Cahit - Equitability of Coronas

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abstract

We prove that the corona graphs $C_n \circ K_1$ are k-equitable as per Cahit's definition of k-equitability, $k = 2, 3, 4, 5, 6$.

1. Introduction

In 1990 Cahit [2] proposed the idea of distributing the vertex and edge labels among $\{0, 1, \cdots, k-1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V, E)$ and any positive integer k, assign vertex labels from $\{0, 1, \cdots, k-1\}$ so that when the edge labels are induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. Cahit called a graph with such an assignment of labels **k-equitable**. Note that a graph $G(V, E)$ is graceful if and only if it is $(|E|+1)$-equitable and $G(V, E)$ is cordial if and only if it is 2-equitable.

Bloom [1] uses the term k-equitable to describe another kind of labeling. Hence we will use the term **Cahit-k-equitable** when the k-equitability is as per Cahit's definition.

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The **corona** $G_1 \circ G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harary[3] as the graph G obtained by taking one copy of G_1 which has p_1 vertices and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. Here we prove that the coronas $C_n \circ K_1, n \geq 3$, are Cahit-k-equitable, $k = 2, 3, 4, 5, 6$.

2. Cahit-i-equitability of coronas, $i = 2, 3$.

All throughout we will use the following notations;

$$V(C_n \circ K_1) = \{u_1, u_2, \ldots, u_n; v_1, \ldots, v_n\}$$

where $u_1 u_2 \ldots u_n u_1$ is the cycle C_n and v_i is the pendant vertex adjacent to u_i,
$1 \leq i \leq n$.

Theorem 1. All coronas are Cahit-2-equitable.

Proof: Give label 0 to all the cycle vertices u_i and give label 1 to all the pendant vertices v_i, $1 \leq i \leq n$. This simple distribtion of labels $0, 1$ is obviously Cahit-2-equitable.

Theorem 2. All coronas are Cahit-3-equitable.

Proof: For Cahit-3-equitability, the label set as well as the edge weight set is $\{0, 1, 2\}$. We have $p(C_n \circ K_1) = q(C_n \circ K_1) = 2n$. We consider three different cases.

Case 1. $2n \equiv 0(mod\ 3)$

Let $p = q = 2n = 3t$, $t \geq 2$. Note that as $3t = 2n$, t is an even number.We give suitable labeling at the end of the proof for $t = 2$. So let $t \geq 4$. For Cahit-3-equitability of $C_n \circ K_1$ each label $0, 1, 2$ will have to be used 't' times, such that each edge weight $0, 1, 2$ will occur 't' times.

Now we describe the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, \}$.

$$\begin{aligned} f(u_i) &= 0, \quad f(v_i) = 1, \quad 1 \leq i \leq \frac{t}{2} + 1; \\ f(u_{\frac{t}{2}+2i}) &= 2, \quad f(v_{\frac{t}{2}+2i}) = 2, \quad 1 \leq i \leq \frac{t}{2}; \\ f(u_{\frac{t}{2}+2i+1}) &= 0, \quad f(v_{\frac{t}{2}+2i+1}) = 1, \quad 1 \leq i \leq \frac{t}{2} - 1. \end{aligned}$$

It can be directly verified that this labeling of $C_n \circ K_1$ is Cahit-3-equitable.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 3$.

Cahit-3-equitable labeling of $C_3 \circ K_1$

Here $p = q = 6, t = 2, n = 3$.

$f(u_1) = f(u_2) = 0, f(v_1) = f(v_2) = 1$ and $f(u_3) = f(v_3) = 2$.

Case 2. $2n \equiv 1 (mod\ 3)$

Let $p = q = 2n = 3t + 1,\ \ t \geq 3$. Note that as $3t = 2n - 1, t$ is an odd number. We give suitable labeling at the end of the proof for $t = 3$. So let $t \geq 5$. For Cahit-3-equitability of $C_n \circ K_1$ two labels will have to be used 't' times and one label will have to be used '$t+1$' times, such that two edge weights will occur 't' times and one edge weight will occur '$t+1$' times.

Now we describe the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2\}$.

$$\begin{aligned} f(u_i) &= 0, \quad f(v_i) = 1, \quad 1 \leq i \leq \frac{t+1}{2}; \\ f(u_{\frac{t+1}{2}+2i}) &= 0, \quad f(v_{\frac{t+1}{2}+2i}) = 1, \quad 1 \leq i \leq \frac{t-1}{2}; \\ f(u_{\frac{t+1}{2}+2i+1}) &= 2, \quad f(v_{\frac{t+1}{2}+2i+1}) = 2, \quad 0 \leq i \leq \frac{t-1}{2}. \end{aligned}$$

It can be directly verified that the vertex labels $0, 1$ occur 't' times each while the vertex label 2 occurs '$t + 1$' times. Also, the edge weights $0, 1$ occur 't' times each while the edge weight 2 occurs '$t + 1$' times.

We give below a suitable labeling for $t = 3$ which corresponds to $n = 5$.

Cahit-3-equitable labeling of $C_5 \circ K_1$.

Here $p = q = 10, t = 3, n = 5$.

v_1	v_2	v_3	v_4	v_5
1	1	2	1	2
0	0	2	0	2
u_1	u_2	u_3	u_4	u_5

Case 3. $2n \equiv 2(mod\ 3)$

Let $p = q = 2n = 3t + 2,\ \ t \geq 2$. Note that as $3t = 2n - 2, t$ is an even number. We give suitable labeling at the end of the proof for $t = 2$. So let $t \geq 4$. For Cahit-3-equitability of $C_n \circ K_1$ two labels will have to be used '$t + 1$' times each and one label will have to be used 't' times, such that two edge weights will occur '$t + 1$' times each and one edge weight will occur 't' times.

