Lah Matrix and Its Algebraic Properties

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Abstract

The $n \times n$ Lah matrix L_n is defined by $(L_n)_{ij} = l(i,j)$,where $l(i,j)$ is the unsigned Lah number. In this paper,we investigate the algebraic properties of L_n , and many important relations between L_n and Pascal matrix and Stirling matrix respectively. In addition , we obtain its exponential expansion and Pascal matrix factorization.Furthermore , we introduce a simple method to find and prove combinatorial identities.

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1. Introduction

The Lah number $\bar{l}(n,k) = (-1)^n l(n,k)$ was originated by Lah in [9] , where n and k are nonegative integers , $l(n,k)=\binom{n}{k}\frac{(n-1)!}{(k-1)!}$ is the unsigned Lah number , $l(n,0)= 0$. Some well known results about $\bar{l}(n,k)$ have been

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obtained in [3]:

$$(-x)_n = (-1)^n < x >_n = \sum_{k=0}^{n} (x)_k \bar{l}(n,k)$$

$$a_n = \sum_{k=0}^{n} \bar{l}(n,k) b_k \text{ is equivalent to } b_n = \sum_{k=0}^{n} \bar{l}(n,k) a_k \qquad (1.1)$$

$$\bar{l}(n,k) = (-1)^n \sum_{j=k}^{n} (-1)^j s(n,j) S(j,k) \qquad (1.2)$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ for $n \geq 1$ and $< x >_n = x(x+1)(x+2)\cdots(x+n-1)$ for $n \geq 1$, $(x)_0 = < x >_0 = 1$, $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two sequences .

For integers n and k with $n \geq k \geq 0$, the Stirling numbers of the first (unsigned) $s(n,k)$ and of the second kind $S(n,k)$ are defined as the coefficients in the following expansion of a variable x(see [3]) :

$(x)_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k) x^k$ and $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$

Let n and k be positive integers , we define the $n \times n$ Lah matrices $L_n, \bar{L}_n, \tilde{L}_n$ as follows respectively :

$$(L_n)_{ij} = \begin{cases} l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}, \quad (\bar{L}_n)_{ij} = \begin{cases} \bar{l}(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases},$$

$$(\tilde{L}_n)_{ij} = \begin{cases} \tilde{l}(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{l}(i,j) = (-1)^{i+j} l(i,j)$, $(A)_{ij}$ denotes the (i,j)-entry of matrix A.

The $n \times n$ Pascal matrix P_n and Stirling matrices s_n , S_n can be defined as follows respectively(see[3]).

$$(P_n)_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}, \quad (s_n)_{ij} = \begin{cases} s(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases},$$

$$(S_n)_{ij} = \begin{cases} S(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

The $n \times n$ generalized Pascal and Stirling matrix with one variable x are defined by (see [1,2]):

$$(P_n[x])_{ij} = \begin{cases} x^{i-j}\binom{i-1}{j-1} & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}, \quad (s_n[x])_{ij} = \begin{cases} x^{i-j} s(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}.$$

$$(S_n[x])_{ij} = \begin{cases} x^{i-j}S(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

For the $k \times k$ Pascal matrix P_k , we define the $n \times n$ matrix $\bar{P}_n$ and $\bar{P}_n[x]$ by :

$$\bar{P}_n = \begin{pmatrix} I_{n-k} & 0 \\ 0 & P_k \end{pmatrix}, \bar{P}_n[x] = \begin{pmatrix} I_{n-k} & 0 \\ 0 & P_k[x] \end{pmatrix}$$

Therefore , $\bar{P}_n = P_n$, and $\bar{P}_1$ is the identity matrix of order n(see [1]).

Lemma 1 [1] . *The Stirling matrix S_n of the second can be factorized by the Pascal matrices $\bar{P}_k$'s:*

(1) $s_n = \bar{P}_n\bar{P}_{n-1}\cdots\bar{P}_2\bar{P}_1$

(2) $S_n^{-1}[x] = s_n[-x]$

(3) $S_n^{-1}[x] = \bar{P}_1[-x]\bar{P}_2[-x]\cdots\bar{P}_{n-1}[-x]\bar{P}_n[-x]$

(4) $S_n[x] = \bar{P}_n[x]\bar{P}_{n-1}[x]\cdots\bar{P}_2[x]\bar{P}_1[x]$

Pascal's matrix and its generalizations are studied in many papers [2, 4, 5, 6, 7] . Stirling matrix is studied carefully in [1] . In this paper , we investigate Lah matrices $L_n, \bar{L}_n, \tilde{L}_n$ and find an interesting fact that they have many similar properties to that of Pascal matrix and Stirling matrix.

