, 94, 98, 384, 385, 545, 553$ and 663 and have been marked with a $*$ in table 1. To obtain these 672 inequivalent matrices we have constructed (using four circulant matrices in the Goethals-Seidel array) and checked in total 2000000 Hadamard matrices.

Finally we have shown that there are at least 1036 inequivalent Hadamard matrices of order 36.

4 The new results

In this section we present the new Hadamard matrices of order 36, we have found. In Table 1 we give the first row of the corresponding circulant $(1, -1)$ matrices of order 9 (in Hex form), which can be used in the Goethals-Seidel array to obtain the 663 new inequivalent Hadamard matrices of order 36. In this table any two digits represent a sequence of length 9. To obtain the sequences in $(1, -1)$ form we convert the two digits from Hex form to binary form. If there exist an overline then the first element of the sequence is 0 otherwise is 1. Then we replace 0 by -1 and we have the desirable $(-1, 1)$ sequences. In the sequel we move to the next two digits (in Hex form) and we do the same thing. Thus we obtain the four sequences which can be used in the Goethals-Seidel array. For details see the following explicit example.

In the next example it is shown how to convert sequences of this form to the $(1, -1)$ sequences of length 9.

Example 3 The last solution in table 1 is given by $\overline{CA}$AC7EC0. The first two digits in Hex form are $\overline{CA}$. Observe that there is an overline and thus the first digit of the sequence is 0. These two digits in Hex form can be written in binary form as 1001010. Thus we have that the sequence is 011001010 and if we replace 0 by -1 we obtain the sequence $A = \{-1, 1, 1, -1, -1, 1, -1, 1, -1\}$.

The second two digits in Hex form are AC. Observe that there is no overline and thus the first digit of the sequence is 1. These two digits in Hex form can be written in binary form as 10101100. Thus we have that the sequence is 110101100 and if we replace 0 by -1 we obtain the sequence $B = \{1, 1, -1, 1, -1, 1, 1, -1, -1\}$.

Fifth and sixth digits in Hex form are 7E. Again we observe that there is no overline and thus the first digit of the sequence is 1. These two digits in Hex form can be written in binary form as 1111110. Since these are less than eight digits, in this case seven, we add one zero at the beginning ($1 = 01 = 001 = 0001 = 0 \ldots 01$ in all arithmetic systems) and we obtain 01111110. Add the fist digit we found and replace 0 by -1 to obtain the sequence $C = \{1, -1, 1, 1, 1, 1, 1, 1, -1\}$.

Finally the last two digits in Hex form are C0. Observe that there is no overline and thus the first digit of the sequence is 1. These two digits in Hex form can be written in binary form as 11000000. Thus we have that the sequence is 111000000 and if we replace 0 by -1 we obtain the sequence $D = \{1, 1, 1, -1, -1, -1, -1, -1, -1\}$.

Then, we use corollary 1, and theorem 1 to obtain the corresponding Hadamard matrices. □

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Table 1: First rows of circulant matrices of order 9 (in Hex form).

