# Hex-triangles with One Interior $H$-point

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**Abstract.** It is known that triangles with vertices in the integral lattice $\mathbb{Z}^2$ and exactly one interior lattice point can have 3,4,6,8 and 9 lattice points on their boundaries. No such triangles with 5, nor 7, nor $n \geq 10$ boundary lattice points exist. The purpose of this note is to study analogous property for Hex-triangles, that is, triangles with vertices in the set $H$ of corners of a tiling of $\mathbb{R}^2$ by regular hexagons of unit edge. We show that any Hex-triangle with exactly one interior $H$-point can have 3,4,5,6,7,8, or 10 $H$-points on its boundary and cannot have 9 nor $n \geq 11$ such points.

**1. Introduction and notation.** There are many interesting results and problems dealing with *lattice polygons*, that is, polygons with vertices in the integral lattice $\mathbb{Z}^2$. The fine expository article by Scott [3] references many of them and is a very good source to get acquainted with the flavor of the subject of lattice polygons. Geometrically, $\mathbb{Z}^2$ is the set of corners of a tiling of $\mathbb{R}^2$ by unit squares. The question arises: How the combinatorics of lattice polygons changes when the square tiling is replaced by another tiling of the plane with congruent regular polygons? Ding and Reay [1] examined the problem in the case of the celebrated Pick's theorem about the area of lattice polygons. From their results we can infer that there are only slight changes when $\mathbb{Z}^2$ is replaced by the corners of a tiling by regular triangles. However, there are significant changes when $\mathbb{Z}^2$ is replaced by the set $H$ of corners of a monohedral tiling of $\mathbb{R}^2$ by regular hexagons with unit edge (so the area of every tile is $3\sqrt{3}/2$). This indicates that the combinatorics of lattice polygons and $H$-*polygons* (simple polygons in $\mathbb{R}^2$ whose vertices

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lie in $H$) are different. We want to check how different they are in the case of lattice triangles and $H$-triangles with exactly one interior lattice point or $H$-point, respectively.

For a planar lattice polygon ($H$-polygon) $P$ in $\mathbb{R}^2$ we denote

$$b = b(P) = |X \cap \partial P| \qquad \text{and} \qquad i = i(P) = |X \cap \text{int}P|,$$

where $X = \mathbb{Z}^2$ or $X = H$.

It is known, see [2, 4], that a lattice triangle in $\mathbb{R}^2$ with exactly one interior lattice point can have 3,4,6,8, or 9 points from $\mathbb{Z}^2$ on its boundary. Moreover, no lattice triangle $\Delta$ with one interior lattice point and with $b(\Delta) = 5$, nor $b(\Delta) = 7$, nor $b(\Delta) \geq 10$ exists.

The main purpose of this paper is to examine analogous results for $H$-triangles. Namely, we show that any $H$-triangle $\Delta$ with exactly one interior $H$-point can have 3,4,5,6,7,8, or 10 $H$-points on its boundary, and cannot have 9 nor $b \geq 11$ such points.

We aim also to comment on the cases of lattice triangles with 5 or 7 boundary lattice points. It is observed here that such triangles exist if and only if the numbers of their interior lattice points are multiples of 3 and 5, respectively.

The main difference between the sets $H$ and $\mathbb{Z}^2$ is such that the latter is a lattice but the former is not. However $H$ can be considered to be the union of a lattice – denoted by $H^+$ – and an "affine" lattice $H^-$. All points in $H^+$ have three tiling edges leaving the point in the same three directions, and all points in $H^-$ have edges which leave in the three opposite directions. Two points $x$ and $y$ are said to have *opposite orientations* if $x \in H^+(H^-)$ and $y \in H^-(H^+)$. Otherwise we say they have the *same orientation*.

Let $A$ denote the set of all centers of the hexagonal tiles which determine $H$. Every element of $A$ will be called an *auxiliary point.* Clearly $H^+ \cup H^- \cup A$ is a *triangular lattice* with the area of each triangular tile $\sqrt{3}/4$. We will denote this lattice by $\mathbb{T}$.

**2. Results.** We prove our results by a sequence of lemmas beginning with a property of the set $H$.

**Lemma 1.** *Assume that the origin of* $\mathbb{R}^2$ *lies in* $H^+$. *If* $x \in A(H^-)$, *then* $2x \in H^-(A)$ *and* $3x \in H^+$.

