FINITE LINEAR SPACES WITH FOUR MORE LINES THAN POINTS

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ABSTRACT. In 1948 de Bruijn and Erdős proved that every finite linear space on v points and with b lines fulfils the inequality $b \geq v$, and the equality holds if the linear space is a (possibly degenerate) projective plane. This result led to the problem of classifying finite linear spaces on v points and with $b = v + s$ lines, $s \geq 1$. This paper contains the classification of finite linear spaces on v points and with $b = v + 4$ lines.

1. INTRODUCTION

A *finite linear space* [5] on v *points* and with b *lines* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set of v points and $\mathcal{L}$ is a family of b subsets (the lines) of $\mathcal{P}$ such that: *any two points are on a unique line, each line contains at least two points and there are at least two lines.*

The *degree* of a point p is the number $[p]$ of lines on p and the *length* of a line ℓ is its size $|\ell|$ [13].

Denote by k the maximal line length and by m the minimum point degree.

The *near-pencil* on v points is the linear space on v points with a line of length $v - 1$ [13].

A (h,k)*-cross*, $3 \leq h \leq k$, is the linear space on $h+k-1$ points, with a point of degree 2 on which there are two lines of length h and k respectively [13].

A linear space is *irreducible* if every line has length at least three.

A *projective* plane is an irreducible linear space such that any two lines meet in a point [13].

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An *affine plane* is a linear space such that for every point–line pair (p, ℓ), with $p \notin \ell$, the number of lines on p missing ℓ is 1 [13].

If $(\mathcal{P}, \mathcal{L})$ is a finite linear space and X is a subset of $\mathcal{P}$, such that $\mathcal{P} \setminus X$ contains at least three non–collinear points, the linear space $(\mathcal{P}', \mathcal{L}')$, where

$$\mathcal{P}' = \mathcal{P} \setminus X$$

$$\mathcal{L}' = \{\ell \setminus \{\ell \cap X\} \mid \ell \in \mathcal{L} \text{ and } |\ell \setminus \ell \cap X| \geq 2\},$$

is called the *complement of X in* $(\mathcal{P}, \mathcal{L})$ [1].

The complement of a line in a projective plane is an affine plane.

A *punctured* linear space $(\mathcal{P}, \mathcal{L})$ is the complement of a point in $(\mathcal{P}, \mathcal{L})$.

A linear space that can be obtained from a projective plane π as a complement is said to be *embeddable* in π [13].

So each affine plane is embeddable in a projective plane.

Two lines ℓ and ℓ' of a linear space are *parallel* if $\ell = \ell'$ or $\ell \cap \ell' = \emptyset$.

If $(\mathcal{P}, \mathcal{L})$ is an affine plane, there is an integer $n \geq 2$, called the *order* of $(\mathcal{P}, \mathcal{L})$, such that $[p] = n + 1$ for all $p \in \mathcal{P}$, $|\ell| = n$ for all $\ell \in \mathcal{L}$, and each line ℓ has exactly $n - 1$ parallel lines different from it. The set of lines parallel to a line ℓ partition $\mathcal{P}$ and is called a *parallel class* of $(\mathcal{P}, \mathcal{L})$.

The notion of parallelism is important to construct new linear spaces from old ones, in particular from affine planes.

Indeed let α be a finite affine plane and let p be a point of α. Let $\ell_1, \ell_2, \ldots, \ell_{n+1}$ the set of lines on p. Each line ℓ_i, $i = 1, 2, \ldots, n+1$ gives rise to a parallel class π_i, $i = 1, 2, \ldots, n+1$, and these $n+1$ parallel classes are all the parallel classes of α.