$\overline{F7}82\overline{A7D8}$	$\overline{F7}B0\overline{7D}15$	$\overline{F7}C074\overline{CD}$	$\overline{F7}C2\overline{BE}66$	$\overline{F7}D0\overline{BE}66$
$\overline{F7}E2\overline{BE}66$	$\overline{F7}E8\overline{BE}66$	$\overline{F7}F2\overline{A7D8}$	$\overline{F7}F4\overline{A7D8}$	$\overline{F7}F874\overline{CD}$
$\overline{DB}DC5E\overline{7E}$	$\overline{DB}F89D\overline{15}$	$\overline{DF}48D1\overline{3B}$	$\overline{DF}A273D0$	$\overline{DF}A80B\overline{C2}$
$\overline{DF}C26\overline{243}$	$\overline{DF}C4\overline{7D}15$	$\overline{DF}D062\overline{43}$*	$\overline{DF}D273D0$	$\overline{DF}D6D1\overline{3B}$
$\overline{DF}D8\overline{7D}15$	$\overline{DF}DAD1\overline{3B}$	$\overline{DF}E262\overline{43}$	$\overline{DF}E4\overline{EC}D5$	$\overline{DF}E6\overline{9DB8}$
$\overline{DF}E862\overline{43}$*	$\overline{DF}EC\overline{9DB8}$	$\overline{DF}F2\overline{3BA7}$	$\overline{DF}F4\overline{3BA7}$	$\overline{DF}F84C\overline{57}$
$\overline{B7}E213BE$	$\overline{BB}C4\overline{F2}02$	$\overline{BB}C6\overline{F2}02$	$\overline{BB}C8\overline{F2}02$	$\overline{BB}D013BE$
$\overline{BB}D8\overline{F2}02$	$\overline{BB}E213BE$	$\overline{BB}F6D1\overline{3B}$	$\overline{BF}84\overline{A7D8}$	$\overline{BF}88D20D$
$\overline{BF}90\overline{A7D8}$	$\overline{BF}9273D0$	$\overline{BF}A473D0$	$\overline{BF}B273D0$	$\overline{BF}B473D0$
$\overline{BF}C6\overline{BE}66$	$\overline{BF}C8\overline{BE}66$	$\overline{BF}DCD20D$	$\overline{BF}E6\overline{A7D8}$	$\overline{BF}EC\overline{A7D8}$
$\overline{6B}84\overline{7A14}$	$\overline{6B}C65F\overline{7D}$	$\overline{6B}D85F\overline{7D}$	$\overline{6B}EC\overline{7A14}$	$\overline{6B}F238\overline{BF}$
$\overline{6B}F438\overline{BF}$	$\overline{6D}88\overline{7A14}$	$\overline{6F}A238\overline{BF}$	$\overline{6F}D238\overline{BF}$	$75B03D\overline{0A}$
$\overline{75}E43D\overline{0A}$	$\overline{77}48\overline{0B}41$	$77A2\overline{F2}02$	$\overline{77}DA\overline{0B}41$	$\overline{77}FAD20D$
$\overline{7B}8A\overline{F2}02$	$\overline{7B}A238\overline{BF}$	$\overline{7B}A8\overline{F2}02$	$\overline{7B}C27E\overline{D6}$	$\overline{7B}D238\overline{BF}$
$\overline{7B}D4\overline{F2}02$	$\overline{7B}E27E\overline{D6}$	$\overline{7B}FA\overline{A7D8}$	$\overline{7D}20\overline{3BA7}$	$\overline{7D}44\overline{2C77}$
$\overline{7D}48\overline{2C77}$	$\overline{7D}804C\overline{57}$	$\overline{7D}84D90A$	$\overline{7D}8A13BE$	$\overline{7D}A2\overline{9C22}$
$\overline{7D}A813BE$	$\overline{7D}CA13BE$	$\overline{7D}D2\overline{9C22}$	$\overline{7D}D413BE$	$\overline{7D}D6\overline{2C77}$
$\overline{7D}DA\overline{2C77}$	$\overline{7D}E6D90A$	$\overline{7D}ECD90A$	$\overline{7D}EE\overline{A7D8}$	$\overline{7D}F6\overline{3BA7}$
$\overline{7D}FC4C\overline{57}$	$\overline{7F}44\overline{A7D8}$	$\overline{7F}884574$	$\overline{7F}94\overline{7D}15$*	$\overline{7F}A26\overline{243}$
$\overline{7F}A4\overline{EC}D5$	$\overline{7F}AA09E4$	$\overline{7F}AC\overline{7D}15$*	$\overline{7F}B2\overline{EC}D5$	$\overline{7F}B4\overline{EC}D5$
$\overline{7F}C2AD94$	$\overline{7F}C475\overline{97}$	$\overline{7F}C675\overline{97}$	$\overline{7F}C875\overline{97}$	$\overline{7F}CA\overline{BE}66$
$\overline{7F}D0AD94$	$\overline{7F}D262\overline{43}$	$\overline{7F}D4\overline{BE}66$	$\overline{7F}D6\overline{A7D8}$	$\overline{7F}D875\overline{97}$
$\overline{7F}DA\overline{A7D8}$	$\overline{7F}DC4574$	$\overline{7F}E2AD94$	$\overline{7F}E4\overline{D7AE}$	$\overline{7F}E8AD94$
