Resolving Acyclic Partitions of Graphs

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Abstract

For a vertex v of a connected graph G and a subset S of $V(G)$, the distance between v and S is $d(v, S) = \min\{d(v, x) | x \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$, the code of v with respect to Π is the k-vector $c_\Pi(v) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. The k-partition Π is a resolving partition if the k-vectors $c_\Pi(v)$, $v \in V(G)$, are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is the partition dimension $\text{pd}(G)$ of G. A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is a resolving-coloring if each S_i $(1 \le i \le k)$ is independent and the resolving-chromatic number $\chi_r(G)$ is the minimum number of colors in a resolving-coloring of G. A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ is acyclic if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \le i \le k)$ is acyclic in G. The minimum k for which there is a resolving acyclic k-partition of $V(G)$ is the resolving acyclic number $a_r(G)$ of G. Thus $2 \le \text{pd}(G) \le a_r(G) \le \chi_r(G) \le n$ for every connected graph G of order $n \ge 2$. We present bounds for the resolving acyclic number of a connected graph in terms of its arboricity, partition dimension, resolving-chromatic number, diameter, girth, and other parameters. Connected graphs of order $n \ge 3$ having resolving acyclic number 2, n, or $n-1$ are characterized.

Key Words: distance, resolving partition, acyclic resolving partition.

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1 Introduction

Let G be a nontrivial connected graph. For a set S of vertices of G and a vertex v of G, the *distance* $d(v, S)$ *between* v *and* S is defined as

$$d(v,S) = \min\{d(v,x) \ : \ x \in S\},$$

where $d(v,x)$ is the distance between v and x. For an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vertex v of G, the *code of* v *with respect to* Π is defined as the k-vector

$$c_\Pi(v) = (d(v,S_1), d(v,S_2), \ldots, d(v,S_k)).$$

The partition Π is a ***resolving partition*** for G if the distinct vertices of G have distinct codes with respect to Π. The minimum k for which there is a resolving k-partition of $V(G)$ is the ***partition dimension*** $\mathrm{pd}(G)$ of G. A resolving partition of $V(G)$ containing $\mathrm{pd}(G)$ elements is called ***a minimum resolving partition***. These concepts were introduced and studied in [4].

Resolving partitions that satisfy certain prescribed properties have been studied. A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is ***independent*** if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \leq i \leq k)$ is independent in G. This topic was introduced and studied from the point of view of graph coloring in [5, 6]. If Π is an independent partition of $V(G)$, then, by coloring the vertices in S_i by i $(1 \leq i \leq k)$, we obtain a proper coloring c of G with k colors that distinguishes all vertices of G in terms of their distances from the color classes. Thus, such a coloring c of a connected graph G is called a ***resolving-coloring***. A ***minimum resolving-coloring*** uses a minimum number of colors and this number is the ***resolving-chromatic number*** $\chi_r(G)$ of G. Since every resolving-coloring is a coloring, $\chi(G) \leq \chi_r(G)$ for each connected graph G. In [5, 6] a resolving-coloring is referred to as a *locating-coloring* and the resolving-chromatic number as the *locating-chromatic number*. We refer to the book [3] for graph theory notation and terminology not described here.

In this paper, we extend resolving-coloring by requiring a property of color classes that is less restrictive than being independent. For a connected graph G, a partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is *acyclic* if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \leq i \leq k)$ is acyclic in G. The *vertex-arboricity* $a(G)$ of G is defined in [1, 2] as the minimum k such that $V(G)$ has an acyclic k-partition. If an acyclic partition Π of $V(G)$ is also a resolving partition, then Π is called a *resolving acyclic partition* of G. The minimum k for which G contains a resolving acyclic k-partition is the *resolving acyclic number* $a_r(G)$ of G. Since every resolving acyclic partition is an acyclic partition, $a(G) \leq a_r(G)$ for each connected graph G. The relationships among $\mathrm{pd}(G)$, $a_r(G)$, and $\chi_r(G)$ are as follows.

Observation 1.1 *For every connected graph G of order $n \geq 2$,*

$$2 \leq \text{pd}(G) \leq a_r(G) \leq \chi_r(G) \leq n.$$

To illustrate these concepts, consider the graph G of Figure 1(a). Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{x\}$, $S_2 = \{u\}$, and $S_3 = \{v, y, z\}$ as shown in Figure 1(b). Then the corresponding codes of vertices of G are

$$r(u|\Pi) = (1,0,1) \quad r(v|\Pi) = (1,2,0) \quad r(x|\Pi) = (0,1,1)$$
$$r(y|\Pi) = (1,1,0) \quad r(z|\Pi) = (2,1,0).$$

Since the codes of the vertices of G with respect to Π are distinct, Π is a resolving partition of G. Because no 2-partition is a resolving partition of G, it follows that Π is a minimum resolving partition of G and so $\text{pd}(G) = 3$. However, Π is not acyclic since $\langle S_3 \rangle = K_3$. On the other hand, let $\Pi' = \{S'_1, S'_2, S'_3, S'_4\}$, where $S'_1 = \{x\}$, $S'_2 = \{u\}$, $S'_3 = \{v, y\}$, and $S'_4 = \{z\}$ as shown in Figure 1(c). It can be verified that Π' is a resolving acyclic partition of G and no 3-partition is a resolving acyclic partition of G. Thus $a_r(G) = 4$. Furthermore, it was shown in [5] that $\chi_r(G) = 5$. Therefore, $\text{pd}(G) < a_r(G) < \chi_r(G)$ for the graph G of Figure 1(a).