Let Π be a set of h parallel classes of α. Let π_i, $i = 1, 2, \ldots, h$, be an element of Π, adding a new point ∞_i to $\mathcal{P}$ and to the lines of π_i, for all $i = 1, 2, \ldots, h$, and by imposing, if $h \geq 2$, to the set $\{\infty_1, \infty_2, \ldots, \infty_h\}$ to be a set of collinear points $\mathcal{L}^*$ or to be a linear space $(\mathcal{P}^*, \mathcal{L}^*)$, one obtains the following new linear space:

$$\mathcal{P}' = \mathcal{P} \cup \{\infty_1, \infty_2, \ldots, \infty_h\},$$

$$\mathcal{L}' = \{\ell \in \mathcal{L} \mid \ell \in \pi_j \notin \Pi\} \cup \{\ell \cup \infty_i \mid \ell \in \mathcal{L} \text{ and } \ell \in \pi_i \in \Pi\} \cup \mathcal{L}^*,$$

called an *inflated affine plane*.

If Π consists of all the parallel classes of $(\mathcal{P}, \mathcal{L})$, then the linear space $(\mathcal{P}', \mathcal{L}')$ is called a *complete* inflated affine plane, and in this case if the points ∞_i are collinear $(\mathcal{P}', \mathcal{L}')$ is a projective plane.

If the linear space on $\{\infty_1, \infty_2, \ldots, \infty_h\}$ is a near–pencil or a projective plane, $(\mathcal{P}', \mathcal{L}')$ is called a *projectively inflated affine plane.*

A famous result of de Bruijn and Erdős [3] states that $b \geq v$, and that $b = v$ only if the linear space is a near–pencil or a projective plane.

Theorem (de Bruijn - Erdős, 1948 [3]) *Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space. Then $b \geq v$. Moreover, equality holds if and only if $(\mathcal{P}, \mathcal{L})$ is a projective plane or a near-pencil.*

This theorem led to the problem of classifying finite linear spaces with $b = v + s$ lines, $s \geq 1$. This question has been studied in some detail; see for instance [[6], Chapter 8 and Chapter 9] and the literature quoted there.

In this paper we study the case $s = 4$. Actually we complete the classification of finite linear spaces on v points and $b = v + 4$ lines, since if $b \geq 16$ one can obtain such classification by a theorem of Metsch [6] on weakly restricted linear spaces. As in [7] we will prove that $k \geq m - 1$, and studying the three different cases $k = m - 1, k = m$, and $k \geq m + 1$ we obtain the required classification.

1.1. **Finite linear spaces with $b - v \leq 3$.** In this section we recall the theorems for $1 \leq s \leq 3$.

The *order* of a finite projective plane π is the integer $n \geq 2$ such $[p] = |\ell| = n + 1$ for all $p \in \mathcal{P}$ and for all $\ell \in \mathcal{L}$ [13].

Theorem I (Bridges, 1972 [2]) *Every linear space with $b - v = 1$ is a punctured projective plane or the affine plane of order two with a point at infinity.*

Theorem II (de Witte, 1976 [14]) *Every finite linear space with two more lines than points is one of the following:*

(1) *A doubly punctured projective plane of order $n \geq 3$.*
(2) *The affine plane of order two.*
(3) *The Fano quasi-plane, which is the affine plane of order two with the near-pencil on 3 points at infinity.*
(4) *The affine plane of order three with one point at infinity.*

Theorem III (Totten, 1976 [10]) *Every linear space with three more lines than points is one of the following spaces:*

(1) *The complement of three points in a finite projective plane of order n, $n \geq 4$.*
(2) *The complement of three non-collinear points in the projective plane of order 3.*
(3) *The affine plane of order three.*
(4) *The affine plane of order 4 with a point at infinity.*
(5) *The punctured affine plane of order 3 with a point at infinity.*
(6) *The linear space on $v = 12$ points obtained from the affine plane of order 3 with a near-pencil on three points at infinity.*

(7) *The affine plane of order* 3 *with a near pencil on four points at infinity.*

(8) *The linear space on* $v = 5$ *points, with a line of length* 3 *and all the other of length* 2.

(9) *The linear space on* $v = 6$ *points, with three lines of length* 3 *and the remaining lines of length* 2.

(10) *The linear space on* $v = 7$ *points, with one line of length* 5, *another of length* 3 *and the remaining of length* 2.