Now we describe the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2\}$.

$$\begin{aligned} f(u_i) &= 0, \quad f(v_i) = 1, \quad 1 \leq i \leq \frac{t}{2} + 2; \\ f(u_{\frac{t}{2}+2+2i}) &= 0, \quad f(v_{\frac{t}{2}+2+2i}) = 1, \quad 1 \leq i \leq \frac{t}{2} - 1; \\ f(u_{\frac{t}{2}+2i+3}) &= 2, \quad f(v_{\frac{t}{2}+2i+3}) = 2, \quad 0 \leq i \leq \frac{t}{2} - 1. \end{aligned}$$

It can be directly verified that two labels and two edge-weights occur $t = 1$ times each and one label and one edge-weights occur t times each.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 4$.

Cahit-3-equitable labeling of $C_4 \circ K_1$

Here $p = q = 8, t = 2, n = 4$.

v_1	v_2	v_3	v_4
1	1	1	2
0	0	0	2
u_1	u_2	u_3	u_4

Illustration

We apply the labeling function f given above in Case 1, for $t = 12$ which correspond to $n = 18$. We describe the labels given to u_i, v_i in the following simple way, thus avoiding the actual drawing of the corona graph involved.

Cahit-3-equitable labeling of $C_{18} \circ K_1$

(v_1)	1	1	$\cdots$	1	(v_7)
(u_1)	0	0	$\cdots$	0	(u_7)

(v_8)	2	1	2	1	$\cdots$	2	(v_{18})
(u_8)	2	0	2	0	$\cdots$	2	(u_{18})

Here we have mentioned the vertices $u_1, v_1; u_{\frac{t}{2}+1}, v_{\frac{t}{2}+1}; u_{\frac{t}{2}+2}, v_{\frac{t}{2}+2}$ and u_n, v_n in brackets to indicate the range of the label sequences $1, 1, \ldots, 1$; $0, 0, \ldots, 0$; $2, 1, 2, 1, \ldots, 2$ and $2, 0, 2, 0, \ldots, 2$ respectively where the upper row gives labels of v_i's and the lower row gives labels of u_i's.

Illustration

We apply the labeling function f given above in Case 2, for $t = 13$ which correspond to $n = 20$.

Cahit-3-equitable labeling of $C_{20} \circ K_1$

$$\begin{array}{lccccccl} (v_1) & 1 & 1 & 1 & \cdots & 1 & (v_7) \\ (u_1) & 0 & 0 & 0 & \cdots & 0 & (u_7) \end{array}$$

$$\begin{array}{lcccccl} (v_8) & 2 & 1 & 2 & 1 & \cdots & 2\ (v_{20}) \\ (u_8) & 2 & 0 & 2 & 0 & \cdots & 2\ (u_{20}) \end{array}$$

Illustration

We apply the labeling function f given above in Case 3 for $t = 12$ which correspond to $n = 19$.

Cahit-3-equitable labeling of $C_{19} \circ K_1$

$$\begin{array}{lccccccl} (v_1) & 1 & 1 & 1 & \cdots & 1 & (v_8) \\ (u_1) & 0 & 0 & 0 & \cdots & 0 & (u_8) \end{array}$$

$$\begin{array}{lcccccl} (v_9) & 2 & 1 & 2 & 1 & \cdots & 2\ (v_{19}) \\ (u_9) & 2 & 0 & 2 & 0 & \cdots & 2\ (u_{19}) \end{array}$$

3. Cahit - 4 - equitability of Coronas

Theorem 3. All coronas are Cahit-4-equitable.

Proof: For Cahit-4-equitability, the label set as well as the edge weight set is $\{0, 1, 2, 3\}$. We have $p(C_n \circ K_1) = q(C_n \circ K_1) = 2n$. We consider the following cases.

Case 1. $2n \equiv 0 (mod\ 4)$
Let $p = q = 2n = 4t$.

Sub-Case 1.1. Suppose t is even. We give suitable labeling at the end of the proof for $t = 2$. So let $t \geq 4$. For Cahit-4-equitability of $C_n \circ K_1$ each label will have to be used 't' times, such that each edge weight will occur 't' times.

We describe the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3\}$.

$$\begin{aligned} f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \leq i \leq t; \\ f(u_{2i}) &= 2, \quad f(v_{2i}) = 2, \quad 1 \leq i \leq \frac{t}{2}; \\ f(u_{2i}) &= 3, \quad f(v_{2i}) = 3, \quad \frac{t}{2} + 1 \leq i \leq t. \end{aligned}$$

It can be directly verified that this labeling of $C_n \circ K_1$ is Cahit-4-equitable.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 4$.

Cahit-4-equitable labeling of $C_4 \circ K_1$

Here $p = q = 8, t = 2, n = 4$.

v_1	v_2	v_3	v_4
1	2	1	3
0	2	0	3
u_1	u_2	u_3	u_4

Sub-Case 1.2. Suppose t is odd. We give suitable labeling at the end of the proof for $t = 3$. So let $t \geq 5$.

In this case each label will have to be used 't' times such that each edge weight will occur 't' times.

We describe the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3\}$.

$$\begin{aligned}
f(u_{2i}) &= 0, \quad f(v_{2i}) = 1, \qquad 1 \leq i \leq t-2; \\
f(u_{2t-2}) &= 0, \quad f(v_{2t-2}) = 3; \\
f(u_{2t}) &= 0, \quad f(v_{2t}) = 2; \\
f(u_{2i-1}) &= 2, \quad f(v_{2i-1}) = 2, \qquad 1 \leq i \leq \frac{t-1}{2}; \\
f(u_{2i-1}) &= 3, \quad f(v_{2i-1}) = 3, \qquad \frac{t+1}{2} \leq i \leq t-1; \\
f(u_{2t-1}) &= 1, \quad f(v_{2t-1}) = 1.
\end{aligned}$$

It can be directly verified that each label and each edge weight occur t times.

We give below a suitable labeling for $t = 3$ which corresponds to $n = 6$.