$\overline{7F}EA\overline{59}63$	$\overline{7F}EC\overline{79AB}$	$\overline{7F}F24C\overline{57}$	$\overline{7F}F44C\overline{57}$	$\overline{49}5011\overline{0E}$
$\overline{49}DC5E\overline{7E}$	$\overline{49}EA11\overline{0E}$	$\overline{49}F89D\overline{15}$	$\overline{4F}20\overline{EC}D5$	$\overline{4F}A83D\overline{0A}$
$\overline{4F}C4\overline{FD94}$	$\overline{4F}C8\overline{FD94}$	$\overline{4F}CA3D\overline{0A}$	$\overline{4F}D43D\overline{0A}$	$\overline{4F}D8\overline{FD94}$
$\overline{4F}DA38\overline{BF}$	$\overline{4F}F6\overline{EC}D5$	$\overline{4F}FA62\overline{43}$	$\overline{53}C65F\overline{7D}$	$\overline{53}D85F\overline{7D}$
$\overline{53}F238\overline{BF}$	$\overline{53}F438\overline{BF}$	$\overline{57}B03D\overline{0A}$	$\overline{57}E43D\overline{0A}$	$\overline{59}C4\overline{BD}F3$
$\overline{59}D0\overline{6250}$	$\overline{59}EEA803$	$\overline{59}FC\overline{EC}D5$	$\overline{5B}88\overline{7A14}$	$\overline{5B}C4\overline{BD}F3$
$\overline{5B}D0\overline{6250}$	$\overline{5B}EEA803$	$\overline{5B}FA73D0$	$\overline{5B}FC\overline{EC}D5$	$\overline{5D}4019\overline{68}$
$\overline{5D}44\overline{C0}13$	$\overline{5D}48\overline{C0}13$	$\overline{5D}82\overline{9C22}$	$\overline{5D}8438\overline{BF}$	$\overline{5D}88\overline{F2}02$
$\overline{5D}8A\overline{32}80$	$\overline{5D}9038\overline{BF}$	$\overline{5D}94\overline{0E84}$	$\overline{5D}985F\overline{7D}$	$\overline{5D}A0\overline{9C22}$
$\overline{5D}A273\overline{7F}$	$\overline{5D}A8\overline{32}80$	$\overline{5D}AC\overline{0E84}$	$\overline{5D}B0\overline{6250}$	$\overline{5D}C43D\overline{0A}$
$\overline{5D}C63D\overline{0A}$	$\overline{5D}C83D\overline{0A}$	$\overline{5D}CA\overline{32}80$	$\overline{5D}CC5F\overline{7D}$	$\overline{5D}D273\overline{7F}$
$\overline{5D}D6\overline{C0}13$	$\overline{5D}D83D\overline{0A}$	$\overline{5D}DA\overline{C0}13$	$\overline{5D}DC\overline{F2}02$	$\overline{5D}E4\overline{6250}$
$\overline{5D}E638\overline{BF}$	$\overline{5D}E8F6B9$	$\overline{5D}EC38\overline{BF}$	$\overline{5D}F2\overline{9C22}$	$\overline{5D}F4\overline{9C22}$
$\overline{5D}F673D0$	$\overline{5D}FA19\overline{68}$	$\overline{5D}FC62\overline{43}$	$\overline{5F}10\overline{A7D8}$	$\overline{5F}20\overline{3BA7}$
$\overline{5F}404B\overline{CE}$	$\overline{5F}44\overline{2C77}$	$\overline{5F}48\overline{2C77}$	$\overline{5F}804C\overline{57}$	$\overline{5F}84D90A$
$\overline{5F}8A13BE$	$\overline{5F}90D90A$	$\overline{5F}984B\overline{28}$	$\overline{5F}A2\overline{9C22}$	$\overline{5F}A813BE$
$\overline{5F}C229BD$	$\overline{5F}C47E\overline{D6}$	$\overline{5F}C67E\overline{D6}$	$\overline{5F}C87E\overline{D6}$	$\overline{5F}CA13BE$
$\overline{5F}CC4B\overline{28}$	$\overline{5F}D029BD$	$\overline{5F}D2\overline{9C22}$	$\overline{5F}D413BE$	$\overline{5F}D6\overline{2C77}$
$\overline{5F}D87E\overline{D6}$	$\overline{5F}DA\overline{2C77}$	$\overline{5F}DCA714$	$\overline{5F}E229BD$	$\overline{5F}E4BD\overline{A7}$
$\overline{5F}E6D90A$	$\overline{5F}E829BD$	$\overline{5F}EAE6\overline{F6}$	$\overline{5F}ECD90A$	$\overline{5F}EE\overline{A7D8}$
$\overline{5F}F6\overline{3BA7}$	$\overline{5F}FA4B\overline{CE}$	$\overline{5F}FC4C\overline{57}$	$\overline{27}945F\overline{7D}$	$\overline{27}A23D\overline{0A}$
$\overline{27}D23D\overline{0A}$	$\overline{2B}10\overline{58}27$	$\overline{2B}B03D\overline{0A}$	$\overline{2B}E43D\overline{0A}$	$\overline{2B}EE\overline{58}27$