We obtain the Pascal-type factorization and beautiful expansion of Lah matrix and the generalized Lah matrix . Furthermore , we apply some results to investigate some combinatorial identities .

2. Some elementary results on Lah matrix

By simple computation,we can obtain the following

$$\tilde{L}_n = \bar{I}_n L_n \tilde{L}_n = \bar{L}_n \bar{I}_n \tag{2.1}$$

$$\bar{L}_n^2 = I_n, L_n\tilde{L}_n = I_n \tag{2.2}$$

$$\tilde{L}_n^{-1} = \tilde{I}_n \bar{L}_n, \qquad \mathrm{L}_n^{-1} = \tilde{I}_n L_n \tilde{I}_n \tag{2.3}$$

where $\tilde{I}_n = diag\{-1, 1, \cdots, (-1)^n\}$, I_n is the identity matrix of order n .

We notice that (2.2) is equivalent to (1.1).

Let all elements of $n \times n$ matrix M be zero except $(M)_{i,i-1} = i(i-1)(i = 2, 3, \cdots, n)$.We can obtain all elements of the matrix $\frac{1}{k!}M^k$ are equal to zero except $(\frac{1}{k!}M^k)_{i,i-1} = l(i+k-1, i+k-2)(i = 2, 3, \cdots, n), k = 1, 2, \cdots, n-1.$

Therefore , we get the following pretty potential expansion of L_n

Let n be a positive integer , $n \times n$ matrix M be defined as above , then we have

$$L_n = I_n + M + \frac{1}{2!}M^2 + \cdots + \frac{1}{(n-1)!}M^{n-1} \tag{2.4}$$

For example, for L_4

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 6 & 1 & 0 \\ 24 & 36 & 12 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 12 & 0 \end{pmatrix}$$
$$+\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 12 & 0 \end{pmatrix}^2 + \frac{1}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 12 & 0 \end{pmatrix}^3$$

By (1.2) ,we obtain

$$L_n = s_n S_n \tag{2.5}$$

Let n be a positive integer and k be a nonnegative integer , and $s(i,j)$, $S(i,j)$ be the Stirling numbers . Because of (2.5) , we have $s_n = L_n S_n^{-1}$ and $S_n = s_n^{-1} L_n$, and this leads to

$$\sum_{i=k}^{n-1} (-1)^{i+k} \binom{n}{i} \frac{(n-1)!}{(i-1)!} s(i,k) = \begin{cases} 2s(n,k) & \text{if } n+k \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

and

$$\sum_{i=k+1}^{n} (-1)^{i+n} \binom{i}{k} \frac{(i-1)!}{(k-1)!} S(n,i) = \begin{cases} 2S(n,k) & \text{if } n+k \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

3. The generalized Lah matrices

As we did for the Pascal triangular matrix , we introduce the generalized Lah matrices as follows:

Definition 1. For any real number x , the $n \times n$ Lah generalized matrices are defined by

$$(L_n[x])_{ij} = \begin{cases} x^{i-j} l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}, (\bar{L}_n[x])_{ij} = \begin{cases} (-1)^i x^{i-j} l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

$$(\tilde{L}_n[x])_{ij} = \begin{cases} (-1)^{i+j} x^{i-j} l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

$$L_n[0] = \tilde{L}_n[0] = \bar{L}_n[0] = I_n$$

By some simple computations , the following theorem holds:

Theorem 1. *For any variables x and y , we have*

(1) $(\bar{L}_n[x])^2 = I_n$

(2) $\bar{L}_n[x+y] = L_n[x] L_n[-y] \tilde{I}_n$

(3) $\tilde{L}_n[x+y] = \tilde{L}_n[x] \tilde{L}_n[y]$

(4) $L_n[x+y] = L_n[x] L_n[y]$

(5) $L_n[x] = \tilde{L}_n[-x]$

(6) $(\tilde{L}_n[x])^{-1} = \tilde{L}_n[-x]$

(7) For any positive integers m and j :

$L_n[mx] = (L_n[x])^m, (L_n[j/m])^m = L_n[j], \tilde{L}_n[mx] = (\tilde{L}_n[x])^m,$

$(\tilde{L}_n[j/m])^m = \tilde{L}_n[j]$

Theorem 2. *$L_n[x]$ is related to the generalized Stirling matrices $s_n[x]$ and $S_n[x]$ for variable x :*

$$L_n[x] = s_n[x] S_n[x] = S_n^{-1}[-x] S_n[x] \tag{3.1}$$

Proof $(s_n[x] S_n[x])_{ij} = \sum_{k=j}^{i} x^{i-k} s(i,k) x^{k-j} S(k,j) = x^{i-j} \sum_{k=j}^{i} s(i,k) S(k,j)$