*Proof:* Assume that the origin of $\mathbb{R}^2$ is placed in the left bottom corner of a hexagonal tile and that it belongs to $H^+$. Let the $x$-axis of $\mathbb{R}^2$ lie along the bottom edge of the tile, and let the $y$-axis be perpendicular to $x$. Denote $\vec{u} = (3/2, \sqrt{3}/2)$ and $\vec{v} = (0, \sqrt{3})$. One can see that the sets $H^+$ (which is now a triangular lattice), $H^-$ and $A$ have the following descriptions:

$$H^+ = \{\alpha\vec{u} + \beta\vec{v} \quad : \quad \alpha, \beta \in \mathbb{Z}\},$$

$$H^- = \{(1,0) + \alpha\vec{u} + \beta\vec{v} \quad : \quad \alpha, \beta \in \mathbb{Z}\},$$

$$A = \{(-1,0) + \alpha\vec{u} + \beta\vec{v} \quad : \quad \alpha, \beta \in \mathbb{Z}\}.$$

Assume that $x \in A$. Then $x = (-1,0) + \alpha(3/2, \sqrt{3}/2) + \beta(0, \sqrt{3})$ for some integers $\alpha$ and $\beta$. Thus

$$\begin{aligned} 2x &= (-2,0) + 2\alpha(3/2, \sqrt{3}/2) + 2\beta(0, \sqrt{3}) \\ &= (1,0) + (-3,0) + 2\alpha(3/2, \sqrt{3}/2) + 2\beta(0, \sqrt{3}) \\ &= (1,0) + (2\alpha - 2)(3/2, \sqrt{3}/2) + (2\beta + 1)(0, \sqrt{3}) \\ &\in H^-. \end{aligned}$$

Similarly

$$\begin{aligned} 3x &= (-3,0) + 3\alpha(3/2, \sqrt{3}/2) + 3\beta(0, \sqrt{3}) \\ &= (3\alpha - 2)(3/2, \sqrt{3}/2) + (3\beta + 1)(0, \sqrt{3}) \\ &\in H^+. \end{aligned}$$

If $x \in H^-$, then similar calculations yield the result. □

A segment with endpoints in $H$ is called an *$H$-segment*. From Lemma 1 we immediately get the following two lemmas.

**Lemma 2.** *If $xy$ is an $H$-segment such that* $\operatorname{relint} xy \cap H = \emptyset$ *and* $\operatorname{relint} xy \cap A = \{a\}$, *then $a$ is the midpoint of $xy$. Moreover, $x$ and $y$ have opposite orientations.*

**Lemma 3.** *If an $H$-segment contains two auxiliary points, then between them there are at least two $H$-points with opposite orientations.*

In subsequent lemmas we will use the notion of $k$th *level* of a Hex-triangle $\Delta = \operatorname{conv}\{x, y, z\}$. We will always label the vertices of $\Delta$ in such a way that $xy$ has the most points from $\mathbb{T} = H \cup A$. The line containing $xy$ is denoted by $l_0$. By $l_1$ denote the line parallel to $l_0$ and passing through the first point from $\mathbb{T}$ reached during shifting $l_0$ towards $z$. We continue shifting $l_0$ and denote by $l_2$ the line passing through the next point from $\mathbb{T}$ met in this way. In a similar way we define $l_j$ for $j \geq 3$. Clearly, the lines are parallel and the distance between $l_j$ and $l_{j+1}$ is the same for every $j$. We say that $\Delta$ *has $k$ levels* if $z \in l_k$. It is obvious that every Hex-triangle with one interior $H$-point has at least two levels.

Let $m_j$ be the *relative length* of $\Delta \cap l_j$, that is, the length in relation to the unit being the distance between two consecutive $\mathbb{T}$-points on $xy$. Denote by $t_j = |l_j \cap \Delta \cap \mathbb{T}|$. Clearly $m_0$ is a positive integer and $m_0 = t_0 - 1$. One can see that for $j > 0$ we have $[m_j] \leq t_j \leq [m_j] + 1$, where $[\cdot]$ denotes the greatest integer function.