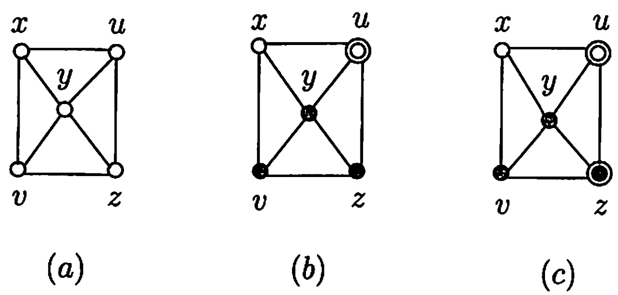

Figure 1: Illustrating concepts

The example just described also illustrates an important point. When determining whether a given partition Π is a resolving partition of a connected graph G, we need only verify that vertices of G belonging to the same subset of $V(G)$ in Π have distinct codes since the codes of two vertices in different subsets in Π have 0 in different coordinates.

2 Bounds on Resolving Acyclic Numbers of Graphs

In this section, we establish bounds for the resolving acyclic numbers of connected graphs in terms of other parameters, beginning with arboricity, partition dimension, and resolving-chromatic number. We present two useful lemmas. Since a proof of the first lemma is straightforward, we omit it.

Lemma 2.1 *If H is an induced subgraph of a nontrivial connected graph G, then $a(H) \le a(G)$.*

Let Π and Π' be two partitions of $V(G)$. Then Π' is called a *refinement* of Π if each element of Π' is a subset of some element of Π.

Lemma 2.2 *Let G be a nontrivial connected graph and let Π and Π' be two partitions of G. If Π is a resolving partition of G and Π' is refinement of Π, then Π' is also a resolving partition of G.*

Proof. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ and $\Pi' = \{S'_1, S'_2, \ldots, S'_\ell\}$, where $k \le \ell$, such that each S'_i $(1 \le i \le \ell)$ is a subset of S_j for some j with $1 \le j \le k$. Let u and v be two distinct vertices of G. We show that $c_{\Pi'}(u) \ne c_{\Pi'}(v)$. Since Π is a resolving partition of G, it follows that $c_\Pi(u) \ne c_\Pi(v)$. Thus $d(u, S_j) \ne d(v, S_j)$ for some j with $1 \le j \le k$, say $d(u, S_1) \ne d(v, S_1)$. If S_1 is an element of Π', then $d(u, S_1) \ne d(v, S_1)$ and so $c_{\Pi'}(u) \ne c_{\Pi'}(v)$. Thus we may assume that $S_1 = S'_{i_1} \cup S'_{i_2} \cup \ldots \cup S'_{i_h}$, where $1 \le i_1 < i_2 < \ldots < i_h \le \ell$ and $h \ge 2$. Observe that at least one of u and v does not belong to S_1, for otherwise, $d(u, S_1) = 0 = d(v, S_1)$. We consider two cases.

Case 1. *Exactly one of u and v is in S_1, say $u \in S_1$ and $v \notin S_1$.* Thus $u \in S'_{i_p}$ for some p with $1 \le p \le h$ and so $d(u, S'_{i_p}) = 0$. Since $v \notin S_1$, it follows that $v \notin S'_{i_p}$ and so $d(v, S'_{i_p}) \ne 0$. Hence $c_{\Pi'}(u) \ne c_{\Pi'}(v)$.

Case 2. $u, v \notin S_1$. Let $x, y \in S_1$ such that $d(u, S_1) = d(u, x)$ and $d(v, S_1) = d(v, y)$, say $d(u, x) < d(v, y)$. If $x, y \in S'_{i_p}$ for some p with $1 \le p \le h$, then $d(u, S'_{i_p}) = d(u, x) < d(v, y) = d(v, S'_{i_p})$, implying that $c_{\Pi'}(u) \ne c_{\Pi'}(v)$. If $x \in S'_{i_p}$ and $y \in S'_{i_q}$, where $1 \le p \ne q \le h$, then $d(u, S'_{i_p}) = d(u, x) < d(v, y) \le d(v, S'_{i_p})$, again, implying that $c_{\Pi'}(u) \ne c_{\Pi'}(v)$.

Therefore, Π' is a resolving partition of G. ∎

Theorem 2.3 *For every nontrivial connected graph G,*

$$\mathrm{pd}(G) \le a_r(G) \le a(G)\,\mathrm{pd}(G).$$

In particular, if G is a tree, then $\mathrm{pd}(G) = a_r(G)$.

Proof. The lower bound follows from Observation 1.1. To verify the upper bound, let G be a nontrivial connected graph with $\mathrm{pd}(G) = k$ and $a(G) = a$. Furthermore, let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving partition of $V(G)$. If Π is acyclic, then Π is a resolving acyclic partition of $V(G)$ and so $a_r(G) \le |\Pi| = k = \mathrm{pd}(G) \le a(G)\,\mathrm{pd}(G)$ since $a(G) \ge 1$. Thus we may assume that Π is not acyclic. Let $a_i = a(\langle S_i \rangle)$ for $1 \le i \le k$. So $1 \le a_i \le a$ by Lemma 2.1. If S_i, where $1 \le i \le k$, is not acyclic, then $a_i \ge 2$ and S_i can be partitioned into a_i nonempty subsets, each of which is acyclic. Define a partition Π' of $V(G)$ from Π by (1) partitioning each nonacyclic element S of Π into $a(\langle S \rangle)$ acyclic subsets of S and (2) keeping each acyclic element of Π the same. So Π' is an acyclic partition of $V(G)$ with at most $\sum_{i=1}^{k} a_i \le ak$ elements. Moreover, Π' is a refinement of Π. By Lemma 2.2, Π' is also a resolving partition of G. Therefore, $a_r(G) \le |\Pi'| \le ak = a(G)\,\mathrm{pd}(G)$. In particular, if G is a tree, then $a(G) = 1$ and so $a_r(G) = \mathrm{pd}(G)$. ∎