(11) *The linear space on* $v = 7$ *points, with a line of length* 4, *three of length* 3 *and the the remaining of length* 2.

(12) *The linear space on* $v = 8$ *points, with a line of length* 4, *six of length* 3, *the remaining of length* 2, *and such that on each point there is a line of length* 3.

(13) *The linear space on* $v = 8$ *points, with a line of length* 4, *six of length* 3, *the remaining of length* 2, *and with a point not on any line of length* 3.

1.2. **Finite linear spaces with $b - v = 4$.** In this section π_n will denote a projective plane of order n, and α_n an affine plane of order n.

Now we give the list of finite linear spaces with $b - v = 4$.

E1. The complement of four points in π_n, $n \geq 5$.

E2. The complement of four non-collinear points in π_4.

E3. The complement of four points no three of which are collinear in π_3.

E4. α_5 with a point at infinity.

E5. α_4.

E6. The punctured affine plane α_4 with a point at infinity.

E7. α_4 with a near–pencil on $v = 3$ points at infinity.

E8. α_4 with a near–pencil on $v = 4$ points at infinity.

E9. α_4 with a near–pencil on $v = 5$ points at infinity.

E10. The punctured affine plane of order 3.

E11. The punctured affine plane of order 3 with a near–pencil on $v = 4$ points at infinity.

E12. The punctured affine plane of order 3 with a near–pencil on $v = 3$ points at infinity.

E13. The linear space on $v = 8$ points, with a single point of degree $m = 3$, on which there are two lines of length 4 and one line of length 2, and with three concurrent lines of length 3 in a point outside of the two lines of length 4.

E14. The linear space on $v = 7$ points, with five pairwise intersecting lines of length $k = 3$, and with $m = 3$.

E15. The linear space on $v = 7$ points, with five lines of length $k = 3$, with a single point of degree $m = 3$ and with two parallel lines of length $k = 3$.

E16. Let p be the point of degree 3 of $E15$. Each line of length 3 on p is parallel to two parallel lines of length 2, and so gives rise to a partition (*parallel class*) of the point set of $E15$.

The linear space obtained from $E15$ by adding the three infinity points of these parallel classes and imposing that these new points form a near–pencil has $v = 10$ points and $b = 14$ lines.

E17. The $(3, 6)$–cross.

E18. The $(4, 4)$–cross.

E19. The linear space on $v = 6$ points, with a line of length $k = 4$ parallel to a line of length 2.

1.3. **The result.** In this paper we prove the following result.

Theorem 1.1. *A finite linear space on v points and with $b = v + 4$ lines is one of the linear spaces described in $E1, \dots, E19$.*

2. Some preliminary results

In this section we recall some classical results on the characterization of finite linear spaces with a prescribed value for the difference $b - v$.

Theorem 2.1 (Totten, 1976 [9]). *Every linear space with $b - v \leq \sqrt{v}$* **(restricted linear space)** *is one of the following spaces:*

(1) *A near-pencil.*
(2) *A projective plane of order n with at most n points deleted but no more than $n - 1$ from the same line.*
(3) *An affine plane, or an affine plane with one point at infinity, or a punctured affine plane with a point at infinity.*
(4) *A complete projectively inflated affine plane.*

(5) *The* $(3,4)$*-cross.*

Theorem 2.2 (Metsch, 1991 [6], Thm. 8.6 pp. 79). *Every linear space with* $b-v \leq \sqrt{b}$ **(weakly restricted linear space)** *is one of the following spaces:*

(1) *A restricted linear space.*
(2) *An affine plane of order* n *with a punctured projective plane on* n *or* $n+1$ *points at infinity.*
(3) *A complete projectively inflated punctured plane.*
(4) *An inflated affine plane of order* 4 *or* 5 *whose space at infinity is the affine plane of order* 2 *with a point at infinity.*
(5) *The linear space obtained from the projective plane of order* 3 *by deleting two lines, their point of intersection, and two more points from each of these two lines.*
(6) *The* $(3,5)$*-cross.*
(7) *The linear space on* $v=7$ *points,* $b=10$ *lines, with a line of length* 4*, three of length* 3 *and the the remaining of length* 2*.*
(8) *The linear space on* $v=8$ *points,* $b=11$ *lines, with a line of length* 4*, six of length* 3*, the remaining of length* 2*, and with a point not on any line of length* 3*.*