**Lemma 4.** *If a Hex-triangle $\Delta$ has $k$ levels, then for $0 \leq j < k$ we have $m_j = m_0 \left(1 - \frac{j}{k}\right)$.*

**Lemma 5.** *If $\Delta$ is an $H$-triangle with one interior $H$-point, then $\Delta$ cannot have:*

(i) *Two auxiliary points on one side and additional auxiliary point on another side,*

(ii) *Three auxiliary points on one side.*

*Proof:* (i) Suppose to the contrary that there is a Hex-triangle $\Delta =$ conv$\{x, y, z\}$ with two auxiliary points on, say $xy$, and another auxiliary point $a$ on, say $xz$. Since $t_1$ does not decrease when the number of levels increase, we can assume that $\Delta$ has two levels and – consequently – that $a \in l_1$. By Lemma 3, $t_0 \geq 6$ and therefore $m_0 \geq 5$. From Lemma 4 it follows that $m_1 \geq 2.5$ Since $a \in l_1 \cap \partial\Delta$ and $m_1 \geq 2.5$, we have at least two interior $\mathbb{T}$-points in $l_1 \cap \Delta$. By Lemma 1 the two interior $\mathbb{T}$-points are $H$-points, a contradiction.

(ii) We again suppose to the contrary that a Hex-triangle $\Delta$ has three auxiliary points on, say $xy$. Lemma 3 implies that $t_0 \geq 9$ and $m_0 \geq 8$. From Lemma 4 we have $m_1 \geq 4$. This guarantees that $t_1 \geq 4$. From among the (at least) four $\mathbb{T}$-points, (at least) three are interior points, and – by Lemma 1 – (at least) two are $H$-points, a contradiction. □

The following observation is an immediate consequence of Lemma 5.

**Lemma 6.** *No $H$-triangle with one interior $H$-point and more than three auxiliary points on its boundary exists.*

It is still possible that a Hex-triangle with one interior $H$-point can have three auxiliary points on its boundary. The next lemma provides a complete description of such triangles.

**Lemma 7.** *If for an $H$-triangle $\Delta$ with one interior $H$-point we have $|A \cap \partial\Delta| = 3$, then every side of $\Delta$ must contain exactly one auxiliary and one $H$-point in its relative interior. Moreover, two auxiliary points cannot lie on the same level of $\Delta$.*

*Proof:* The assumption $|A \cap \partial\Delta| = 3$ in conjunction with Lemma 5 implies that every side of $\Delta$ must contain one auxiliary point. First, let notice that at least one side of $\Delta$ must contain a point from $H$ in its relative interior. Indeed, if no side of $\Delta$ contained any $H$-point, then by Lemma 1 the auxiliary points would have to be the midpoints of their sides and the endpoints of every side would have to have opposite orientations. Thus one point would have double orientation, which – of course – is impossible. So, one side of $\Delta$ must contain an $H$-point in its relative interior. Suppose it is $xy$. Thus $t_0 \geq 4$. Clearly, the remaining two auxiliary points cannot lie on the same level of $\Delta$, otherwise there would be – by Lemma 3 – two $H$-points in the interior of $\Delta$. It follows that $\Delta$ has at least 3 levels and therefore – by Lemma 4 – we have $m_1 \geq 2$ and $m_2 \geq 1$. Since there is an auxiliary point in $l_1 \cap \partial\Delta$, we obviously cannot have $m_1 > 2$. So $m_1 = 2$ and $t_1 = 3$. Hence, in addition to an auxiliary boundary point in $l_1 \cap \Delta$ we have two $H$-points, one in the interior and the other one on the boundary of $\Delta$. Similarly, we obtain $m_2 = 1$ and $t_2 = 2$. This leads to one $H$-point and one auxiliary point on the boundary of $\Delta$ and ends the proof. An example of an $H$-triangle satisfying the conditions of Lemma 7 is shown below. ☐

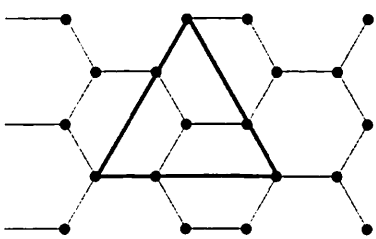

Fig.1: Hex-triangle satisfying conditions of Lemma 7.

**Lemma 8.** *No $H$-triangle with one interior $H$-point and four interior auxiliary points exists.*

*Proof.* Assume to the contrary that an $H$-triangle $\Delta$ contains four interior auxiliary points. First, we shall show that the auxiliary points cannot lie on one line. If they were on one line then the line could not pass through any $H$-point, otherwise between any two auxiliary points there were two $H$-points, which is impossible. Moreover, the line would have to be – as one can easily check – parallel to one side of $\Delta$, say $xy$. Consequently, we can assume that the auxiliary points lie on $l_1$. Another consequence of the above observations is that on $l_0$ and $l_3$ we have only $H$-points with the same orientation, and on $l_2$ there are only $H$-points but with opposite orientations. We can also assume that $\Delta$ has three levels.