$= x^{i-j}l(i,j) = (L_n[x])_{ij}$

This completes the proof. □

Theorem 3. *Lah matrix and its generalized matrix can be factorized by the Pascal matrices $\bar{P}_k$'s:*

(1) $L_n[x] = \bar{P}_1[x]\bar{P}_2[x]\cdots\bar{P}_{n-1}[x]\bar{P}_n[x]\bar{P}_n[x]\bar{P}_{n-1}[x]\cdots\bar{P}_2[x]\bar{P}_1[x]$ *(3.2)*

(2) $L_n = \bar{P}_1\bar{P}_2\cdots\bar{P}_{n-1}\bar{P}_n\bar{P}_n\bar{P}_{n-1}\cdots\bar{P}_2\bar{P}_1$ *(3.3)*

Proof (1) holds because of (3.1) and Lemma 1. (2) holds for (1) when $x = 1$. □ .

Clearly , all $\bar{L}_n[x]$, $\tilde{L}_n[x]$, $\bar{L}_n$, $\tilde{L}_n$ and their inverse matrices matrices have similar factorizations. For example :

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 6 & 1 & 0 \\ 24 & 36 & 12 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Theorem 4. *For arbitrary variables x and y , let $\{a_n(x,y)\}_{n\geq 1}$ and $\{b_n(x,y)\}_{n\geq 1}$ be two functional sequences , if $\Phi_n[x,y]$ is a lower triangular functional matrix such that $\Phi_n^2[x,y] = I_n$, then the following two formulas are equivalent :*

$$a_n(x,y) = \sum_{k=1}^{n} \varphi_{n,k}(x,y) b_k(x,y) \quad \textit{and} \quad b_n(x,y) = \sum_{k=1}^{n} \varphi_{n,k}(x,y) a_k(x,y) \tag{3.4}$$

where $\varphi_{i,j}(x,y) = (\Phi_n[x,y])_{ij}$.

Proof We notice that this proof is equivalent to proving $\Phi_n^2[x,y] = I_n$. □.

Clearly , (1.1) is a special case in (3.4). In fact , this is a inverse relation . We can find and prove some interesting combinatorial identities by Theorem 4.

Example 3.1 (Corollary 3 in [2]). For any given sequence $\{x_n\}_{n\geq 0}$ and integer with $m \geq 0$, considering the following system of equations:

$$y_i = \binom{i}{0}x_0 - \binom{i}{1}x_1 + \binom{i}{2}x_2 - \cdots + (-1)^i\binom{i}{i}x_i \quad (i = 0, 1, 2, \cdots, m)$$

then it follows that

$$x_i = \binom{i}{0}y_0 - \binom{i}{1}y_1 + \binom{i}{2}y_2 - \cdots + (-1)^i\binom{i}{i}y_i \quad (i = 0, 1, 2, \cdots, m)$$

here $\varphi_{i,j}(x, y) = (-1)^j\binom{i}{j}$,$\{a_n(x, y)\} = \{x_n\}$ and $\{b_n(x, y)\} = \{y_n\}$

Example 3.2 (Ljunggren,also see [8,(3.18)])n, k are integers and x, y, z are real numbers , then

$$\sum_{k=0}^{n}\binom{n}{k}\binom{z+k}{k}(x-y)^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}\binom{z}{k}x^{n-k}y^k$$

Let $\varphi_{i,j}(x, y) = (-1)^j\binom{i}{j}(x-y)^{i-j}$ if $i \geq j$,$b_k(x, y) = \sum_{i=0}^{k}\binom{k}{i}\binom{z}{i}x^{k-i}y^i$,

then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-y)^{n-k}\sum_{i=0}^{k}\binom{k}{i}\binom{z}{i}x^{k-i}y^i$$

$$=\sum_{i=0}^{n}\binom{z}{i}y^i(-1)^n\sum_{k=i}^{n}\binom{n}{i}\binom{n-i}{k-i}(y-x)^{n-k}x^{k-i}$$

$$= (-1)^n\sum_{i=0}^{n}\binom{z}{i}y^iy^{n-i}\binom{n}{i} = (-1)^n\binom{z+n}{n}y^n = a_n(x, y)$$