Theorem 2.3 (Vanstone, [11]). *A finite linear space with maximum point degree* $n+1$ *and with* $v \geq n^2$ *points is embeddable in a finite projective plane of order* n*.*

Theorem 2.4 (Doyen, [4]). *The finite linear spaces with* $b-v=4$ *and* $v \leq 9$ *are those described in* $E3, E10, E13, E14, E15, E17, E18, E19$*.*

3. Proof of Theorem 1.1

Throughout this section $(\mathcal{P}, \mathcal{L})$ will denote a finite linear space with $b-v=4$, k will denote the maximal line length, m the minimum point degree, and if L is a line, δ_L will denote the number of lines parallel to L and different from L.

Proposition 3.1. *If there are two lines* ℓ *and* ℓ' *such that* $\mathcal{P} = \ell \cup \ell'$*, then* $(\mathcal{P}, \mathcal{L})$ *is one of the linear space described in* $E17, E18$ *and* $E19$*.*

PROOF. Put $|\ell| = k$ and $|\ell'| = h$, so $h \leq k$.

If $\ell \cap \ell' = \emptyset$, then $v = h+k$ and $b = h+k+4$. On the other hand, counting the lines meeting ℓ and ℓ' we have $b = hk+2$. So,

$$h+k+4 = hk+2,$$

that is

$$(h-1)k = h+2.$$

If $h \geq 3$, then $2h \leq 2k \leq h+2$ and so $h = 2$, that is impossible. So $h = 2$ and $k = 4$ and $(\mathcal{P}, \mathcal{L})$ is the linear space $E19$.

If $\ell \cap \ell' \neq \emptyset$, then $v = h + k - 1$, and so $b = h + k + 3$. But $b = (h-1)(k-1) + 2$, thus

$$(h-1)(k-1) = h + k + 1,$$

that is

$$(k-1)(h-2) = h + 2 - 2 + 2$$

from which it follows that

$$(k-2)(h-2) = 4$$

and so $h, k \geq 3$ and since $k \geq h$ we have $(h-2)^2 \leq 4$, hence

$$h = 3, \; k = 6$$

or

$$h = 4, \; k = 4$$

and so $(\mathcal{P}, \mathcal{L})$ is the $(3,6)$–cross or the $(4,4)$–cross, that is one of the linear spaces $E17$ and $E18$. □

From now on we may suppose that given two lines ℓ and ℓ' there is a point outside of them, and in particular that $m \geq 3$.

Proposition 3.2. $k \geq m - 1$.

PROOF. Counting in double way the point–line pairs (p, ℓ), with $p \in \ell$, we have:

$$(*) \qquad vm \leq \sum_{p \in \mathcal{P}} [p] = \sum_{\ell \in \mathcal{L}} |\ell| \leq bk = (v+4)k,$$

and so

$$v(m-k) \leq 4k.$$

Assume by way of contradiction that $k \leq m - 2$, then $m \geq 4$ and $2v \leq 4k$. If $m \geq 5$, from $b \geq k(m-1) + 1 \geq 4k + 1$ it follows that $b \geq 2v + 1$, and so $4 = b - v \geq v + 1 \geq m + 1 \geq 6$, that is impossible! Hence $m = 4$ and so $k = 2$. Therefore $[x] = 4$ for every $x \in \mathcal{P}$ and $v = 5$, and so from equation $(*)$ it follows that $20 \leq 18$, a contradiction!

Hence $k \geq m - 1$ and the assertion is proved. □

3.1. $k = m - 1$. In this section we are going to prove the following result.