Consider the line $p$ passing through the vertex $z$ and the existing interior $H$-point (lying on $l_2$). Since the two $H$-points have opposite orientations, Lemma 3 implies that $p$ intersects $l_1$ at an auxiliary point and $l_0$ at an

$H$-point. Clearly, on one side of $p$ there are two interior auxiliary points of $\Delta$. We denote the part of $l_1$ containing two interior auxiliary points by $l_{1,2}$ and the relative length of $\Delta \cap l_{1,2}$ by $m_{1,2}$. Similarly, the part of $l_2$ lying on the same side of $p$ as $l_{1,2}$ does is denoted by $l_{2,2}$ and the relative length of $\Delta \cap l_{2,2}$ is denoted by $m_{2,2}$.

The definition of $l_{1,2}$ implies that $m_{1,2} > 2$. By similar triangles we have

$$\frac{2}{m_{1,2}} = \frac{1}{m_{2,2}}.$$

Thus $m_{2,2} = m_{1,2} \cdot \frac{1}{2} > 2 \cdot \frac{1}{2} > 1$. This means that $l_{2,2} \cap \text{int}\Delta$ contains an $H$-point. So we obtained two $H$-points in $l_2 \cap \text{int}\Delta$, a contradiction.

Now we consider the case when the four interior auxiliary points do not lie on one line. From the fact that there are four such points it follows that two of them, say $a$ and $b$, lie on a line that is not parallel to any side of $\Delta$. The points $a$ and $b$ cannot be collinear with any vertex of $\Delta$, otherwise by Lemma 3 we would have two interior $H$-points lying between $a$ and $b$, which is impossible. If we translate the line passing through $a$ and $b$ to every vertex of $\Delta$ then at one vertex, say $x$, the translate will intersect the interior of $\Delta$. We can assume that the translation carries $a$ to $x$ and $b$ to an interior point $x'$. Since $a$ and $b$ are both auxiliary points, $x$ and $x'$ are both $H$-points with the same orientations. Consider the triangle $xx''c$ (see Fig. 2), where

$$x'' = x + 2(x' - x) \qquad \text{and} \qquad c = x + 2(a - x).$$

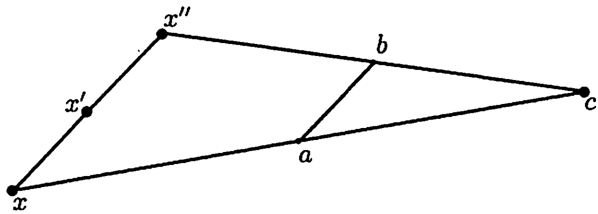

Fig.2: Triangle $xx''c$.

One can see that $x''$ and $c$ are both $H$-points (with opposite orientations) and $b$ is the midpoint of $x''c$. Since $b \in \text{int}\Delta$, at least one of the two $H$-points $x''$ or $c$ must lie in $\text{int}\Delta$. The one point from among $x''$ and $c$ plus $x'$ give us two interior $H$-points lying in $\Delta$, a contradiction. □

In the next lemma we summarize the relationships between the set $A$ of auxiliary points and a Hex-triangle with one interior $H$-point.

**Lemma 9.** *For any Hex-triangle $\Delta$ with one interior $H$-point we have $|A \cap \partial\Delta| \leq 3$ and $|A \cap \text{int}\Delta| \leq 3$.*

Now we shall study sufficient conditions for the existence of Hex-triangles with exactly one interior $H$-point. We express the sufficient conditions in terms of the numbers of $\mathbb{T}$-points on each side of the Hex-triangle. Denote by $\alpha$, $\beta$ and $\gamma$ the relative lengths of the sides of $\Delta$, that is, the lengths with respect to the units that are distances between consecutive $\mathbb{T}$-points on the lines containing respective sides of $\Delta$. We can assume that $\alpha \geq \beta \geq \gamma$. Thus $\alpha = m_0$ and $t_0 = \alpha + 1$. Also $\beta + 1$ and $\gamma + 1$ represent the numbers of $\mathbb{T}$-points on the other two sides of $\Delta$.