Cahit-4-equitable labeling of $C_6 \circ K_1$

Here $p = q = 12, t = 3, n = 6$.

v_1	v_2	v_3	v_4	v_5	v_6
2	1	3	3	1	2
2	0	3	0	1	0
u_1	u_2	u_3	u_4	u_5	u_6

Case 2. $2n \equiv 2(mod\ 4)$

Let $p = q = 2n = 4t + 2$.

Sub-Case 2.1 Suppose t is even. We give suitable labeling at the end of the proof for $t = 2$. So let $t \geq 4$. For Cahit-4-equitability of $C_n \circ K_1$ two labels will have to be used 't' times each, and two labels will have to be used '$t+1$' times each such that two edge weights will occur 't' times each and two edge weights will occur '$t+1$' times each.

We describe the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0,1,2,3\}$.

$$\begin{aligned} f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \le i \le t+1; \\ f(u_{2i}) &= 2, \quad f(v_{2i}) = 2, \quad 1 \le i \le \frac{t}{2}; \\ f(u_{2i}) &= 3, \quad f(v_{2i}) = 3, \quad \frac{t}{2}+1 \le i \le t. \end{aligned}$$

It can be directly verified that two labels and two edge-weights occur t times each and two labels and two edge-weights occur $t+1$ times each.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 5$.

Cahit-4-equitable labeling of $C_5 \circ K_1$

Here $p = q = 10, t = 2, n = 5$.

v_1	v_2	v_3	v_4	v_5
1	2	1	3	1
0	2	0	3	0
u_1	u_2	u_3	u_4	u_5

Sub-Case 2.2 Suppose t is odd. We give suitable labeling at the end of the proof for $t = 1$. So let $t \geq 3$. For Cahit-4-equitability of $C_n \circ K_1$ two labels will have to be used 't' times each, and two labels will have to be used '$t+1$' times each such that two edge weights will occur 't' times each and two edge weights will occur '$t+1$' times each.

We describe the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0,1,2,3\}$.

$$\begin{aligned} f(u_{2i-1}) &= 2, \quad f(v_{2i-1}) = 2, \quad 1 \le i \le \frac{t+1}{2}; \\ f(u_{2i-1}) &= 3, \quad f(v_{2i-1}) = 3, \quad \frac{t+1}{2}+1 \le i \le t+1; \\ f(u_{2i}) &= 0, \quad f(v_{2i}) = 1, \quad 1 \le i \le t. \end{aligned}$$

It can be directly verified that two labels and two edge-weights occur 't' times each and two labels and two edge-weights occur '$t+1$' times each.

We give below a suitable labeling for $t = 1$ which corresponds to $n = 3$.

Cahit-4-equitable labeling of $C_3 \circ K_1$

Here $p = q = 6, t = 1, n = 3$.

v_1	v_2	v_3
2	1	3
2	0	3
u_1	u_2	u_3

Illustration

We apply the labeling function f given above in Sub-Case 1.1, for $t = 8$ which corresponds to $n = 16$.

Cahit-4-equitable labeling of $C_{16} \circ K_1$

$v_1 \quad v_3 \quad v_5 \quad \dots \quad v_{15}$ each labeled 1
$u_1 \quad u_3 \quad u_5 \quad \dots \quad u_{15}$ each labeled 0

$v_2 \quad v_4 \quad v_6 \quad v_8$ each labeled 2
$u_2 \quad u_4 \quad u_6 \quad u_8$ each labeled 2

$v_{10} \quad v_{12} \quad v_{14} \quad v_{16}$ each labeled 3
$u_{10} \quad u_{12} \quad u_{14} \quad u_{16}$ each labeled 3

Illustration

We apply the labeling function f given above in Sub-Case 1.2, for $t = 9$ which corresponds to $n = 18$.

Cahit-4-equitable labeling of $C_{18} \circ K_1$

$v_2 \quad v_4 \quad v_6 \quad \dots \quad v_{14}$ each labeled 1
$u_2 \quad u_4 \quad u_6 \quad \dots \quad u_{14}$ each labeled 0

v_{16} labeled 3, v_{18} labeled 2
u_{16} labeled 0, u_{18} labeled 0

$v_1 \quad v_3 \quad v_5 \quad v_7$ each labeled 2
$u_1 \quad u_3 \quad u_5 \quad u_7$ each labeled 2

$v_9 \quad v_{11} \quad v_{13} \quad v_{15}$ each labeled 3
$u_9 \quad u_{11} \quad u_{13} \quad u_{15}$ each labeled 3

v_{17} labeled 1, u_{17} labeled 1 .

Illustration

We apply the labeling function f given above in Sub-Case 2.1, for $t = 8$ which corresponds to $n = 17$.

Cahit-4-equitable labeling of $C_{17} \circ K_1$

v_1	v_3	v_5	$\ldots$	v_{17}	each labeled	1
u_1	u_3	u_5	$\ldots$	u_{17}	each labeled	0

v_2	v_4	v_6	v_8	each labeled	2
u_2	u_4	u_6	u_8	each labeled	2

v_{10}	v_{12}	v_{14}	v_{16}	each labeled	3
u_{10}	u_{12}	u_{14}	u_{16}	each labeled	3

Illustration

We apply the labeling function f given above in Sub-Case 2.2, for $t = 9$ which corresponds to $n = 19$.

Cahit-4-equitable labeling of $C_{19} \circ K_1$

v_1	v_3	v_5	v_7	v_9	each labeled	2
u_1	u_3	u_5	u_7	u_9	each labeled	2

v_{11}	v_{13}	v_{15}	v_{17}	v_{19}	each labeled	3
u_{11}	u_{13}	u_{15}	u_{17}	u_{19}	each labeled	3

v_2	v_4	v_6	$\ldots$	v_{18}	each labeled	1
u_2	u_4	u_6	$\ldots$	u_{18}	each labeled	0

4. Cahit - 5 - equitability of Coronas

Theorem 4. All coronas are Cahit - 5 - equitable.