Proposition 3.3. *If $k = m - 1$, then $(\mathcal{P}, \mathcal{L})$ is either the punctured affine plane of order 3, or the affine plane of order 4, (that is $E10$ or $E5$).*

First we observe that $m \geq 4$. Indeed if $m = 3$, then $k = 2$ and so $v = m + 1 = 4$. Since on 4 points there are only two linear spaces, the near-pencil and the affine plane of order 2 and they fulfill the inequality $b - v \leq 2$, we have a contradiction.

If $m \geq 5$, then $b \geq (m-1)^2 + 1 \geq 17 > 4^2$, and so by Theorem 2. 2, we have that $m = 5$ and $(\mathcal{P}, \mathcal{L})$ is the affine plane of order $m - 1 = 4$.

If $m = 4$, then $v \leq (m-1)^2 \leq 9$, and so by Theorem 2.4 we have that $(\mathcal{P}, \mathcal{L})$ is the punctured affine plane of order 3.

3.2. $k = m$. In this section we will prove the following result.

Proposition 3.4. *Let $(\mathcal{P}, \mathcal{L})$ be a finite linear space with $b - v = 4$, then $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces described in $E1, E2, E3, E7, E8, E9, E11, E12, E14$ and $E15$.*

The following lemmas are the proof of Proposition 3.1.

Before to start with the proof of the lemmas, we recall that the projective plane of order 2 is also called the *Fano* plane.

Lemma 3.5. *If all points have constant point degree m, then $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces $E1, E2$ and $E3$.*

PROOF. Since each point has degree m, then each line of length m has no parallel line, and so $b = m(m-1) + 1 = m^2 - m + 1$. Put $n = m - 1$, then $v = n^2 + n - 3$, $b = n^2 + n + 1$, and all points have degree $n + 1$. Since $(\mathcal{P}, \mathcal{L})$ is not a near-pencil, it follows that $n \geq 3$. From Theorem 2.3 it follows that $(\mathcal{P}, \mathcal{L})$ is embeddable in a projective plane of order $n \geq 3$, and so it is the complement of four points in a projective plane of order $n \geq 3$. It follows that $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces described in $E1, E2$ and $E3$. □

Hence we have to consider the non-constant point degree case.

So from now on, there is at least a point x of degree at least $m + 1$. Hence $b \geq m(m-1) + 1 + 1$. Throughout this section $n = m - 1$, and so $b \geq n^2 + n + 2$ and $v \leq n^2 + n + 1$.

Lemma 3.6. *If $n \geq 4$, then $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces described in $E7, E8$ and $E9$.*

PROOF. Since $n \geq 4 = b - v$, by Theorem 2.2 it follows that $(\mathcal{P}, \mathcal{L})$ is an inflated affine plane of order 4, that is one of the linear spaces described in $E7, E8$ and $E9$. □

Lemma 3.7. *If* $n = 3$ *then* $(\mathcal{P}, \mathcal{L})$ *is one of the linear spaces* $E11$ *and* $E12$.

PROOF. From $n = 3$ it follows that $b \geq 9+3+2 = 14$ and $v \leq 13$, and so

$$10 \leq v \leq 13 \text{ and } 14 \leq b \leq 17.$$

From Theorem 2.2 we have that if $b \geq 16$ then $v = 12$ and $(\mathcal{P}, \mathcal{L})$ is the linear space E11.

Hence we have to study the cases $v \in \{10, 11\}$.

Consider the case $v = 10$.
If there is a line L of length k with all points of degree m, then from $b = 14$ it follows that L has exactly one parallel line. Let H be this parallel line to L. By Proposition 3.1 there is a point x outside $L \cup H$, and $[x] = m$. Since $v = 10$, on x there is a line T of length k meeting L. Let y a point of $H \setminus \{H \cap T\}$, the parallel line on y to T meets L in a point of degree $m+1$, a contradiction! So each line of length k has exactly one point of degree $m+1$, and so each line of length k has no parallel line. It follows that either there is a single point of degree $m+1$ on which there are all lines of length k, or there is a single line of length $k = m$. In both cases there is a point of degree m on which there is no line of length k, and so counting v on a point of degree m we have: $v \leq 4 \cdot 2 + 1 = 9$, a contradiction!