Table 1: cont.

$\overline{2D}$8438$\overline{BF}$	$\overline{2D}$88$\overline{F2}$02	$\overline{2D}$985F$\overline{7D}$	$\overline{2D}$C43D$\overline{0A}$	$\overline{2D}$C63D$\overline{0A}$
$\overline{2D}$C83D$\overline{0A}$	$\overline{2D}$CC5F$\overline{7D}$	$\overline{2D}$D83D$\overline{0A}$	$\overline{2D}$DC$\overline{F2}$02	$\overline{2D}$EC38$\overline{BF}$
$\overline{2F}$4813BE	$\overline{2F}$8ADB37	$\overline{2F}$A8DB37	$\overline{2F}$CADB37	$\overline{2F}$D4DB37
$\overline{2F}$D613BE	$\overline{35}$B03D$\overline{0A}$	$\overline{35}$E43D$\overline{0A}$	$\overline{37}$107C74	$\overline{37}$20$\overline{7D}$15
$\overline{37}$504D$\overline{FE}$	$\overline{37}$8075$\overline{97}$	$\overline{37}$945F$\overline{7D}$	$\overline{37}$A23D$\overline{0A}$	$\overline{37}$C25B$\overline{FB}$
$\overline{37}$D05B$\overline{FB}$	$\overline{37}$D23D$\overline{0A}$	$\overline{37}$E85B$\overline{FB}$	$\overline{37}$EE7C74	$\overline{37}$FC75$\overline{97}$
$\overline{39}$44$\overline{F2}$02	$\overline{39}$8075$\overline{97}$	$\overline{39}$945F$\overline{7D}$	$\overline{39}$A23D$\overline{0A}$	$\overline{39}$AC5F$\overline{7D}$
$\overline{39}$B0$\overline{FD94}$	$\overline{39}$D23D$\overline{0A}$	$\overline{39}$DA$\overline{F2}$02	$\overline{39}$E4$\overline{FD94}$	$\overline{39}$FA$\overline{BE}$66
$\overline{39}$FC75$\overline{97}$	$\overline{3B}$44$\overline{F2}$02	$\overline{3B}$8075$\overline{97}$	$\overline{3B}$8ACB6F	$\overline{3B}$945F$\overline{7D}$
$\overline{3B}$A07E$\overline{D6}$	$\overline{3B}$A23D$\overline{0A}$	$\overline{3B}$A8CB6F	$\overline{3B}$AC5F$\overline{7D}$	$\overline{3B}$B0$\overline{FD94}$
$\overline{3B}$CACB6F	$\overline{3B}$D23D$\overline{0A}$	$\overline{3B}$D4CB6F	$\overline{3B}$DA$\overline{F2}$02	$\overline{3B}$E4$\overline{FD94}$
$\overline{3B}$EA4D$\overline{FE}$	$\overline{3B}$F47E$\overline{D6}$	$\overline{3B}$F6$\overline{7D}$15	$\overline{3B}$FA$\overline{BE}$66	$\overline{3B}$FC75$\overline{97}$
$\overline{3D}$40$\overline{314D}$	$\overline{3D}$4413BE	$\overline{3D}$4813BE	$\overline{3D}$80AD94	$\overline{3D}$8ADB37
$\overline{3D}$907E$\overline{D6}$	$\overline{3D}$92$\overline{6250}$	$\overline{3D}$A2F6B9	$\overline{3D}$A4$\overline{6250}$	$\overline{3D}$A8DB37
$\overline{3D}$AC3D$\overline{0A}$	$\overline{3D}$B2$\overline{6250}$	$\overline{3D}$B4$\overline{6250}$	$\overline{3D}$C45B$\overline{FB}$	$\overline{3D}$C65B$\overline{FB}$