Proof: For Cahit - 5 - equitability, the label set as well as the edge weight set is $\{0, 1, 2, 3, 4\}$. We have $p(C_n \circ K_1) = q(C_n \circ K_1) = 2n$. We consider five different cases.

Case 1. $2n \equiv 0 (mod\ 5)$

Let $p = q = 2n = 5t,\ \ t \geq 2$. Note that $5t = 2n$, therefore 't' is even. We give suitable labeling at the end of the proof for $t = 2$. So let $t \geq 4$. For Cahit - 5 - equitability of $C_n \circ K_1$ each label will have to be used 't' times such that each edge weight will occur 't' times.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4\}$.

$$\begin{aligned}
f(u_{2i-1}) &= 0,\ \ f(v_{2i-1}) = 1,\ \ 1 \leq i \leq \frac{t}{2};\\
f(u_{2i}) &= 4,\ \ f(v_{2i}) = 2,\ \ 1 \leq i \leq \frac{t}{2} - 1;\\
f(u_t) &= 4 = f(v_t);\\
f(u_{t+2i-1}) &= 0,\ \ f(v_{t+2i-1}) = 1,\ \ 1 \leq i \leq \frac{t}{2} - 1;\\
f(u_{t+2i}) &= 3 = f(v_{t+2i}),\ \ 1 \leq i \leq \frac{t}{2} - 1;\\
f(u_{2t-1}) &= 0,\ \ f(v_{2t-1}) = 3;\\
f(u_{2t}) &= 3,\ \ f(v_{2t}) = 1;\\
f(u_{2t+1}) &= 2 = f(v_{2t+1});\\
f(u_{2t+1+i}) &= 2,\ \ f(v_{2t+1+i}) = 4,\ \ 1 \leq i \leq \frac{t}{2} - 1.
\end{aligned}$$

It can be directly verified that each label and each edge-weight occurs exactly 't' times.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 5$.

Cahit - 5 - equitable labeling of $C_5 \circ K_1$

Here $p = q = 10, t = 2, n = 5$.

(v_1)	1	4	3	1	2	(v_5)
(u_1)	0	4	0	3	2	(u_5)

Case 2. $2n \equiv 1 (mod\ 5)$

Let $p = q = 2n = 5t + 1,\ \ t \geq 1$. Note that as $2n = 5t + 1$, t is an odd number. We give suitable labelings at the end of the proof for $t = 1, 3, 5$. So let $t \geq 7$. For Cahit - 5 - equitability of $C_n \circ K_1$ four labels will have to be used 't' times each and one label will have to be used '$t + 1$' times, such that four edge weights will occur 't' times each and one edge weight will occur '$t + 1$' times.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4\}$.

$$
\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \le i \le \frac{t+1}{2};\\
f(u_{2i}) &= 4, \quad f(v_{2i}) = 2, \quad 1 \le i \le \frac{t-3}{2};\\
f(u_{t-1}) &= 4 = f(v_{t-1});\\
f(u_{t+1}) &= 4 = f(v_{t+1});\\
f(u_{2i}) &= 0, \quad f(v_{2i}) = 1, \quad \frac{t+3}{2} \le i \le t-1;\\
f(u_{2i-1}) &= 3, \quad f(v_{2i-1}) = 3, \quad \frac{t+3}{2} \le i \le t;\\
f(u_{2i}) &= 0, \quad f(v_{2i}) = 2, \quad i = t, t+1;\\
f(u_{2t+1}) &= 3, \quad f(v_{2t+1}) = 1;\\
f(u_{2t+3}) &= 2 = f(v_{2t+3});\\
f(u_i) &= 2, \quad f(v_i) = 4, \quad 2t+4 \le i \le \frac{5t+1}{2}.
\end{aligned}
$$

It can be directly verified that four labels and four edge weights occur 't' times each and one label and one edge weight occurs '$t+1$' times each.

We give below suitable labelings for $t = 1, 3, 5$ which corresponds to $n = 3, 8, 13$ respectively.

Cahit - 5 - equitable labeling of $C_3 \circ K_1$

Here $p = q = 6, t = 1, n = 3$.

v_1	v_2	v_3
2	4	1
0	4	3
u_1	u_2	u_3

Cahit - 5 - equitable labeling of $C_8 \circ K_1$

Here $p = q = 16, t = 3, n = 8$.

(v_1)	1	2	1	4	3	2	1	2	(v_8)
(u_1)	0	4	0	4	3	0	3	0	(u_8)

Cahit - 5 - equitable labeling of $C_{13} \circ K_1$

Here $p = q = 26, t = 5, n = 13$.

(v_1)	1	2	1	4	1	(v_5)
(u_1)	0	4	0	4	0	(u_5)

(v_6)	4				
(u_6)	4				

(v_7)	3	1	3	(v_9)
(u_7)	3	0	3	(u_9)

(v_{10})	2	1	2	2	(v_{13})
(u_{10})	0	3	0	2	(u_{13})

Case 3. $2n \equiv 2(mod\ 5)$.

Let $p = q = 2n = 5t + 2, \quad t \geq 2$. Note that as $2n = 5t + 2, t$ is an even number.We give a suitable labeling at the end of the proof for $t = 2$. So let $t \geq 4$. For Cahit-5-equitability of $C_n \circ K_1$ three labels will have to be used 't' times each and two labels will have to be used '$t + 1$' times each such that three edge weights will occur 't' times each and two edge weights will occur '$t + 1$' times each.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4\}$.

$$\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \leq i \leq \frac{t}{2}; \\
f(u_{2i}) &= 4, \quad f(v_{2i}) = 2, \quad 1 \leq i \leq \frac{t}{2} - 1; \\
f(u_t) &= 4 = f(v_t); \\
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad \frac{t}{2} + 1 \leq i \leq t + 1; \\
f(u_{2i}) &= 3 = f(v_{2i}), \quad \frac{t}{2} + 1 \leq i \leq t; \\
f(u_{2t+2}) &= 2 = f(v_{2t+2}); \\
f(u_i) &= 2, \quad f(v_i) = 4, \quad 2t + 3 \leq i \leq \frac{5t+2}{2}.
\end{aligned}$$

It can be directly verified that three labels and three edge weights occur 't' times each and two labels and two edge weights occur '$t + 1$' times each.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 6$.