Now consider the case $v = 11$.
From $b = 15$ it follows that a line of length k has either two points of degree $m+1$, or a point of degree $m+2$, or a point of degree $m+1$ and a parallel line, or all points of degree m and two parallel lines.

On each point of degree m there are at least two lines of length $k = m$. Each line of length $k = m$ has at least two points of degree m, and so any two lines of length m meet in a point.

Claim: *each line of length k has at least a point of degree at least* $m+1$.

Let L be a line of length k with all points of degree m. Let H be a line parallel to L, and T be a line of length m meeting L. If H meets T, then the parallel line to T on a point of H different from $T \cap H$ meets L in a point of degree $m+1$, a contradiction! So H is parallel to each line of length m, it follows that $|H| = 2$ and the points of H have degree $m+2 = 6$. Each line of length k has all points of degree m, on a point of degree m there are two lines of length k and two lines of length 3, otherwise H would have length 1. Thus there are 5 lines of length k, either 8 of length 3 and three of length 2, or a single point of degree $m+2$, and so on a point of degree $m+1$ there are a line of length 2 and four of length 3. In the former case $b=16$, in the second $v=10$, and so in both cases a contradiction!

If a line L of length k has a point of degree $m+2$ then each line of length k meeting L in a point of degree m has two parallel lines and so L has a parallel, that is impossible!

So each line of length k has one point of degree $m+1$ and one parallel line. Thus there is no point of degree $m+2$. Let L be a line of length k, x a point of degree m of L and $H(\neq L)$ be a line of length k on x. Let y be the point of degree $m+1$ of L, and z be the point of degree $m+1$ of H. Let ℓ the line on y parallel to H, then $|\ell|=2$, otherwise L has two parallel lines, let p be the second point of ℓ, the parallel line T on p to L meets H in z, and $|T|=2$. The line px has length at least 3, otherwise counting v via the lines on p one gets: $v \leq 2+1+1+3+3=10$, that is impossible! Let u and w be the other two points of L, then pu has length 3 and pw has length 4 (or vice versa), so on y there is a parallel to pw and this parallel meets h in z, and so it is the line yz, and so three points p, y and z give rise to three lines of length 2. On each of these three points there are two lines of length 2, two of length 4 and one of length 3. Consider the linear space obtained from $(\mathcal{P}, \mathcal{L})$ by deleting the three points p, y and z and so the three lines py, pz, yz, it has 8 points and 12 lines, each point has degree 4 and lines have length 2 or 3. There are four lines of length 2 which partition the point set of this linear space, and so adding a point at infinity to this parallel class we get the affine plane of order 3, and so $(\mathcal{P}, \mathcal{L})$ is the punctured affine plane of order 3 with a triangle at infinity, that is $E12$. □

Lemma 3.8. *If $n=2$ then $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces described in $E14$ and $E15$.*

PROOF. If $n=2$, then $v \leq 7$, and so the assertion easily follows from Theorem 2.4. □

3.3. $k \geq m+1$. In this section we are going to prove the following proposition.

Proposition 3.9. *If $k \geq m+1$ then $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces $E4, E6, E13$ and $E16$.*

First consider the case $m \geq 4$.

From $b \geq k(m-1)+1 \geq m^2$ it follows that $b \geq 16$, and so $b-v \leq \sqrt{b}$, and by Theorem 2.2 we have that there is a single point of degree m and $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces $E4$ and $E6$.

So we may assume that $m = 3$, and the following lemmas are the proof of Proposition 3.5.

We distinguish two cases.