Example 3.3. Let x, y be two real numbers , and $\varphi_{i,j}(x, y) = (-1)^j\binom{i}{j}$(if $i \geq j$) , $b_k(x, y) = (-1)^k\sum_{k=0}^{k}\binom{k}{i}\binom{x+i}{k}(y-1)^i$

we can get

$$\sum_{k=0}^{n}\varphi_{n,k}(x, y)b_k(x, y) = \binom{n+x}{n}y^n$$

Therefore , a combinatorial identity is derived :

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+x}{k}y^k = (-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{x+k}{n}(y-1)^k$$

In addition , we notice that there are other applications by Theorem 4 if $a_n(x, y) = \sum_{k=1}^{n}\varphi_{n,k}(x, y)b_k(x, y)$ or $b_n(x, y) = \sum_{k=1}^{n}\varphi_{n,k}(x, y)a_k(x, y)$ is easy to be calculated .

Theorem 5. *For any real number x , it follows that*

$$L_n[x] = I_n + \sum_{k=1}^{n-1} \frac{L_0^{(k)}}{k!} x^k \tag{3.5}$$

where $(L_0^{(k)})_{ij} = \begin{cases} k!(i-j)_k l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}, k = 1, 2, \cdots, n-1.$

This theorem holds because of (2.4).

$\tilde{L}_n[x]$ and $\bar{L}_n[x]$ have similar results like this.

Theorem 6. *For any real number x and an integer $n \geq 2$, we have*

$$(L_n[x] - I_n)^{n-1} = M_n \tag{3.6}$$

where M_n is a matrix with order n , in which all elements are equal to zero except its $(n, 1)$-entry is $n!(n-1)!x^{n-1}$.

Proof . We argue by induction on n .

It is clear that for $n = 2$. Suppose it holds for $n - 1 (n \geq 3)$. Let

$$L_n[x] = \begin{pmatrix} L_{n-1}[x] & 0 \\ Q_n[x] & 0 \end{pmatrix}, \quad L_n[x] - I_n = \begin{pmatrix} L_{n-1}^{(0)}[x] & 0 \\ Q_n[x] & 0 \end{pmatrix}$$

where $L_{n-1}^{(0)}[x] = L_{n-1}[x] - I_{n-1}$, and a row matrix

$Q_n[x] = (l(n,1)x^{n-1}, l(n,2)x^{n-2}, \cdots, l(n, n-1)x)$, then it follows

$$(L_n[x] - I_n)^n = \begin{pmatrix} L_{n-1}^{(0)} & 0 \\ Q_n[x] & 0 \end{pmatrix}^n = \begin{pmatrix} (L_{n-1}^{(0)})^n & 0 \\ Q_n[x](L_{n-1}^{(0)}[x])^{n-1} & 0 \end{pmatrix}$$

$$\stackrel{*}{=} \begin{pmatrix} 0 & 0 \\ n!(n-1)!x^{n-1} & 0 \end{pmatrix} = M_n$$

where $\stackrel{*}{=}$ holds by considering the inductive assumption . □

Definition 2[2]. Let x and λ be two real numbers and n be a positive integer , we define the notation $x^{n|\lambda}$ as follows

$$x^{n|\lambda} = \begin{cases} x(x+\lambda)\cdots(x+(n-1)\lambda) & \text{if } n>0 \\ 1 & \text{if } n=0 \end{cases}$$

Lemma 2[2]. *With any real numbers x, y, λ and a positive integer n , then*

$$(x+y)^{n|\lambda} = \sum_{i=0}^{n} \binom{n}{i} x^{(n-i)|\lambda} y^{i|\lambda}$$

Definition 3. The $(n+1)\times(n+1)$ matrix $L_{n,\lambda}[x]$, in which n is a natural number and λ, x are real numbers , is defined by

$$(L_{n,\lambda}[x])_{ij} = \begin{cases} x^{(i-j)|\lambda} l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

Clearly , $L_{n,0}[x] = L_{n+1}[x]$, $L_{n,\lambda}[0] = I_{n+1}$.

Theorem 7. *For any real numbers x, y, λ and a natural number n , it follows that :*

$$L_{n,\lambda}[x+y] = L_{n,\lambda}[x] L_{n,\lambda}[y] \tag{3.7}$$

Proof By using Lemma 2 , we get

$(L_{n,\lambda}[x]L_{n,\lambda}[y])_{ij} = \sum\limits_{k=j}^{i} l(i,k) x^{(i-k)|\lambda} l(k,j) y^{(k-j)|\lambda}$

$= l(i,j) \sum\limits_{k=j}^{i} \binom{i-j}{k-j} x^{(i-k)|\lambda} y^{(k-j)|\lambda} = l(i,j)(x+y)^{(i-j)|\lambda} = (L_{n,\lambda}[x+y])_{ij}$

This completes the proof . □

The theorem can lead to the following :

$$L_{n,\lambda}^{-1}[x] = L_{n,\lambda}[-x]$$

and for any integers j and $k (k>0)$, we obtain

$L_{n,\lambda}^{j}[1] = L_{n,\lambda}[j]$ and $L_{n,\lambda}^{k}[j/k] = L_{n,\lambda}^{j}[1]$

In addition , it follows the following as Theorem 5 .