Theorem 2.4 *For every nontrivial connected graph G,*

$$\frac{\chi_r(G)}{2} \le a_r(G) \le \chi_r(G).$$

Proof. The upper bound follows from Observation 1.1. To verify the lower bound, let G be a nontrivial connected graph with $a_r(G) = k$ and let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving acyclic partition of $V(G)$. If Π is independent, then $\chi_r(G) \le |\Pi| = k = a_r(G)$. It then follows by Observation 1.1 that $a_r(G) = \chi_r(G) \ge \chi_r(G)/2$. Thus we may assume that Π is not independent. If an element S_i of Π, where $1 \le i \le k$, is not independent, then $\langle S_i \rangle$ is acyclic and so $\chi(\langle S_i \rangle) = 2$. Hence S_i can be partitioned into two nonempty independent sets, namely, the two color classes of any proper minimum coloring of $\langle S_i \rangle$. Define a partition Π' of $V(G)$ from Π by (1) partitioning each nonindependent element of Π into two independent subsets and (2) keeping each independent element of Π the same. Thus Π' is an independent partition of $V(G)$ with at most $2k$ elements. Furthermore, Π' is a refinement of Π. By Lemma 2.2, Π' is also a resolving partition of G. Therefore, $\chi_r(G) \le |\Pi'| \le 2k = 2a_r(G)$ and so $a_r(G) \ge \chi_r(G)/2$. ∎

A related parameter was studied in [7, 8, 9]. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices in a connected graph G, and let $v \in V(G)$. The k-vector $c_W(v)$ of v with respect to W is defined by

$$c_W(v) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)).$$

The set W is called *a resolving set* if the k-vectors $c_W(v)$, $v \in V(G)$, are distinct. The minimum cardinality of a resolving set is the *dimension* $\dim(G)$ of G. Next we present bounds for $a_r(G)$ of a connected graph G in terms of $a(G)$ and $\dim(G)$.

Proposition 2.5 *For every connected graph G,*

$$a(G) \leq a_r(G) \leq a(G) + \dim(G).$$

Proof. We have seen that $a_r(G)$ is bounded below by $a(G)$ for every connected graph G. Thus it remains to verify the upper bound. Let $\dim(G) = k$, let $W = \{w_1, w_2, \ldots, w_k\}$ be a resolving set of G, and let $H = G - W$. Suppose that $\{V_1, V_2, \ldots, V_{a(H)}\}$ is an acyclic partition of $V(H)$. Then the partition $\Pi = \{S_1, S_2, \ldots, S_k, V_1, V_2, \ldots, V_{a(H)}\}$, where $S_i = \{w_i\}$, $1 \leq i \leq k$, is acyclic. Since the k-vectors $c_W(v)$, $v \in V(G)$, are distinct, the codes $c_\Pi(v)$, $v \in V(G)$, are distinct as well. It then follows by Lemma 2.1 that $a_r(G) \leq |\Pi| = a(H) + \dim(G) \leq a(G) + \dim(G)$. ■

We have seen that $2 \leq a_r(G) \leq n$ for every nontrivial connected graph G of order n. We now determine all connected graphs of order $n \geq 2$ with resolving acyclic number 2 or n. The following observation (see [5]) is useful.

Observation 2.6 *Let Π be a resolving partition in a connected graph G. If u and v are distinct vertices of G such that $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u,v\}$, then u and v belong to distinct elements of Π.*

Theorem 2.7 *Let $n \geq 2$ and let G be a connected graph of order n. Then*

(*a*) $a_r(G) = 2$ *if and only if* $G = P_n$.

(*b*) $a_r(G) = n$ *if and only if* $G = K_n$.

Proof. We first verify (a). Let $P_n : v_1, v_2, \ldots, v_n$ be a path of order n. Since the partition $\Pi = \{S_1, S_2\}$, where $S_1 = \{v_1\}$ and $S_2 = \{v_2, v_3, \ldots, v_n\}$, is a resolving acyclic partition of $V(P_n)$, it follows that $a_r(P_n) = 2$. For the converse, if G is a connected graph of order n with $a_r(G) = 2$. By Observation 1.1, it follows that $\mathrm{pd}(G) = 2$. However, it was shown in [4] that P_n is the only nontrivial connected graph of order n with partition dimension 2. Therefore, $G = P_n$.

Next, we verify (b). By Observation 2.6, $a_r(K_n) = n$. For the converse, let G be a connected graph of order n with $a_r(G) = n$ and let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Assume, to the contrary, that $G \neq K_n$. Since G is connected, $n \geq 3$. Moreover, we may assume that $d(v_1, v_3) = 1$ and $d(v_2, v_3) = 2$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-1}\}$ be the partition of $V(G)$, where $S_1 = \{v_1, v_2\}$ and $S_i = \{v_{i+1}\}$ for $2 \leq i \leq n-1$. Then Π is acyclic. Since the second coordinate of $c_\Pi(v_1)$ is 1 and the second coordinate of $c_\Pi(v_2)$ is 2, it follows that $c_\Pi(v_1) \neq c_\Pi(v_2)$. Thus Π is a resolving acyclic partition. Therefore, $a_r(G) \leq |\Pi| = n - 1$, producing a contradiction. ■

By Theorem 2.7, if G is a connected graph of order $n \geq 3$ that is neither P_n nor K_n, then

$$3 \leq a_r(G) \leq n - 1. \tag{1}$$

However, the bounds in (1) can be improved, as we show next. The *diameter* of a connected graph G is the largest distance between two vertices in G. Since the complete graph K_n is the only connected graph of order n with diameter 1 and $a_r(K_n) = n$, we will only consider connected graphs of order $n \geq 3$ with diameter $d \geq 2$.