CASE 1. *There is a single point p of degree m.*

Let p be the point of degree 3, counting v via the lines on p we obtain $v \leq 3(k-1)+1$. Since $b \geq 3k$, from $b - v \leq 4$ it follows that

$$3k \leq b \leq 3k+2, \ \ 3k-4 \leq v \leq 3k-2.$$

Lemma 3.10. *If $m = 3$, then $(\mathcal{P}, \mathcal{L})$ is one of the linear spaces $E13$ and $E16$.*

PROOF. If on p there is a single line of length k, then counting v on p we have

$$3k-4 \leq v \leq k+k-2+k-2,$$

and so $v = 3k-4$, and on p there are two lines of length $k-1$ and one line of length k. From $b = 3k$ it follows that the line of length k has no parallel lines and that its points different from p have degree $m+1 = 4$. Let x be a point of a line of length $k-2$, x has degree k, on it there are one line of length $k-1$, $k-2$ lines of length 3 and one line of length 2. So there are $2k-4$ lines of length 2, one line of length k, two of length $k-1$ and $(k-2)^2$ of length 3. Thus

$$3k = b = 1+2+2k-4+k^2-4k+4,$$

and so

$$k^2-5k+3 = 0.$$

Since k is an integer we have a contradiction.

Hence on p there are at least two lines of length k.

If on p there are two lines of length k, then $3k-4 \leq v \leq 3k-3$.

If $v = 3k-4$, then on p there are two lines of length k and one of length $k-2$. From $3k = b = 3+(k-1)m$, it follows $[x] = m+1$ for each x on a line of length k, and lines of length k have no parallel line. Thus $k = m+1 = 4$, and so on p there are two lines of length 4 and one of length 2. Let x be the second point of the line of length 2 on p, then on x there are $k-1$ lines of length 3. If y belongs to a line of length k, then on y there are one line of length 3, one of length k and $k-2 = m-1 = 2$ lines of length 2. Hence $v = 8$, $b = 12$ and $(\mathcal{P}, \mathcal{L})$ is the linear space $E13$.

If $v = 3k-3$, then $b = 3k+1$. A line of length k contains either a point of degree $m+2$ or a parallel line. If L is a line of length k with a parallel line ℓ, since ℓ has length 2 it meets the other line of length k that has a point of degree $k+1$ and a parallel line a contradiction. So lines of length k have no parallel line. So they have a point of degree $m+2$, and so $k = m+2$, hence each point of a line of length k different from p has degree $m+2$, a contradiction since $b = 3k+1$.

So $v = 3k-2$, $b = 3k+2$, and on p there are three lines of length k. Let L, L_1 and L_2 be the three lines on p.

If $\delta_L = 0$, then for each point not in L $[x] = k$, and so each point of $(\mathcal{P}, \mathcal{L})$ has degree k. It follows that on each point x different from p there are one line of length k and $k-1$ lines of length 3. Let b_i denote the number of lines of length i. Hence

$$(v-1)(k-1) = 3b_3 = \frac{(3k-2)(k-1)}{2}.$$

From $b = b_3 + b_k = 3k$, it follows that

$$3k^2 - 14k + 5 = 0,$$

a contradiction since k is an integer.

So $\delta_L \geq 1$. Thus there is a line ℓ parallel to L, and so both L_2 and L_3 have a point of degree at least $k+1$ and so they have a parallel line, hence each of them has exactly one parallel line and exactly one point of degree $k+1$, one of degree m and the remaining of degree $m+1$. So $k = m+1 = 4$. Each line of length k has a point of degree $k+1 = 5$. It follows that $v = 3k-2 = 10$ and $b = v+4 = 14$.

On a point of degree $k+1 = 5$ there are a line of length k, two lines of length 2 and two of length 3. On a point of degree k there is no line of length 2. Hence there are three lines of length 2, eight of length 3 and three of length 4. There are three points of degree $k+1 = 5$. Let us denote these three points by ∞_1, ∞_2 and ∞_3.