Theorem 8. *For any real numbers x and $\lambda \neq 0$*

$$L_{n,\lambda}[x] = I_{n+1} + \sum_{k=1}^{n} \frac{L_{\lambda}^{(k)}}{k!} x^k \tag{3.8}$$

where $(L_{\lambda}^{(k)})_{ij} = \begin{cases} k!s(i-j,k)l(i,j)\lambda^{i-j-k} & \text{if } i \geq j+k \\ 0 & \text{otherwise} \end{cases}$

See Theorem 5 for case $\lambda = 0$.

Clearly , $\tilde{L}_{n,\lambda}[x]$ has similar results as $L_{n,\lambda}[x]$.

Definition 4. The $n \times n$ matrix $L_n[x,y]$, where n is a natural number and x, y are real numbers , is defined by

$$(L_n[x,y])_{ij} = \begin{cases} x^{i-j}y^{i+j-2}l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

$\tilde{L}_n[x,y]$ and $\bar{L}_n[x,y]$ have similar definitions .

It is easy to see that the following theorem holds by the similar arguments for $L_n[x]$ by Definition 4.

Theorem 9. *Let x and y be any real numbers , then*

(a) $\bar{L}_n^{-1}[x,y] = \bar{L}_n[x,1/y]$

(b) $L_n[x,y]L_n[z,1/y] = L_n[(x+z)y]$

(c) $L_n^{-1}[x,y] = L_n[-x,1/y]$

Definition 5. The $(n+1) \times (n+1)$ matrix $L_{n,\lambda}[x]$, in which n is a natural number and x, y, λ are real numbers , is defined by

$$(L_{n,\lambda}[x,y])_{ij} = \begin{cases} x^{(i-j)|\lambda}y^{(j-1)|\lambda)}l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3. *The matrix $L_{n,\lambda}[x,y]$ can be factorized :*

$$L_{n,\lambda}[x,y] = L_{n,\lambda}[x]diag(1, y^{1|\lambda}, y^{2|\lambda}, \cdots, y^{n|\lambda})$$

Theorem 10. *For any real numbers x, y, z, λ and a natural number n .*

$$L_{n,\lambda}[x+y,z] = L_{n,\lambda}[x]L_{n,\lambda}[y,z]$$

Considering the previous discussions , we want to generalize the matrix $L_{n,\lambda}[x,y]$ in two variables associated with an arbitrary sequence $\bar{a} =$

$\{a_n\}_{n\geq 0}$.

Definition 6 . Suppose λ, x, y are three real numbers , n is a natural number and $\bar{a} = \{a_n\}_{n\geq 0}$ is an arbitrary sequence , then we define

$$(L_{n,\lambda}[x,y,\bar{a}])_{ij} = \begin{cases} a_{j-1}x^{(i-j)|\lambda}y^{(j-1)|\lambda)}l(i,j) & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

Lemma 4. *The matrix $L_{n,\lambda}[x,y,\bar{a}]$ has the multiplicative factorization :*

$$L_{n,\lambda}[x,y,\bar{a}] = L_{n,\lambda}[x,y]diag(a_0, a_1, \cdots, a_n)$$

Proof The proof is clear by mathematical induction and Theorem 10 . □

Theorem 11. *For any real numbers x, y, z, λ and any sequence $\bar{a} = \{a_n\}_{n\geq 0}$, we have*

(1) $L_{n,\lambda}[x+y,z,\bar{a}] = L_{n,\lambda}[x]L_{n,\lambda}[y,z,\bar{a}]$

(2) $L_{n,\lambda}[x+y,z,\bar{a}] = L_{n,\lambda}[x]L_{n,\lambda}[y]diag(a_0, a_1z^{1|\lambda}, \cdots, a_nz^{n|\lambda})$

4 . Conclusion

In this paper , we find some basic interesting properties and applications of Lah matrix and think there is further study to be made in the future . For example,various generalized Lah matrices and their properties and applications , and how to find more combinatorial identities by using (1.1) , Theorem 1(1) , and (3.4).

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