Cahit - 5 - equitable labeling of $C_6 \circ K_1$

Here $p = q = 12, t = 2, n = 6$.

v_1	v_2	v_3	v_4	v_5	v_6
1	4	1	3	1	2
0	4	0	3	0	2
u_1	u_2	u_3	u_4	u_5	u_6

Case 4. $2n \equiv 3(mod\ 5)$.

Let $p = q = 2n = 5t + 3, \quad t \geq 1$. Note that as $2n = 5t + 3, t$ is an odd number. We give suitable labelings at the end of the proof for $t = 1, 3$. So let $t \geq 5$. For Cahit-5-equitability of $C_n \circ K_1$ two labels will have to be used 't' times each and three labels will have to be used '$t + 1$' times each such that two edge weights will occur 't' times each and three edge weights will occur '$t + 1$' times each.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4\}$.

$$\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \leq i \leq \frac{t+1}{2}; \\
f(u_{2i}) &= 4, \quad f(v_{2i}) = 2, \quad 1 \leq i \leq \frac{t-1}{2}; \\
f(u_{t+1}) &= 4 = f(v_{t+1}); \\
f(u_{2i-1}) &= 3 = f(v_{2i-1}), \quad \frac{t+3}{2} \leq i \leq t + 1; \\
f(u_{2i}) &= 0, \quad f(v_{2i}) = 1, \quad \frac{t+3}{2} \leq i \leq t; \\
f(u_{2t+2}) &= 0, \quad f(v_{2t+2}) = 2; \\
f(u_{2t+3}) &= 2 = f(v_{2t+3}); \\
f(u_i) &= 2, \quad f(v_i) = 4, \quad 2t + 4 \leq i \leq \frac{5t+3}{2}.
\end{aligned}$$

It can be directly verified that two labels and two edge weights occur 't' times each and three labels and three edge weights occur '$t + 1$' times each.

We give below suitable labelings for $t = 1, 3$ which correspond to $n = 4, 9$.

Cahit - 5 - equitable labeling of $C_4 \circ K_1$

Here $p = q = 8, t = 1, n = 4$.

v_1	v_2	v_3	v_4
1	2	3	2
0	4	3	0
u_1	u_2	u_3	u_4

Cahit - 5 - equitable labeling of $C_9 \circ K_1$

Here $p = q = 18, t = 3, n = 9$.

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
1	2	1	4	3	1	3	2	2
0	4	0	4	3	0	3	0	2
u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9

Case 5. $2n \equiv 4(mod\ 5)$.

Let $p = q = 2n = 5t+4, \quad t \geq 2$. Note that as $2n = 5t+4, t$ is an even number. We give suitable labeling at the end of the proof for $t = 2$. So let $t \geq 4$. For Cahit-5-equitability of $C_n \circ K_1$ one label will have to be used 't' times and four labels will have to be used '$t+1$' times each such that one edge weight will occur 't' times each and four edge weights will occur '$t+1$' times each.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0,1,2,3,4\}$.

$$
\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \leq i \leq \frac{t}{2}; \\
f(u_{2i}) &= 4, \quad f(v_{2i}) = 2, \quad 1 \leq i \leq \frac{t}{2} - 1; \\
f(u_t) &= 4 = f(v_t); \\
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad \frac{t}{2} + 1 \leq i \leq t; \\
f(u_{2i}) &= 3 = f(v_{2i}), \quad \frac{t}{2} + 1 \leq i \leq t; \\
f(u_{2t+1}) &= 0, \quad f(v_{2t+1}) = 2; \\
f(u_{2t+2}) &= 3, \quad f(v_{2t+2}) = 1; \\
f(u_{2t+3}) &= 2 = f(v_{2t+3}); \\
f(u_i) &= 2, \quad f(v_i) = 4, \quad 2t+4 \leq i \leq \frac{5t+4}{2}.
\end{aligned}
$$

It can be easily verified that one label and one edge weight occur 't' times each and four labels and four edge weights occur '$t+1$' times each.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 7$.

Cahit - 5 - equitable labeling of $C_7 \circ K_1$

Here $p = q = 14, t = 2, n = 7$.

v_1	v_2	v_3	v_4	v_5	v_6	v_7
1	4	1	3	2	1	2
0	4	0	3	0	3	2
u_1	u_2	u_3	u_4	u_5	u_6	u_7.

Illustration

We apply the labeling function f given above in Case 1, for $t = 10$ which corresponds to $n = 25$.

Cahit - 5 - equitable labeling of $C_{25} \circ K_1$

(v_1)	1	2	1	2	1	2	1	2	1	(v_9)
(u_1)	0	4	0	4	0	4	0	4	0	(u_9)

(v_{10})	4
(u_{10})	4

(v_{11})	1	3	1	3	$\cdots$	1	3	(v_{18})
(u_{11})	0	3	0	3	$\cdots$	0	3	(u_{18})

v_{19}	v_{20}	v_{21}
3	1	2
0	3	2
u_{19}	u_{20}	u_{21}

(v_{22})	4	4	$\cdots$	4	(v_{25})
(u_{22})	2	2	$\cdots$	2	(u_{25})

Illustration

We apply the labeling function f given above in Case 2, for $t = 11$ which corresponds to $n = 28$.