The linear space obtained from $(\mathcal{P}, \mathcal{L})$ by deleting the three points ∞_i, $i = 1, 2, 3$ is a linear space on $v = 7$ points, $b = 11$ lines, $m = 3 = k$, with few lines of length 3 and with two parallel lines of length 3. Its points have degree m and $m+1$, and so by the previous section it follows that it is the linear space $E15$. Let $\{1,2,3,4,5,6,7\}$ be the points of $E15$, and $\{123, 345, 561, 276, 147, 46, 37, 24, 57, 36, 25\}$ be the lines of $E15$. The triple of lines $\{123, 46, 57\}$, $\{561, 24, 37\}$ and $\{147, 36, 25\}$ are three parallel classes of $E15$, and so adding ∞_1 to first triple, ∞_2 to the second and ∞_3 to the third triple, and adding the lines $\infty_1\infty_2$, $\infty_1\infty_3$ and $\infty_2\infty_3$ we obtain $(\mathcal{P}, \mathcal{L})$ and so $(\mathcal{P}, \mathcal{L})$ is the linear space $E15$ with a near-pencil on 3 points at infinity, that is the linear space $E16$. □

CASE 2. *There are at least two points of degree m.*

In this case $v \leq k+4$ and there is a single line of length k. Denote by L the unique line of length k.

Since $b \geq 2k+1$, from $b-v \leq 4$ it follows that

$$k \leq 7.$$

Now we are going to prove the following result.

Lemma 3.11. *There is no finite linear space with $b-v=4$, two points of degree m, $k \geq m+1$, and $m=3$.*

PROOF. If $k \leq 5$, then $v \leq 9$ and by Theorem 2.4 it follows that there is no finite linear space with $m=3$, $4 \leq k \leq 5$ and $b-v \leq 4$.

If $k=7$, then $b=15$, $v=11$ and the points of the line of length $k=7$ have all degree $m=3$. Since on a point of degree 3 there are one line of length k and two lines of length 3, then there are at least fourteen lines of length 3, and this is in contradiction with the fact that the four points outside the line of length k give rise to six lines of length 3.

If $k=6$ then

$$9 \leq v \leq 10, \quad \text{and } 13 \leq b \leq 14.$$

By Theorem 2.4 it follows that $v=9$ cannot occur.

If $v=10$, $b=14$, on a point of degree m there are one line of length k and two lines of length 3. The line of length k contains at least five points of degree m, and so there are ten lines of length 3 contradicting the fact that the four points outside of the line of length k give rise to six lines of length 3. □

References

[1] L. M. Batten, A. Beutelspacher, The Theory of Finite Linear Spaces, *Cambridge University Press*, Cambridge, (1993).

[2] W.G. Bridges, Near 1-designs, *J. Combin. Theory* Ser. A 13, (1972).

[3] N.G. de Bruijn, P. Erdős, On a combinatorial problem, *Indag. Math.* 10, (1948).

[4] J. Doyen, Sur le nombre d'espaces linéaires non isomorphes de n points, *Bull. Soc. Math. Belg.* 19, (1967), 421 – 437.

[5] P. Libois, Quelques espaces linéaires, *Bull. Soc. Math. Belg.* 16, (1964), 13–22.

[6] K. Metsch, Finite linear spaces with few lines, *Lecture Notes in Mathematics*, 1490, Springer – Verlag, Berlin – Heidelberg, (1991).

[7] V. Napolitano, Una dimostrazione unificata dei teoremi di Bridges, de Witte e del teorema fondamentale degli spazi lineari finiti, *Rend. Circ. Mat. Palermo* 50, (2001) (to appear).

[8] G. Tallini, Spazi di rette e k-insiemi di $PG(r,q)$, *Conf. Sem. Mat. Univ. Bari*, 192, (1984).

[9] J. Totten, Classification of restricted linear spaces, *Can. J. Math.* (A) 28, (1976).

[10] J. Totten, Finite linear space with three more lines than points, *Simon Stevin*, 51, (1976).
[11] S. A. Vanstone, The extendability of $(r, 1)$ - designs, Proc. Third Southeastern Conf. on Num. Math. *Utilitas Math.*, 409 - 418, (1973).
[12] P. de Witte, A new proof of a theorem of Bridges, *Simon Stevin* 47, (1973).
[13] P. de Witte, Combinatorial properties of finite linear spaces II, *Bull. Soc. Math. Belg.*, 27, (1975).
[14] P. de Witte, Finite linear spaces with two more lines than points, *J. Reine Ang. Math.* (*Crelle*), 288, (1976).

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