Since we want to apply Pick's theorem, it will be convenient to use a linear transformation to map $\mathbb{T}$ into $\mathbb{Z}^2$ and $\Delta$ into a lattice triangle $\Delta'$. Clearly, we can assume that the linear transformation carries $x$ into the origin. Obviously, the numbers of $\mathbb{T}$-points on the sides of $\Delta$ and lattice points on the respective sides of $\Delta'$ are the same. Thus

$$\alpha + \beta + \gamma = b(\Delta') = b.$$

The area argument presented below is very similar to that in Lemma 3 from [4]. We repeat it here hoping that this will be with a convenience to the reader.

The two sides of $\Delta'$ meeting at the origin are of the form $\alpha(a_1, a_2)$ and $\beta(b_1, b_2)$ for some integers $a_1, a_2, b_1, b_2$. Computing the area of $\Delta'$, $\mu(\Delta')$, by means of the well-known determinant formula we get

$$\mu(\Delta') = \frac{1}{2}\left|\begin{vmatrix} \alpha a_1 & \alpha a_2 \\ \beta b_1 & \beta b_2 \end{vmatrix}\right| = \frac{1}{2}\alpha\beta\left|\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right| = \frac{1}{2}\alpha\beta t$$

for some $t \in \mathbb{N}$. On the other hand, using Pick's theorem we have $\mu(\Delta') = \frac{b}{2} + i(\Delta') - 1$. The interior lattice points in $\Delta'$ are images of some auxiliary points and one $H$-point in $\Delta$. So $i(\Delta') = 1 + i_A$, where $i_A$ is the number of images of interior auxiliary points in $\Delta$. Thus

$$\mu(\Delta') = \frac{b}{2} + i_A,$$

where – by Lemma 8 – $i_A \in \{0, 1, 2, 3\}$. Equating the two expressions for the area of $\Delta'$ we see that

$$b + 2i_A = \alpha\beta t.$$

This implies that $b + 2i_A$ is divisible by $\alpha, \beta$ and $\alpha\beta$. If we choose another vertex of $\Delta$ to be carried into the origin then we will similarly obtain that $b + 2i_A$ must be also divisible by $\gamma, \alpha\gamma$ and $\beta\gamma$.

Summarizing the above observations we can see that the existence of Hex-triangles with one interior $H$-point has been reduced to solving the problem:

for given $b \geq 3$ and $i_A \in \{0,1,2,3\}$ find triples $(\alpha, \beta, \gamma)$ of positive integers satisfying the following conditions

(1) $\alpha + \beta + \gamma = b$,

(2) $\alpha \geq \beta \geq \gamma$,

(3) $\alpha \geq \frac{b}{3}$,

(4) all the numbers $\alpha$, $\beta$, $\gamma$, $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$ divide $b + 2i_A$.

**Lemma 10.** *No lattice triangle with $b \geq 15$ boundary points and $1 + k$, $k \in \{0,1,2,3\}$, interior lattice points exists.*

*Proof:* If a lattice triangle with $b \geq 15$ and $i = 1 + k$, where $k = 0, 1, 2$, or 3, existed, then its triple would satisfy conditions (1)-(4) above. By condition (4) we would have for some $n \in \mathbb{N}$

$$b + 2k = n\alpha\beta. \tag{1}$$

Using condition (3) we would get

$$b + 2k = n\alpha\beta \geq n\frac{b}{3}\beta,$$

which can be rewritten as

$$b(n\beta - 3) \leq 6k, \qquad k = 0, 1, 2, 3. \tag{2}$$

Since $b \geq 15$, inequality (2) could be true only when $n\beta - 3 \leq 1$ or $n\beta \leq 4$. This inequality is true when

$$\begin{cases} n = 1 \\ \beta = 1 \end{cases} \text{or} \quad \begin{cases} n = 2 \\ \beta = 1 \end{cases} \text{or} \quad \begin{cases} n = 3 \\ \beta = 1 \end{cases} \text{or} \quad \begin{cases} n = 4 \\ \beta = 1 \end{cases} \text{or}$$

$$\begin{cases} n = 1 \\ \beta = 2 \end{cases} \text{or} \quad \begin{cases} n = 1 \\ \beta = 3 \end{cases} \text{or} \quad \begin{cases} n = 1 \\ \beta = 4 \end{cases} \text{or} \quad \begin{cases} n = 2 \\ \beta = 2. \end{cases}$$

First, we show that $\beta = 1$ cannot happen. Indeed, if $\beta = 1$, then by condition (2) also $\gamma = 1$, and by condition (1) $\alpha = b - 2$ and equality (1) would have the following form

$$b + 2k = n\alpha\beta = n(b - 2)$$

which clearly makes sense only when $n > 1$. Solving the latter equality for $b$ and estimating we obtain

$$b = \frac{2k + 2n}{n - 1} \leq 10$$

because $k \in \{0,1,2,3\}$ and $n \in \{2,3,4\}$.