$\overline{3D}$C85B$\overline{FB}$	$\overline{3D}$CADB37	$\overline{3D}$D2F6B9	$\overline{3D}$D4DB37	$\overline{3D}$D613BE
$\overline{3D}$D85B$\overline{FB}$	$\overline{3D}$DA13BE	$\overline{3D}$E67E$\overline{D6}$	$\overline{3D}$EC7E$\overline{D6}$	$\overline{3D}$F662$\overline{43}$
$\overline{3D}$FA$\overline{314D}$	$\overline{3D}$FCAD94	$\overline{3F}$204C$\overline{57}$	$\overline{3F}$44$\overline{ED}$CD	$\overline{3F}$50$\overline{31}$B6
$\overline{3F}$84$\overline{EDB7}$	$\overline{3F}$880A$\overline{DB}$	$\overline{3F}$8A$\overline{EF}$34	$\overline{3F}$90$\overline{EDB7}$	$\overline{3F}$92BD$\overline{A7}$
$\overline{3F}$947E$\overline{D6}$	$\overline{3F}$A4BD$\overline{A7}$	$\overline{3F}$A8$\overline{EF}$34	$\overline{3F}$B2BD$\overline{A7}$	$\overline{3F}$B4BD$\overline{A7}$
$\overline{3F}$CA$\overline{EF}$34	$\overline{3F}$D229BD	$\overline{3F}$D4$\overline{EF}$34	$\overline{3F}$DA$\overline{ED}$CD	$\overline{3F}$DC0A$\overline{DB}$
$\overline{3F}$E6$\overline{EDB7}$	$\overline{3F}$EA$\overline{31}$B6	$\overline{3F}$EC$\overline{EDB7}$	$\overline{3F}$EE74$\overline{CD}$	$\overline{3F}$F64C$\overline{57}$
$\overline{3F}$FA$\overline{D6}$59	$\overline{455E}$13BE	$\overline{13}$A238$\overline{BF}$	$\overline{13}$D238$\overline{BF}$	$\overline{14}$84$\overline{A7D8}$
$\overline{14}$88D20D	$\overline{14}$90$\overline{A7D8}$	$\overline{14}$9273D0	$\overline{14}$A473D0	$\overline{14}$B273D0
$\overline{14}$B473D0	$\overline{14}$DCD20D	$\overline{14}$E6$\overline{A7D8}$	$\overline{14}$EC$\overline{A7D8}$	$\overline{16}$44$\overline{2C77}$
$\overline{16}$48$\overline{2C77}$	$\overline{16}$804C$\overline{57}$	$\overline{16}$8A13BE	$\overline{16}$A2$\overline{9C22}$	$\overline{16}$A813BE
$\overline{16}$CA13BE	$\overline{16}$D2$\overline{9C22}$	$\overline{16}$D413BE	$\overline{16}$D6$\overline{2C77}$	$\overline{16}$DA$\overline{2C77}$
$\overline{16}$ECD90A	$\overline{16}$EE$\overline{A7D8}$	$\overline{16}$F2DA$\overline{46}$	$\overline{16}$F6$\overline{3BA7}$	$\overline{17}$40$\overline{314D}$
$\overline{17}$4413BE	$\overline{17}$4813BE	$\overline{17}$80AD94	$\overline{17}$8ADB37	$\overline{17}$A8DB37
$\overline{17}$B0$\overline{A4}$DE	$\overline{17}$C45B$\overline{FB}$	$\overline{17}$C65B$\overline{FB}$	$\overline{17}$C85B$\overline{FB}$	$\overline{17}$CADB37
$\overline{17}$D4DB37	$\overline{17}$D613BE	$\overline{17}$D85B$\overline{FB}$	$\overline{17}$DA13BE	$\overline{17}$E4$\overline{A4}$DE