Theorem 2.8 *If G is a connected graph of order $n \geq 3$ and diameter $d \geq 2$, then*

$$\log_{d+1} n \leq \text{pd}(G) \leq a_r(G) \leq n - d + 1.$$

Proof. First, we show that $a_r(G) \leq n - d + 1$. Let u and v be vertices of G for which $d(u,v) = d$, and let $P_{d+1} : u = v_1, v_2, \ldots, v_{d+1} = v$ be a $u - v$ path of length d in G. Let $\Pi = \{S_1, S_2, \ldots, S_{n-d+1}\}$ be the partition of $V(G)$, where $S_1 = \{v_1, v_2, \ldots, v_d\}$, $S_2 = \{v_{d+1}\}$, and each set S_i $(3 \leq i \leq n - d + 1)$ contains exactly one vertex from $V(G) - V(P_{d+1})$. Then Π is acyclic. Since, for j with $1 \leq j \leq d$, the second coordinate of $c_\Pi(v_j)$ is $d - j + 1$, the codes $c_\Pi(v_j)$ $(1 \leq j \leq d)$ are distinct. Thus Π is a resolving acyclic partition of G and so $a_r(G) \leq |\Pi| = n - d + 1$.

Next we show that $\text{pd}(G) \geq \log_{d+1} n$. Let $\text{pd}(G) = k$ and let Π be a resolving k-partition of $V(G)$. Since each coordinate of the code of a vertex in G with respect to Π is a nonnegative integer not exceeding d and all codes are distinct, it follows that $(d+1)^k \geq n$. Hence $\log_{d+1} n \leq k = \text{pd}(G)$. The result then follows by Observation 1.1. ∎

The *girth* of a graph is the length of its shortest cycle. Next, we provide bounds for the resolving acyclic number of a connected graph in terms of its order and girth.

Theorem 2.9 *If G is a connected graph of order $n \geq 3$ and girth $\ell \geq 3$, then*

$$3 \leq a_r(G) \leq n - \ell + 3.$$

In particular, if G is a cycle of order $n \geq 3$, then $a_r(G) = 3$.

Proof. Since $\ell \geq 3$, it follows that G is not a path and so $a_r(G) \geq 3$ by Theorem 2.7. To verify the upper bound, let $C_\ell : v_1, v_2, \ldots, v_\ell, v_1$ be a cycle of length ℓ in G, let $d = \lfloor \ell/2 \rfloor$, and let $\Pi = \{S_1, S_2, \ldots, S_{n-\ell+3}\}$ be the partition of $V(G)$, where $S_1 = \{v_1\}$, $S_2 = \{v_2, v_3, \ldots, v_d\}$, $S_3 = \{v_{d+1}, v_{d+2}, \ldots, v_\ell\}$, and each of S_i $(4 \leq i \leq n - \ell + 3)$ contains exactly one vertex in $V(G) - V(C_\ell)$. Since C_ℓ is a cycle of smallest length in G, it follows that $\langle S_2 \rangle$ and $\langle S_3 \rangle$ are acyclic, implying that Π is acyclic. Furthermore,

$c_\Pi(v_1) = (0, 1, 1, \ldots)$, $c_\Pi(v_i) = (i-1, 0, \min\{i, d-i+1\}, \ldots)$ for $2 \le i \le d$, and $c_\Pi(v_i) = (\ell - i + 1, \min\{i-d, \ell-i+2\}, 0, \ldots)$ for $d+1 \le i \le \ell$. Since the codes of vertices of G are distinct, Π is a resolving acyclic partition of $V(G)$. Thus $a_r(G) \le |\Pi| = n - \ell + 3$. Observe that if G is a cycle of order n, then $\ell = n$ and so $a_r(G) = 3$ by (1). ∎

Since the girth of K_n is 3 and the girth of C_n is n, by Theorems 2.7 and 2.9, $a_r(G) = n - \ell + 3$ for $G = K_n$ or $G = C_n$. In fact, K_n and C_n are the only connected graphs G of order $n \ge 3$ and girth $\ell \ge 3$ such that $a_r(G) = n - \ell + 3$, as we show next.

Theorem 2.10 *Let G be a connected graph of order $n \ge 3$ and girth $\ell \ge 3$. Then $a_r(G) = n - \ell + 3$ if and only if $G = K_n$ or $G = C_n$.*

Proof. We have seen that $a_r(G) = n - \ell + 3$ for $G = K_n$ or $G = C_n$. Thus it remains to verify the converse. Assume that G be a connected graph of order $n \ge 3$ with girth $\ell \ge 3$ such that $a_r(G) = n - \ell + 3$. If $\ell = 3$, then $a_r(G) = n$ and, by Theorem 2.7, $G = K_n$. Thus we may assume that $\ell \ge 4$. We show in this case $G = C_n$.

Assume, to the contrary, that $G \ne C_n$. Let $C_\ell : v_1, v_2, \ldots, v_\ell, v_1$ be a smallest cycle in G, where $\ell < n$. Since G is connected and $G \ne C_n$, there exists a vertex $v \in V(G) - V(C_\ell)$ such that v is adjacent to a vertex of C_ℓ, say $vv_1 \in E(G)$. We consider two cases.