Second, we shall show that $n = 1$ cannot happen either. Assume that $n = 1$ and $\beta \in \{2, 3, 4\}$. By condition (1), $\alpha = b - \beta - \gamma$. Hence equality (1) has now the form

$$b + 2k = (b - \beta - \gamma)\beta.$$

After solving it for $b$ and evaluating for different values of $\beta$, keeping in mind that $\gamma \leq \beta$, we obtain

$$b = \frac{\beta^2 + \beta\gamma + 2k}{\beta - 1} \leq \begin{cases} 14 & \text{when} \quad \beta = 2 \\ 12 & \text{when} \quad \beta = 3 \\ \frac{38}{3} & \text{when} \quad \beta = 4 \end{cases}$$

Lastly, consider the case when $n = 2$ and $\beta = 2$. Now equality (1) has the form

$$b + 2k = 4\alpha$$

where $\alpha = b - 2 - \gamma$ and $1 \leq \gamma \leq 2$. Thus

$$b + 2k = 4b - 8 - 4\gamma$$

and since $1 \leq \gamma \leq 2$ and $0 \leq k \leq 3$ we get

$$b = \frac{8 + 2k + 4\gamma}{3} \leq \frac{22}{3}.$$

In either of the three cases considered we get $b < 15$, a contradiction. The proof is complete. □

In view of Lemmas 9 and 10 it is obvious that for finding all possible triples $(\alpha, \beta, \gamma)$ satisfying conditions (1)-(4) we can restrict the domains of $b$ and $i_A$ to: $3 \leq b \leq 14$ and $0 \leq i_A \leq 3$. One can check that the table below provides the only solutions to the system (1)-(4). From the prospective of our main result the most important part of the table is the one when $b \geq 9$. However, to have a full account of possible solutions of the system (1)-(4) we provide also the triples for $3 \leq b \leq 8$.

| | $b$ | | | | | |
|---|---|---|---|---|---|---|
| $i_A$ | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | (1,1,1) | (2,1,1) | – | (3,2,1) | – | (4,2,2) |
| 1 | (1,1,1) | (2,1,1) | – | (2,2,2)<br>(4,1,1) | (3,3,1) | (5,2,1) |
| 2 | (1,1,1) | (2,1,1) | (3,1,1) | – | – | (4,3,1)<br>(6,1,1) |
| 3 | (1,1,1) | (2,1,1) | – | (2,2,2)<br>(3,2,1)<br>(4,1,1) | – | – |

| $i_A$ | $b$ | | | | | |
|---|---|---|---|---|---|---|
| | 9 | 10 | 11 | 12 | 13 | 14 |
| 0 | (3,3,3) | - | - | - | - | - |
| 1 | - | (6,2,2) | - | - | - | - |
| 2 | - | (7,2,1) | - | (4,4,4)<br>(8,2,2) | - | - |
| 3 | (5,3,1) | (4,4,2)<br>(8,1,1) | - | (6,3,3)<br>(9,2,1) | - | (10,2,2) |

Having the above table we are in a position to prove the main result of this paper.

**Theorem.** *If $\Delta$ is a Hex-triangle with one interior $H$-point, then $b(\Delta) \in \{3, 4, 5, 6, 7, 8, 10\}$.*

*Proof:* Figures 4 and 5 provide Hex-triangles with one interior $H$-point and the required numbers of boundary points. Therefore, in order to prove our theorem it is enough to show that no other numbers of boundary points are possible.

Let $(\alpha, \beta, \gamma)$ be the triple of a Hex-triangle $\Delta$ with one interior $H$-point. Obviously

$$\alpha + \beta + \gamma \geq b(\Delta).$$

This in conjunction with Lemma 10 implies that no $\Delta$ with $b(\Delta) \geq 15$ exists.

If a Hex-triangle with one interior $H$-point and $b(\Delta) < 15$ exists, then its triple is among the triples provided by the above table. We shall check which of these triples have indeed a realization. Since the crucial part of the table is the one when $b \geq 9$ we examine every triple in this part. We eliminate some of these triples and describe possible realizations of others.