Cahit - 5 - equitable labeling of $C_{28} \circ K_1$

(v_1)	1	2	1	2	$\cdots$	1	2	1	(v_9)
(u_1)	0	4	0	4	$\cdots$	0	4	0	(u_9)

(v_{10})	4	1	4	(v_{12})
(u_{10})	4	0	4	(u_{12})

(v_{13})	3	1	3	1	$\cdots$	1	3	(v_{21})
(u_{13})	3	0	3	0	$\cdots$	0	3	(u_{21})

(v_{22})	2	1	2	2	(v_{25})
(u_{22})	0	3	0	2	(u_{25})

(v_{26})	4	4	4	(v_{28})
(u_{26})	2	2	2	(u_{28})

Illustration

We apply the labeling function f given in Case 3, above in for $t = 8$ which corresponds to $n = 21$.

Cahit - 5 - equitable labeling of $C_{21} \circ K_1$

(v_1)	1	2	1	2	1	2	1	(v_7)		
(u_1)	0	4	0	4	0	4	0	(u_7)		
(v_8)	4									
(u_8)	4									
(v_9)	1	3	1	3	1	3	1	3	1	(v_{17})
(u_9)	0	3	0	3	0	3	0	3	0	(u_{17})
(v_{18})	2									
(u_{18})	2									
(v_{19})	4	4	4	(v_{21})						
(u_{19})	2	2	2	(u_{21})						

Illustration

We apply the labeling function f given above in Case 4, for $t = 9$ which corresponds to $n = 24$.

Cahit - 5 - equitable labeling of $C_{24} \circ K_1$

(v_1)	1	2	1	2	1	2	1	2	1	(v_9)
(u_1)	0	4	0	4	0	4	0	4	0	(u_9)
(v_{10})	4									
(u_{10})	4									
(v_{11})	3	1	3	1	3	1	3	1	3	(v_{19})
(u_{11})	3	0	3	0	3	0	3	0	3	(u_{19})
(v_{20})	2									
(u_{20})	0									

(v_{21})	2			
(u_{21})	2			

(v_{22})	4	4	4	(v_{24})
(u_{22})	2	2	2	(u_{24})

Illustration

We apply the labeling function f given in Case 5, above for $t = 8$ which corresponds to $n = 22$.

Cahit - 5 - equitable labeling of $C_{22} \circ K_1$

(v_1)	1	2	1	2	1	2	1	(v_7)
(u_1)	0	4	0	4	0	4	0	(u_7)

(v_8)	4
(u_8)	4

(v_9)	1	3	1	3	1	3	1	3	(v_{16})
(u_9)	0	3	0	3	0	3	0	3	(u_{16})

(v_{17})	2	(v_{18})	1	(v_{19})	2
(u_{17})	0	(u_{18})	3	(u_{19})	2

(v_{20})	4	4	4	(v_{22})
(u_{20})	2	2	2	(u_{22})

5. Cahit - 6 - equitability of Coronas

Theorem 5. All coronas are Cahit - 6 - equitable.

Proof: For Cahit - 6 - equitability, the label set as well as the edge weight set is $\{0, 1, 2, 3, 4, 5\}$. We have $p(C_n \circ K_1) = q(C_n \circ K_1) = 2n$. We consider three different cases.

Case 1. $2n \equiv 0 (mod\ 6)$

Let $p = q = 2n = 6t,\ \ t \geq 2$. So for Cahit - 6 - equitability of $C_n \circ K_1$ each label will have to be used 't' times such that each edge weight will occur 't' times.

Sub-Case 1.1. Supose n is odd. Hence t is odd, $t \geq 3$.

We give suitable labeling at the end of the proof for $t = 3$. So let $t \geq 5$.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4, 5\}$.

$$
\begin{aligned}
f(u_{2i-1}) &= 5, \quad f(v_{2i-1}) = 2, \quad 1 \leq i \leq \frac{t-1}{2}; \\
f(u_{2i}) &= 0, \quad f(v_{2i}) = 1, \quad 1 \leq i \leq \frac{t-1}{2}; \\
f(u_t) &= 5 = f(v_t); \\
f(u_{2i}) &= 0, \quad f(v_{2i}) = 2, \quad \frac{t+1}{2} \leq i \leq t-1; \\
f(u_{2i+1}) &= 4, \quad f(v_{2i+1}) = 1, \quad \frac{t+1}{2} \leq i \leq t-1; \\
f(u_{2t}) &= 0, \quad f(v_{2t}) = 3; \\
f(u_{2t+1}) &= 4 = f(v_{2t+1}); \\
f(u_{2t+2}) &= 3, \quad f(v_{2t+2}) = 1; \\
f(u_{2t+3}) &= 3, \quad f(v_{2t+3}) = 2; \\
f(u_i) &= 3, \quad f(v_i) = 4, \quad 2t+4 \leq i \leq \frac{5t+3}{2}; \\
f(u_i) &= 3, \quad f(v_i) = 5, \quad \frac{5t+5}{2} \leq i \leq 3t.
\end{aligned}
$$

It can be directly verified that each label and each edge weight occurs exactly 't' times.

We give below a suitable labeling for $t = 3$ which corresponds to $n = 9$.

Cahit - 6 - equitable labeling of $C_9 \circ K_1$

Here $p = q = 18, t = 3, n = 9$.

(v_1)	2	1	5	2	1	3	4	1	2	(v_9)
(u_1)	5	0	5	0	4	0	4	3	3	(u_9)

Sub-Case 1.2. Supose n is even. Hence t is even, $t \geq 2$.