Case 1. $\ell = 4$. Then G contains a subgraph obtained from the 4-cycle v_1, v_2, v_3, v_4, v_1 by adding an edge vv_1. Since $\ell = 4$, it follows that $vv_2, vv_4 \notin E(G)$; while the edge vv_3 may or may not be present. Let $\Pi = \{S_1, S_2, \ldots, S_{n-\ell+2}\}$ be a partition of $V(G)$, where $S_1 = \{v, v_1\}$, $S_2 = \{v_2, v_3\}$, $S_3 = \{v_4\}$, and each of S_i $(4 \le i \le n - \ell + 2)$ contains exactly one vertex from $V(G) - (V(C_\ell) \cup \{v\})$. Then Π is acyclic. Since $d(v, S_3) = 2$, $d(v_1, S_3) = 1$, $d(v_2, S_3) = 2$, and $d(v_3, S_3) = 1$, it follows that $c_\Pi(v) \ne c_\Pi(v_1)$ and $c_\Pi(v_2) \ne c_\Pi(v_3)$. Thus Π is an acyclic resolving partition of G and so $a_r(G) \le |\Pi| = n - \ell + 2$, which is a contradiction. Therefore, $G = C_4$.

Case 2. $\ell \ge 5$. Since C_ℓ is a smallest cycle in G, it follows that v is adjacent exactly one vertex of C_ℓ. Let $d = \lfloor \ell/2 \rfloor$ and let $\Pi = \{S_1, S_2, \ldots, S_{n-\ell+2}\}$ be a partition of $V(G)$, where $S_1 = \{v, v_1\}$, $S_2 = \{v_2, v_3, \ldots, v_d\}$, $S_3 = \{v_{d+1}, v_{d+2}, \ldots, v_\ell\}$, and each of S_i $(4 \le i \le n-\ell+2)$ contains exactly one vertex from $V(G) - (V(C_\ell) \cup \{v\})$. Since C_ℓ is a smallest cycle in G, it follows that $\langle S_2 \rangle$ and $\langle S_3 \rangle$ are acyclic and so Π is an acyclic partition of $V(G)$. Since $c_\Pi(v) = (0, 2, 2, \ldots)$, $c_\Pi(v_i) = (i-1, 0, \min\{i, d-i+1\}, \ldots)$ for $2 \le i \le d$, and $c_\Pi(v_i) = (\ell - i + 1, \min\{i-d, \ell-i+2\}, 0, \ldots)$ for $d + 1 \le i \le \ell$, it follows that Π is a resolving partition of G. Thus, $a_r(G) \le |\Pi| = n - \ell + 2$, which is a contradiction. Therefore, $G = C_n$. ∎

We have seen that if G is a connected graph of order $n \geq 2$, then $2 \leq \text{pd}(G) \leq a_r(G) \leq \chi_r(G) \leq n$. It was shown in [4, 5] that $\text{pd}(K_n) = \chi_r(K_n) = n$ and $\text{pd}(P_n) = \chi_r(P_n) = 2$. Moreover, for $n \geq 3$, $\text{pd}(C_n) = 3$; while $\chi_r(C_n) = 3$ if n is odd and $\chi_r(C_n) = 4$ if n is even. It then follows by Theorems 2.7 and 2.9 that $\text{pd}(G) = a_r(G) = \chi_r(G)$ if $G = K_n, P_n$ for $n \geq 2$ or $G = C_n$ for each odd integer $n \geq 3$.

3 A Characterization of Connected Graphs of Order n with Resolving Acyclic Number n − 1

Connected graphs of order $n \geq 3$ with partition dimension $n-1$ are characterized in [4] and connected graphs of order $n \geq 4$ with resolving-chromatic number $n-1$ are characterized in [6]. In this section, we determine all nontrivial connected graphs of order n with resolving acyclic number $n-1$. In order to do this, we first study the resolving acyclic numbers of connected bipartite graphs.

Theorem 3.1 *Let r, s be positive integers. If G is a connected bipartite graph with partite sets of cardinalities r and s, then*

$$a_r(G) \leq \begin{cases} r+1 & \text{if } r = s \\ \max\{r, s\} & \text{if } r \neq s. \end{cases} \tag{2}$$

Moreover, the equality in (2) holds if $G = K_{r,s}$.

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_r\}$ and $V_2 = \{v_1, v_2, \ldots, \ldots, v_s\}$ be the partite sets of G, where $1 \leq r \leq s$, say. First, we assume that $G \neq K_{r,s}$. There are two cases.

Case 1. $r = s$. Since $G \neq K_{r,r}$, we may assume that $u_{r-1}v_r \in E(G)$ and $u_r v_r \notin E(G)$. If $r = s = 2$, then $G = P_4$ and, by Theorem 2.7, $a_r(G) = 2 < 3 = r + 1$. If $r = s \geq 3$, let $\Pi = \{S_1, S_2, \ldots, S_r\}$ be the partition of $V(G)$, where $S_i = \{u_i, v_i\}$ for $1 \leq i \leq r-2$, $S_{r-1} = \{u_{r-1}, v_{r-1}, u_r\}$, and $S_r = \{v_r\}$. Observe that $d(u_i, S_r)$ is odd and $d(v_i, S_r)$ is even for $1 \leq i \leq r-2$. Furthermore, $d(u_{r-1}, S_r) = 1$, $d(v_{r-1}, S_r)$ is even, and $d(u_r, S_r)$ is odd but different from 1. Thus the codes of vertices of G with respect to Π are distinct and so Π is a resolving acyclic partition of G. Hence $a_r(G) \leq |\Pi| = r$.

Case 2. $r < s$. Since $G \neq K_{r,s}$, we may assume that $s \geq 3$ and $u_r v_s \in E(G)$ and $u_r v_{s-1} \notin E(G)$. Let $\Pi = \{S_1, S_2, \ldots, S_s\}$ be the partition of $V(G)$, where $S_i = \{u_i, v_i\}$ if $1 \leq i \leq r-1$, $S_i = \{v_i\}$ if $r \leq i \leq s-2$,

$S_{s-1} = \{v_{s-1}, v_s\}$, and $S_s = \{u_r\}$. Observe that $d(u_i, S_{s-1})$ is odd and $d(v_i, S_{s-1})$ is even for $1 \le i \le r$. Also, $d(v_s, S_r) = 1$ and $d(v_{s-1}, S_r) \ne 1$. Thus, the acyclic codes of vertices of G with respect to Π are distinct. Therefore, Π is a resolving acyclic partition and $a_r(G) \le |\Pi| = s$.