We start with eliminating all triples with $\alpha \geq 8$. If $\alpha = m_0 \geq 8$, then on $l_0$ we can only have $H$-points with the same orientation, otherwise we would have three auxiliary points, which by Lemma 5 is impossible. If $l_0$ contains only $H$-points with the same orientations then on $l_1$ there are only auxiliary points and on $l_2$ $H$-points with opposite orientations or vice-versa. Consequently, $\Delta$ has at least three levels and by Lemma 4 we have $m_1 = m_0(1 - 1/k) \geq 8(1 - 1/3) > 5$. This guarantees too many auxiliary or $H$-points in $l_1 \cap \Delta$. So, no Hex-triangle with one interior $H$-point having triple $(\alpha, \beta, \gamma)$ in which $\alpha \geq 8$ exists.

Now we shall eliminate the triple (7,2,1). Here $\beta = 2$ and this guarantees that in the relative interior of one side of $\Delta$, say $xz$, there is a $\mathbb{T}$-point $u$. If $x$ and $z$ have the same orientation, then $u$ is an $H$-point with the same orientation as $x$ and $z$ and, as it is easy to check, the line $l_k$ passing through $u$ contains at least two interior $H$-points. When $x$ and $z$ have opposite orientations, then $u$ is an auxiliary point and $\Delta$ has two levels. Any $\Delta$ with such a triple has $m_1 > 3$ with different types of points on $l_1$. This means that there are three interior $\mathbb{T}$-points in $l_1 \cap \Delta$ and two of them are $H$-points. Hence, no $\Delta$ with such a triple exists.

Elimination of the triple (4,4,2). We can even show that no lattice triangle with such a triple exists. If a lattice triangle with such a triple existed, then its sides would be of the form $4\vec{a}$, $4\vec{b}$ and $\vec{c} = 4\vec{b} - 4\vec{a}$ for some *lattice vectors* (vectors with integer coordinates) $\vec{a}$ and $\vec{b}$. Since $\vec{b} - \vec{a} = 1/4\vec{c}$ and $\vec{b} - \vec{a}$ is obviously a lattice vector, so must be $1/4\vec{c}$. But then $1/2\vec{c}$ and $3/4\vec{c}$ would be also lattice vectors, which would imply $\gamma \geq 4$. Thus no Hex-triangle with triple (4,4,2) exists.

The last triple which we eliminate is (4,4,4). Clearly, we can assume that $\Delta$ with such a triple has four levels. Hence, by Lemma 4, $m_1 = 3$ and $m_2 = 2$. If all points on $l_0$ were $H$-points with the same orientation then - since $m_1 = 3$ - on $l_1$ we would have to have auxiliary points, on $l_2$ $H$-points with opposite orientations, on $l_3$ $H$-points with the same orientation as on $l_0$ and eventually on $l_4$ again auxiliary points. As $\Delta$ is a Hex-triangle this cannot happen. So we can assume that on $l_0$ there are both $H$-points and auxiliary points. It can be easily checked that this can happen only when an auxiliary point is the midpoint of $xy$, and $x$ and $y$ have opposite orientations. If $z$ has the same orientation as $x$, then on $xz$ there can be only $H$-points with the same orientation. On $yz$, in turn, the midpoint is an auxiliary point. Now one can check that the only interior $\mathbb{T}$-point on $l_2$ is an $H$-point (with the same orientation as $y$), and there is an additional $H$-point (also with the same orientation as $y$) on $l_1$. Hence no Hex-triangle with one interior $H$-point and triple (4,4,4) exists.

Triple (6,2,2). Reasoning similar to that in the case of triple (7,2,1) leads to the following $H$-triangle which is the only realization of the triple.

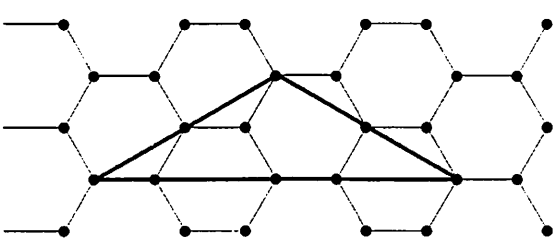

Fig.3: Hex-triangle with triple (6,2,2).