We give suitable labelings at the end of the proof for $t = 2, 4$. So let $t \geq 6$.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0,1,2,3,4,5\}$.

$$\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \le i \le \frac{t-2}{2};\\
f(u_{2i}) &= 5, \quad f(v_{2i}) = 2, \quad 1 \le i \le \frac{t-2}{2};\\
f(u_{t-1}) &= 0, \quad f(v_{t-1}) = 2;\\
f(u_t) &= 5 = f(v_t);\\
f(u_{t+1}) &= 0, \quad f(v_{t+1}) = 2;\\
f(u_{2i}) &= 4, \quad f(v_{2i}) = 1, \quad \frac{t+2}{2} \le i \le t-1;\\
f(u_{2i+1}) &= 0, \quad f(v_{2i+1}) = 2, \quad \frac{t+2}{2} \le i \le t-2;\\
f(u_{2t-1}) &= 0, \quad f(v_{2t-1}) = 4;\\
f(u_{2t}) &= 4, \quad f(v_{2t}) = 1;\\
f(u_{2t+1}) &= 3, \quad f(v_{2t+1}) = 1;\\
f(u_{2t+2}) &= 3, \quad f(v_{2t+2}) = 2;\\
f(u_i) &= 3, \quad f(v_i) = 4, \quad 2t+3 \le i \le \frac{5t+2}{2};\\
f(u_i) &= 3, \quad f(v_i) = 5, \quad \frac{5t+4}{2} \le i \le 3t.
\end{aligned}$$

It can be directly verified that each label and each edge weight occurs exactly 't' times.

We give below suitable labelings for $t = 2, 4$ which correspond to $n = 6, 12$ respectively.

Cahit - 6 - equitable labeling of $C_6 \circ K_1$

Here $p = 12 = q, t = 2, n = 6$.

(v_1)	2	5	2	1	3	1	(v_6)
(u_1)	0	5	0	4	3	4	(u_6)

Cahit - 6 - equitable labeling of $C_{12} \circ K_1$

Here $p = 24 = q, t = 4, n = 12$.

(v_1)	1	2	(v_2)
(u_1)	0	5	(u_2)

(v_3)	2	5	2	(v_5)
(u_3)	0	5	0	(u_5)

(v_6)	1	4	1	1	2	(v_{10})
(u_6)	4	0	4	3	3	(u_{10})

(v_{11})	4	5	(v_{12})
(u_{11})	3	3	(u_{12})

Case 2. $2n \equiv 2(mod\ 6)$

Let $p = q = 2n = 6t + 2$. So for Cahit - 6 - equitability of $C_n \circ K_1$ four labels will have to be used 't' times each and two labels will have to be used '$t + 1$' times each so that four edge weights will occur 't' times each and two edge weights will occur '$t + 1$' times each.

Sub-Case 2.1. Suppose n is odd. Hence t is even, $t \geq 2$.

We give suitable labelings at the end of the proof for $t = 2, 4, 6$. So let $t \geq 8$.

We describe below the labeling function. $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4, 5\}$.

$$\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \leq i \leq \frac{t}{2}; \\
f(u_{2i}) &= 5, \quad f(v_{2i}) = 2, \quad 1 \leq i \leq \frac{t}{2} - 1; \\
f(u_t) &= 5 = f(v_t); \\
f(u_{t+1}) &= 0, \quad f(v_{t+1}) = 1; \\
f(u_{t+2}) &= 4 = f(u_{t+4}), f(v_{t+2}) = 2 = f(v_{t+4}); \\
f(u_{t+3}) &= 0, \quad f(v_{t+3}) = 2; \\
f(u_{2i+1}) &= 0, \quad f(v_{2i+1}) = 2, \quad \frac{t}{2} + 2 \leq i \leq t - 2; \\
f(u_{2i+2}) &= 4, \quad f(v_{2i+2}) = 1, \quad \frac{t}{2} + 2 \leq i \leq t - 2; \\
f(u_{2t-1}) &= 0 = f(u_{2t+1}), \quad f(v_{2t-1}) = 3 = f(v_{2t+1}); \\
f(u_{2t}) &= 4, \quad f(v_{2t}) = 1; \\
f(u_{2t+2}) &= 4 = f(v_{2t+2}); \\
f(u_{2t+3}) &= 3, \quad f(v_{2t+3}) = 2; \\
f(u_{2t+4}) &= 3, \quad f(v_{2t+4}) = 1; \\
f(u_{2t+5+i}) &= 3, \quad 0 \leq i \leq t - 4; \\
f(v_{2t+5+i}) &= 4, \quad 0 \leq i \leq \frac{t}{2} - 3; \\
f(v_{2t+i}) &= 5, \quad \frac{t}{2} + 3 \leq i \leq t + 1.
\end{aligned}$$

It can be directly verified that four lables and four edge-weights occur

't' times each and two labels and two edge weights occur '$t+1$' times each.

We give below suitable labelings for $t = 2,4,6$ which correspond to $n = 7,13,19$ respectively.

Cahit - 6 - equitable labeling of $C_7 \circ K_1$

Here $p = q = 14, t = 2, n = 7$.

(v_1)	1	5	1	2	3	2	3	(v_7)
(u_1)	0	5	0	4	0	4	3	(u_7)

Cahit - 6 - equitable labeling of $C_{13} \circ K_1$

Here $p = q = 26, t = 4, n = 13$.

(v_1)	1	2	1	5	1	(v_5)
(u_1)	0	5	0	5	0	(u_5)

(v_6)	2	3	2	3	4	(v_{10})
(u_6)	4	0	4	0	4	(u_{10})

(v_{11})	2	1	5	(v_{13})
(u_{11})	3	3	3	(u_{13})

Cahit - 6 - equitable labeling of $C_{19} \circ K_1$

Here $p = q = 38, t = 6, n = 19$.

(v_1)	1	2	1	2	1	5	1	(v_7)
(u_1)	0	5	0	5	0	5	0	(u_7)

(v_8)	2	2	2	3	1	3	4	(v_{14})
(u_8)	4	0	4	0	4	0	4	(u_{14})

(v_{15})	2	1	4	5	5	(v_{19})
(u_{15})	3	3	3	3	3	(u_{19})

Sub-Case 2.2. Suppose n is even. Hence t is odd, $t \geq 1$.

We give suitable labelings at the end of the proof for $t = 1,3,5,7$. So let $t \geq 9$.