$\overline{17}$FA$\overline{314D}$	$\overline{17}$FCAD94	$\overline{18}$44$\overline{A7D8}$	$\overline{18}$94$\overline{7D}$15*	$\overline{18}$AC$\overline{7D}$15*
$\overline{18}$D6$\overline{A7D8}$	$\overline{18}$DA$\overline{A7D8}$	$\overline{18}$E4$\overline{D7AE}$	$\overline{19}$A238$\overline{BF}$	$\overline{19}$A8$\overline{F2}$02
$\overline{19}$C0$\overline{EDB7}$	$\overline{19}$C27E$\overline{D6}$	$\overline{19}$D07E$\overline{D6}$	$\overline{19}$D238$\overline{BF}$	$\overline{19}$E87E$\overline{D6}$
$\overline{19}$EA$\overline{19}$60	$\overline{19}$FA$\overline{A7D8}$	$\overline{1A}$10$\overline{A7D8}$	$\overline{1A}$20$\overline{3BA7}$	$\overline{1A}$404B$\overline{CE}$
$\overline{1A}$44$\overline{2C77}$	$\overline{1A}$48$\overline{2C77}$	$\overline{1A}$8A13BE	$\overline{1A}$A2$\overline{9C22}$	$\overline{1A}$A813BE
$\overline{1A}$C67E$\overline{D6}$	$\overline{1A}$C87E$\overline{D6}$	$\overline{1A}$CA13BE	$\overline{1A}$D2$\overline{9C22}$	$\overline{1A}$D413BE
$\overline{1A}$D6$\overline{2C77}$	$\overline{1A}$DA$\overline{2C77}$	$\overline{1A}$EAE6$\overline{F6}$	$\overline{1A}$FA4B$\overline{CE}$	$\overline{1B}$20$\overline{EC}$D5
$\overline{1B}$5013BE	$\overline{1B}$80$\overline{D7AE}$	$\overline{1B}$82BD$\overline{A7}$	$\overline{1B}$884B$\overline{28}$	$\overline{1B}$90F95A
$\overline{1B}$94$\overline{BD}$F3	$\overline{1B}$A0BD$\overline{A7}$	$\overline{1B}$A2$\overline{6250}$	$\overline{1B}$A83D$\overline{0A}$	$\overline{1B}$AC$\overline{BD}$F3
$\overline{1B}$B07A$\overline{84}$	$\overline{1B}$C2$\overline{A4}$DE	$\overline{1B}$C4$\overline{FD94}$	$\overline{1B}$C8$\overline{FD94}$	$\overline{1B}$CA3D$\overline{0A}$
$\overline{1B}$D0$\overline{A4}$DE	$\overline{1B}$D2$\overline{6250}$	$\overline{1B}$D43D$\overline{0A}$	$\overline{1B}$D8$\overline{FD94}$	$\overline{1B}$DA38$\overline{BF}$
$\overline{1B}$DC4B$\overline{28}$	$\overline{1B}$E2$\overline{A4}$DE	$\overline{1B}$E47A$\overline{84}$	$\overline{1B}$E8$\overline{A4}$DE	$\overline{1B}$EA13BE
$\overline{1B}$ECF95A	$\overline{1B}$F2BD$\overline{A7}$	$\overline{1B}$F4BD$\overline{A7}$	$\overline{1B}$F6$\overline{EC}$D5	$\overline{1B}$FA62$\overline{43}$
$\overline{1B}$FC$\overline{D7AE}$	$\overline{1C}$84$\overline{EDB7}$	$\overline{1C}$92BD$\overline{A7}$	$\overline{1C}$A4BD$\overline{A7}$	$\overline{1C}$B2BD$\overline{A7}$
$\overline{1C}$B4BD$\overline{A7}$	$\overline{1C}$E6$\overline{EDB7}$	$\overline{1D}$40$\overline{314D}$	$\overline{1D}$4413BE	$\overline{1D}$4813BE