Triples (6,3,3) and (5,3,1). Here $\beta = 3$ which implies that $\Delta$ has at least three levels. For similar reasons as above we can assume that $\Delta$ has three levels. Since in both cases $m_1 > 3$ we cannot have $H$-points and auxiliary points on $l_0$ (and therefore on $l_1$, $l_2, \ldots$, either), otherwise there were two interior $H$-points on $l_1$. Thus on $l_0$ there have to be $H$-points with the same orientation, on $l_1$ auxiliary points, on $l_2$ $H$-points with opposite orientations, and $z \in l_3$ has the same orientation as $x$ does. One can check that every triangle with triple (5,3,1) has exactly one auxiliary point on its boundary and therefore $b(\Delta) = 8$ in this case. Triangle $xyz_2$ on Fig. 4 shows that triple (5,3,1) has a realization. For any triangle with triple (6,3,3) we have $|l_1 \cap \partial\Delta| = 2$, which means that there are two auxiliary points on its boundary. Hence $b(\Delta) = 10$ and triangle $xyz_3$ on Fig. 4 illustrates that triple (6,3,3) has indeed a realization.

Triple (3,3,3). If all points on $l_0$ are $H$-points with the same orientation, then on $l_1$ we have to have only $H$-points but with opposite orientations. Then on $l_2$ there are two boundary auxiliary points and $b(\Delta) = 7$. If on $l_0$ there is an auxiliary point then $b(\Delta) \leq 8$. It can be shown that in this case we either have $b(\Delta) = 7$ or $b(\Delta) = 6$. The triangle on Fig.1 is a realization of the latter case. Now the proof is complete. □

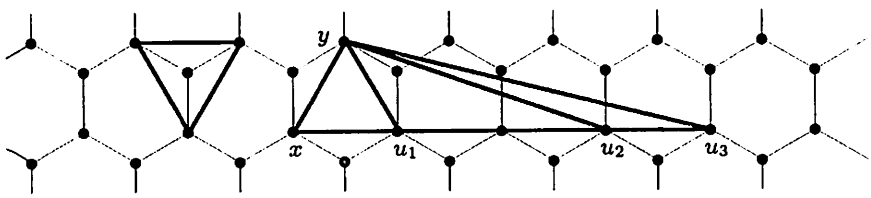


Fig.4: $H$-triangles with $i = 1$ and $b = 3$, $b(yu_1u_2) = 4$, $b(xyu_2) = 5$, $b(xyu_3) = 6$.

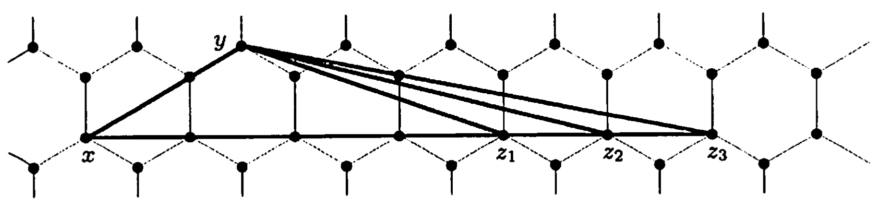


Fig.5: $H$-triangles with $i = 1$ and $b(xyz_1) = 7$, $b(xyz_2) = 8$, $b(xyz_3) = 10$.

Figures 3, 4 and 4 do not cover the entire spectrum of Hex-triangles with one interior $H$-point. The reader is invited to construct other examples. Obviously, for every number $b \geq 3$ there exists a Hex-triangle without interior $H$-points and exactly $b$ boundary $H$-points.

**Remark.** It is probably of interest to notice that lattice triangles with $b = 5$ and $b = 7$ exist if and only if $i = 3t$ and $i = 5s$, respectively.

First we comment on the case $b = 5$. Triple (3,1,1) is the only possible one, no matter what is $i$. By Pick's theorem the area of such triangles is equal to $3/2 + i$. On the other hand, see Lemma 3 in [4], the area must be a multiple of 3/2. So, for some $n \in \mathbb{N}$, $3/2 + i = n3/2$, or equivalently $2i = 3(n-1)$. This implies $i = 3t$ for some $t \in \mathbb{N}$. One can check that the lattice triangles with vertices $(0,0), (3,0)$ and $(2, 3+2j)$, $j \geq 0$, are of the type (3,1,1) and contain $3(j+1)$ interior lattice points.

For $b = 7$ the above table provides the triple (3,3,1). In a similar way as in the case of triple (4,4,2) it can be shown that no lattice triangle with such a triple exists. The other way of splitting 7 (not provided by the table) is (5,1,1). Proceeding in a similar way as above one can check that the triple has a realization if and only if $i = 5s$. Note that the lattice triangles with vertices $(0,0), (5,0)$ and $(4, 3+2j)$, $j \geq 0$, are of the type (5,1,1) and have $5(j+1)$ interior lattice points. □

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