Finally, we show that the equality in (2) holds for $K_{r,s}$. It was shown in [4] that $\mathrm{pd}(K_{r,r}) = r+1$ and $\mathrm{pd}(K_{r,s}) = s$ if $r < s$. It then follows by Observation 1.1 that $a_r(K_{r,r}) \ge r+1$ and $a_r(K_{r,s}) \ge s$ if $r < s$. On the other hand, if $r = s$, then $\Pi = \{S_1, S_2, \ldots, S_{r+1}\}$, where $S_i = \{u_i, v_i\}$ for $1 \le i \le r-1$, $S_r = \{u_r\}$, $S_{r+1} = \{v_r\}$ is a resolving acyclic partition $V(K_{r,r})$; while if $r < s$, say, then $\Pi' = \{S'_1, S'_2, \ldots, S'_r\}$, where $S'_i = \{u_i, v_i\}$ for $1 \le i \le r$ and $S'_i = \{v_i\}$ for $r+1 \le i \le s$, is a resolving acyclic partition $V(K_{r,s})$. Thus $a_r(K_{r,r}) = r+1$ and $a_r(K_{r,s}) = s$ if $r < s$. ∎

The following corollary is a consequence of Theorems 2.8 and 2.9.

Corollary 3.2 *If G is a connected graph of order $n \ge 3$ with $a_r(G) = n-1$, then the diameter of G is 2 and the girth of G is at most 4.*

We are now prepared to determine all connected graphs of order $n \ge 3$ with resolving acyclic number $n-1$. If $n = 3$, then $G = P_3$ or $G = K_3$. Since $a_r(P_3) = 2$ and $a_r(K_3) = 3$, it follows that P_3 is the only connected graph of order 3 with resolving acyclic number 2. If $n = 4$, then, by Theorem 2.7 and (1), any connected graph G of order 4 such that $G \ne P_4, K_4$ has $a_r(G) = 3$. For $n \ge 5$, by the proof technique used in [4], we have the following characterization. For a vertex v in a graph G, let $N(v)$ denote the set of all vertices of G that are adjacent to v.

Theorem 3.3 *Let G be a connected graph of order $n \ge 5$. Then $a_r(G) = n-1$ if and only if $G \in \{C_4 + K_1, K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-2})\}$.*

Proof. It is straightforward to verify that the graphs mentioned in the theorem have resolving acyclic number $n-1$. For the converse, assume that G is a connected graph of order $n \ge 5$ with resolving acyclic number $n-1$. If G is bipartite, then the diameter of G is 2 by Corollary 3.2 and so $G = K_{r,s}$ for some integers r and s with $n = r+s \ge 5$. It then follows by Theorem 3.1 that $G = K_{1,n-1}$. If G is not bipartite, let Y be the vertex set of a maximum clique of G. Since G is not bipartite, G contains an odd cycle C. By Corollary 3.2, the girth of G is at most 4 and so $C = C_3$. Therefore, $|Y| \ge 3$. Let $U = V(G) - Y$ and then $|U| \ge 1$ since G is not complete.

Assume first that $|U| = 1$. Then $G = K_s + (K_1 \cup K_t)$ for some integers s and t. Since G is connected and G is not complete, $s \ge 1$ and $t \ge 1$. Let $V(K_s) = \{u_1, u_2, \ldots, u_s\}$, $V(K_t) = \{v_1, v_2, \ldots, v_t\}$, and $V(K_1) = \{w\}$. If $s \ge t$, let $\Pi = \{S_1, S_2, \ldots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \le i \le t)$, $S_i = \{u_i\}$ $(t+1 \le i \le s)$, and $S_{s+1} = \{w\}$. Then Π is acyclic. Since

$d(u,w) = 1$ for $u \in V(K_s)$ and $d(v,w) = 2$ for $v \in V(K_t)$, it follows that Π is a resolving acyclic $(s+1)$-partition of $V(G)$. Hence $a_r(G) \leq s+1$. By Observation 2.6, $a_r(G) \geq s$. However, $a_r(G) \neq s$, for otherwise $s = n-1$ and $G = K_n$. Therefore, $a_r(G) = s+1$. Since $a_r(G) = n-1$, it follows that $s = n-2$ and $t = 1$. Therefore, $G = K_{n-2} + (K_1 \cup K_1) = K_n - e$. If $s < t$, let $\Pi = \{S_1, S_2, \ldots, S_{t+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \leq i \leq s)$, $S_i = \{v_i\}$ $(s+1 \leq i \leq t)$, and $S_{t+1} = \{w\}$, is a resolving acyclic partition of $V(G)$. Thus $a_r(G) \leq t+1$. By Observation 2.6, $a_r(G) \geq t$. However, $a_r(G) \neq t$, for otherwise $t = n-1$ and $s = 0$, implying that G is disconnected. Therefore, $a_r(G) = t+1$. Since $a_r(G) = n-1$, we have $t = n-2$ and $s = 1$. Therefore, $G = K_1 + (K_1 \cup K_{n-2})$

Next we assume that $|U| \geq 2$. If $n = 5$, then $|Y| = 3$ and $|U| = 2$. It is routine to verify that $C_4 + K_1$ is the only graph with the desired properties. For $n \geq 6$, we claim that U is an independent set of vertices. Assume, to the contrary, that U contains two adjacent vertices u and w. Since Y is the vertex set of a maximum clique of G, there exist $v \in Y$ such that $uv \notin E(G)$ and $v' \in Y$ such that $wv' \notin E(G)$, where v and v' are not necessarily distinct. We also consider these two cases.