Table 1: cont.

1D80AD94	1D8229BD	1D8ADB37	1D907ED6	1D926250
1DA2F6B9	1DA46250	1DA8DB37	1DAC3D0A	1DB26250
1DB46250	1DC45BFB	1DC65BFB	1DC85BFB	1DCADB37
1DD2F6B9	1DD4DB37	1DD613BE	1DD85BFB	1DDA13BE
1DE67ED6	1DEC7ED6	1DFA314D	1DFCAD94	1E4829BD
1E94A4DE	1EACA4DE	1EEEAD94	22441BC5	22481BC5
22506CE4	2282A7D8	2284A22F	228A5827	2292A803
22A4A803	22A85827	22B07D15	22B2A803	22B4A803
22C074CD	22C2BE66	22C47C74	22C67C74	22C87C74
22CA5827	22D0BE66	22D20BC2	22D45827	22D61BC5
22D87C74	22DA1BC5	22E0AD94	22E2BE66	22E6A22F
22E8BE66	22EA6CE4	22ECA22F	22F0AD94	22F2A7D8
22F4A7D8	22F874CD	23245E7E	23480B41	23804574
23927A14	23A0A714	23A2F202	23A47A14	23B04B28
23B27A14	23B47A14	23B65E7E	23D60B41	23DA0B41
23E44B28	23E8FB5D	23EA491C	23F2A714	23F4A714
23F6A22F	23F80ADB	23FAD20D	23FC4574	2448D13B
24A273D0	24A80BC2	24C26243	24C47D15	24D06243*
24D273D0	24D6D13B	24D87D15	24DAD13B	24E26243
24E4ECD5	24E69DB8	24E86243*	24EC9DB8	24F23BA7
24F43BA7	24F84C57	25101BC5	2544F3E3	25822C77
258329BD	258513BE	258729BD*	2589F202	258A0DE7
258B13BE	2590F8D6	2591F202	259338BF	2594C017
25950DE7	25972C77	25A02C77	25A113BE	25A2C013
25A313BE	25A5C013	25A72C77	25A80DE7	25A90DE7
25ACC017	25AF3CE4	25B038BF	25B1F202	25B3F8D6
25B7D13B	25B8F202	25B90B41	25BB1BC5	893C7ABE
89367ABE	8DF27ABE	8D867ABE	D2C47418	D2DC7418
51C45F24	51C65F24	51C85F24	51D85F24	58C45F24
58C65F24	58C85F24	58D85F24	5C8E5F24	5CEC5F24
5C135F24	5C1B5F24	B62A11E3	B6EA11E3	B744641C
B75B641C	B7DA641C	BA128980	BADA8980	D1488980
D14A8980	D1BA8980	D1DA8980	D748641C	D74A641C
D7A28980	D7D28980	D7D6641C	D7DA641C	51E98980
51EB641C	51ED641C	5112641C	922A8980	92EA8980
96858980	A42A8980	A4AF8980	9D4A360C	9D29360C
9DA2D9F9	9DD2D9F9	9DD6360C	9DDA360C	A3A2360C
A3A4360C	A3B4D9F9	A3B7360C	A3BAD9F9	D62A360C
D6EA360C	5A3CD9F9	5A36D9F9	5A86360C	5A6F360C
32EBE2E7	32EDE2E7	3248E2E7	3225E2E7	B32B979F
B3A8979F	B3A9979F	B3CA979F	CAA7979F	CAB1979F
CA35979F	CAB9979F	CAAC7EC0		