Case 1. There exists a vertex $v \in Y$ such that $uv, wv \notin E(G)$. We now consider two subcases.

Subcase 1.1. There exists a vertex $x \in Y$ that is adjacent to exactly one of u and w, say u. Since $|Y| \geq 3$, there exist a vertex $y \in Y$ that is distinct from v and x. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, there $S_1 = \{u, w\}$, $S_2 = \{v, x\}$, $S_3 = \{y\}$, and each of remaining sets S_i $(4 \leq i \leq n-2)$ contains exactly one vertex from $V(G) - \{u, v, w, x, y\}$. Then $\langle S_i \rangle$ is acyclic for all $1 \leq i \leq n-2$. Since $c_\Pi(u)=(0,1,\ldots)$, $c_\Pi(v)=(2,0,\ldots)$, $c_\Pi(w)=(0,2,\ldots)$, and $c_\Pi(x)=(1,0,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Subcase 1.2. Every vertex of Y is adjacent to either both u and w or to neither u nor w. If u and w are adjacent to every vertex in $Y - \{v\}$, then the induced subgraph $\langle (Y - \{v\}) \cup \{u, w\} \rangle$ is complete in G, contradicting the defining property of Y. Thus, there exists a vertex $y \in Y$ such that y is distinct from v, and y is adjacent to neither u nor w. Since the diameter of G is 2, there is a vertex x of G that is adjacent to both u and v. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{x, y, w\}$, $S_2 = \{u\}$, $S_3 = \{v\}$, and each of the remaining sets S_i $(4 \leq i \leq n-2)$ contains only one vertex from $V(G) - \{u, v, w, x, y\}$. Since y is not adjacent to w, it follows that $\langle S_1 \rangle$ is acyclic and so Π is acyclic. Since $c_\Pi(x)=(0,1,1,\ldots)$, $c_\Pi(y)=(0,2,1,\ldots)$, and $c_\Pi(w)=(0,1,2,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Case 2. There exist distinct vertices v and v' in Y such that $uv, wv' \notin E(G)$. Necessarily, then $vw, v'u \in E(G)$. Since $|Y| \geq 3$, there exists a vertex y in Y distinct from v and v'. Also, at least one of the edges yu and yw must be present in G, say yu. If $yw \notin E(G)$, let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{u, w, y\}$, $S_2 = \{v\}$, $S_3 = \{v'\}$, and each of the remaining sets S_i ($4 \leq i \leq n-2$) contains only one vertex from $V(G) - \{u, v, v', w, y\}$. Since $yw \notin E(G)$, it follows that Π is acyclic. Because $c_\Pi(u)=(0,2,1,\ldots)$, $c_\Pi(w)=(0,1,2,\ldots)$, and $c_\Pi(y)=(0,1,1,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction. Thus, we assume that $yw \in E(G)$. Since $n \geq 6$, there exists $x \in V(G) - \{u, v, v', w, y\}$. We consider two subcases, according to whether $x \in Y$ or $x \in U$.

Subcase 2.1. $x \in Y$. Then $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{v\}$, $S_2 = \{v'\}$, $S_3 = \{x, w\}$, $S_4 = \{u, y\}$, and each of the remaining sets S_i ($5 \leq i \leq n-2$) contains only one vertex from $V(G) - \{u, v, v', w, x, y\}$. Thus Π is acyclic. Since $c_\Pi(x)=(1,1,0,\ldots)$, $c_\Pi(w)=(1,2,0,\ldots)$, $c_\Pi(u)=(2,1,1,0,\ldots)$, and $c_\Pi(y)=(1,1,1,0,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Subcase 2.2. $x \in U$. Then there exists $y' \in Y$ that is not adjacent to x, for otherwise, $x \in Y$. We consider four subcases, according to whether y' is one of v, v', y or not.

Subcase 2.2.1. $y' = v$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{v\}$, $S_2 = \{u, v'\}$, $S_3 = \{x, w\}$, $S_4 = \{y\}$, and each of remaining sets S_i ($5 \leq i \leq n-2$) contains exactly one vertex from $V(G) - \{u, v, v', w, x, y\}$. Since $c_\Pi(u)=(2,0,\ldots)$, $c_\Pi(v')=(1,0,\ldots)$, $c_\Pi(x)=(2,*,0,\ldots)$, where $*$ is either 1 or 2, and $c_\Pi(w)=(1,1,0,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Subcase 2.2.2. $y' = v'$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{y\}$, $S_2 = \{v'\}$, $S_3 = \{u, w\}$, $S_4 = \{x, v\}$, and each of remaining sets S_i ($5 \leq i \leq n-2$) contains exactly one vertex from $V(G) - \{u, v, v', w, x, y\}$. Since $c_\Pi(u) = (1,1,0,\ldots)$, $c_\Pi(w) = (1,2,0,\ldots)$, $c_\Pi(x) = (*,2,*,0,\ldots)$, where $*$ is either 1 or 2, and $c_\Pi(v)=(1,1,1,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Subcase 2.2.3. $y' = y$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{y\}$, $S_2 = \{v\}$, $S_3 = \{v', x\}$, $S_4 = \{u, w\}$, and each of remaining sets S_i ($5 \leq i \leq n-2$) contains exactly one vertex from $V(G) - \{u, v, v', w, x, y\}$. Since $c_\Pi(v')=(1,1,0,\ldots)$, $c_\Pi(x)=(2,*,0,\ldots)$, $c_\Pi(u) = (1, 2, 1, 0, \ldots)$, and $c_\Pi(w) = (1,1,*,\ldots)$, where $*$ is either 1 or 2, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Subcase 2.2.4. $y' \notin \{v, v', y\}$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{v\}$, $S_2 = \{v'\}$, $S_3 = \{y', w\}$, $S_4 = \{u, y\}$, and each of remaining sets S_i ($5 \leq$

$i \leq n-2$) contains exactly one vertex from $V(G) - \{u, v, v', w, x, y\}$. Since $c_\Pi(y')=(1,1,0,\ldots)$, $c_\Pi(w)=(1,2,0,\ldots)$, $c_\Pi(u)=(2,1,1,0,\ldots)$, and $c_\Pi(y) = (1,1,1,0,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Therefore, in any case, U is an independent set. Next we claim the $N(u) = N(w)$ for all $u, w \in U$. It suffices to show that if $uv \in E(G)$, then $vw \in E(G)$. Suppose that $uv \in E(G)$ for some vertex v of G. Necessarily $v \in Y$. Assume, to the contrary, that $vw \notin E(G)$. Since Y is the vertex set of a maximum clique, there exists $y \in Y$ such that $uy \notin E(G)$. Since G is connected and U is independent, w is adjacent to some vertex of Y. First assume that w is adjacent only to y. Since w and y are not adjacent to u, it follows that $d(w,u) = 3$, which contradicts the fact that the diameter of G is 2. Thus, there exists a vertex x in Y distinct from y such that $wx \in E(G)$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{w, x\}$, $S_2 = \{u, v\}$, $S_3 = \{y\}$, and each of the remaining sets S_i $(4 \leq i \leq n-2)$ contains only one vertex of $V(G) - \{u, w, x, v, y\}$. Then $\langle S_i \rangle$ is acyclic for all $1 \leq i \leq n-2$. Since $c_\Pi(u)=(*,0,2,\ldots)$, where $*$ is either 1 or 2, $c_\Pi(v)=(1,0,1,\ldots)$, $c_\Pi(w)=(0,2,1,\ldots)$, and $c_\Pi(x)=(0,1,1,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, contradicting the fact that $a_r(G) = n-1$.

So far, we have, for $n \geq 6$, $V(G) = Y \cup U$, where $\langle Y \rangle$ is complete, U is independent, $|Y| \geq 3$, $|U| \geq 2$, and $N(u) = N(w)$ for all $u, w \in U$. Next we show that for each $u \in U$, there exist at most one vertex of Y not contained in $N(u)$. Assume, to the contrary, that there are two vertices $x, y \in Y$ not in $N(u)$. Let w be a vertex of U that is distinct from u. Thus $wx, wy \notin E(G)$. Since G is connected, there exists $z \in Y$ such that $z \in N(u) = N(w)$. Let $\Pi = \{S_1, S_2, \ldots, S_{n-2}\}$, where $S_1 = \{y, z, w\}$, $S_2 = \{u\}$, $S_3 = \{x\}$, and each of the remaining sets S_i $(4 \leq i \leq n-2)$ contains only one vertex of $V(G) - \{y, z, w, u, x\}$. Since $wy \notin E(G)$, it follows that $\langle S_1 \rangle$ is acyclic and so Π is acyclic. Since $c_\Pi(y)=(0,2,1,\ldots)$, $c_\Pi(z)=(0,1,1,\ldots)$, and $c_\Pi(w)=(0,2,2,\ldots)$, it follows that Π is a resolving acyclic $(n-2)$-partition of $V(G)$, a contradiction.

Now either $N(u) = Y$ or $N(u) = Y - \{v\}$ for some $v \in Y$. If $N(u) = Y$, then $G = K_s + \overline{K}_t$ for $s = |Y| \geq 3$ and $t = |U| \geq 2$. If $N(u) = Y - \{v\}$, then $G = K_s + (K_1 \cup \overline{K}_t)$, where $V(K_1) = \{v\}$, $s = |Y| - 1 \geq 2$, and $t = |U| \geq 2$. However, $K_s + (K_1 \cup \overline{K}_t) = K_s + \overline{K}_{t+1}$. In either case, $G = K_s + \overline{K}_t$, where $t \geq 3$ and so $s \leq n-3$. Let $V(K_s) = \{u_1, u_2, \ldots, u_s\}$ and $V(\overline{K}_t) = \{v_1, v_2, \ldots, v_t\}$. If $s = t$, let $\Pi = \{S_1, S_2, \ldots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \leq i \leq s-1)$, $S_s = \{u_s\}$, and $S_{s+1} = \{v_s\}$. Since $d(u, v_s) = 1$ $(u \in V(K_s))$ and $d(v, v_s) = 2$ $(v \in V(K_t))$, it follows that Π is a resolving acyclic $(s+1)$-partition of $V(G)$. Hence $a_r(G) \leq s+1 \leq n-3+1 = n-2$, which is a contradiction. If $s > t$, let $\Pi = \{S_1, S_2, \ldots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \leq i \leq t-1)$, $S_i = \{u_i\}$ $(t+1 \leq i \leq s)$, and

$S_{s+1} = \{v_t\}$. Since $d(u, v_t) = 1$ $(u \in V(K_s))$ and $d(v, v_t) = 2$ $(v \in V(K_t))$, it follows that Π is a resolving acyclic $(s+1)$-partition of $V(G)$. Hence $a_r(G) \le s + 1 \le n - 3 + 1 = n - 2$, which is a contradiction. If $s < t$, let $\Pi = \{S_1, S_2, \ldots, S_t\}$, where $S_i = \{u_i, v_i\}$ $(1 \le i \le s)$ and $S_i = \{v_i\}$ $(s+1 \le i \le t)$. Since Π is a resolving acyclic t-partition of $V(G)$, it follows that $a_r(G) \le t \le n - 2$, which is a contradiction. ■

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