We describe below the labeling function. $f : V(C_n \circ K_1) \longrightarrow \{0,1,2,3,4,5\}$.

$$
\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \le i \le \frac{t+3}{2};\\
f(u_{2i}) &= 5, \quad f(v_{2i}) = 2, \quad 1 \le i \le \frac{t+1}{2};\\
f(u_{t+3}) &= 4, \quad f(v_{t+3}) = 2;\\
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 2, \quad \frac{t+5}{2} \le i \le t;\\
f(u_{2i}) &= 4, \quad f(v_{2i}) = 1, \quad \frac{t+5}{2} \le i \le t-1;\\
f(u_{2i}) &= 4 = f(v_{2i}), i = t, t+1;\\
f(u_{2t+1}) &= 0, \quad f(v_{2t+1}) = 3;\\
f(u_{2t+3}) &= 3, \quad f(v_{2t+3}) = 2;\\
f(u_{2t+4}) &= 3, \quad f(v_{2t+4}) = 1;\\
f(u_{2t+4+i}) &= 3, \quad 1 \le i \le t-3;\\
f(v_{2t+4+i}) &= 4, \quad 1 \le i \le \frac{t-5}{2};\\
f(v_{2t+i}) &= 5, \quad \frac{t+5}{2} \le i \le t+1.
\end{aligned}
$$

It can be directly verified that four labels and four edge weights occur 't' times each and two labels and two edge weights occur '$t+1$' times each.

We give below suitable labelings for $t = 1,3,5,7$ which correspond to $n = 4, 10, 16, 22$ respectively.

Cahit - 6 - equitable labeling of $C_4 \circ K_1$

Here $p = q = 8, t = 1, n = 4$.

v_1	v_2	v_3	v_4
1	2	3	4
0	5	3	4
u_1	u_2	u_3	u_4

Cahit - 6 - equitable labeling of $C_{10} \circ K_1$

Here $p = q = 20, t = 3, n = 10$.

(v_1)	1	2	1	2	1	2	2	4	3	5	(v_{10})
(u_1)	0	5	0	5	0	4	0	4	3	3	(u_{10})

Cahit - 6 - equitable labeling of $C_{16} \circ K_1$

Here $p = q = 32, t = 5, n = 16$.

(v_1)	1	2	1	2	1	2	1	(v_7)
(u_1)	0	5	0	5	0	5	0	(u_7)

(v_8)	2	2	4	3	4	2	1	(v_{14})
(u_8)	4	0	4	0	4	3	3	(u_{14})

(v_{15})	5	5	(v_{16})
(u_{15})	3	3	(u_{16})

Cahit - 6 - equitable labeling of $C_{22} \circ K_1$

Here $p = q = 44, t = 7, n = 22$.

(v_1)	1	2	1	2	$\cdots$	1	(v_9)
(u_1)	0	5	0	5	$\cdots$	0	(u_9)

(v_{10})	2
(u_{10})	4

(v_{11})	2	1	2	(v_{13})
(u_{11})	0	4	0	(u_{13})

(v_{14})	4	3	4	2	1	(v_{18})
(u_{14})	4	0	4	3	3	(u_{18}).

(v_{19})	4
(u_{19})	3

(v_{20})	5	5	5	(v_{22})
(u_{20})	3	3	3	(u_{22}).

Case 3. $2n \equiv 4(mod\ 6)$

Let $p = q = 2n = 6t + 4$. So for Cahit - 6 - equitability of C_nOK_1 two labels will have to be used 't' times each and four labels will have to be used '$t + 1$' times each so that two edge weights will occur 't' times each and four edge weights will occur '$t + 1$' times each.

Sub-Case 3.1. Suppose n is odd. Hence t is odd, $t \geq 1$.

We give suitable labeling at the end of the proof for $t = 1$. So let $t \geq 3$.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4, 5\}$.

$$
\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \le i \le \frac{t+1}{2};\\
f(u_{2i}) &= 5, \quad f(v_{2i}) = 2, \quad 1 \le i \le \frac{t-1}{2};\\
f(u_{t+1}) &= 5 = f(v_{t+1});\\
f(u_{2i+1}) &= 0, \quad f(v_{2i+1}) = 2, \quad \frac{t+1}{2} \le i \le t;\\
f(u_{2i}) &= 4, \quad f(v_{2i}) = 1, \quad \frac{t+3}{2} \le i \le t;\\
f(u_{2t+2}) &= 4 = f(v_{2t+2});\\
f(u_{2t+3}) &= 3, \quad f(v_{2t+3}) = 1;\\
f(u_i) &= 3, \quad f(v_i) = 4, \quad 2t+4 \le i \le \frac{5t+5}{2};\\
f(u_i) &= 3, \quad f(v_i) = 5, \quad \frac{5t+7}{2} \le i \le 3t+2.
\end{aligned}
$$

It can be directly verified that two labels and two edge weights occur 't' times each and four labels and four edge weights occur '$t+1$' times each.

We give below a suitable labeling for $t = 1$ which corresponds to $n = 5$.

Cahit - 6 - equitable labeling of $C_5 \circ K_1$

Here $p = q = 10, t = 1, n = 5$.

(v_1)	1	5	2	4	1	(v_5)
(u_1)	0	5	0	4	3	(u_5)

Sub-Case 3.2. Suppose n is even. Hence t is even, $t \ge 2$.

We give suitable labeling at the end of the proof for $t = 2$. So let $t \ge 4$.

We describe below the labeling function $f : V(C_n \circ K_1) \longrightarrow \{0, 1, 2, 3, 4, 5\}$.

$$
\begin{aligned}
f(u_{2i-1}) &= 0, \quad f(v_{2i-1}) = 1, \quad 1 \le i \le \frac{t}{2};\\
f(u_{2i}) &= 5, \quad f(v_{2i}) = 2, \quad 1 \le i \le \frac{t}{2} - 1;\\
f(u_t) &= 5 = f(v_t);\\
f(u_{2i+1}) &= 0, \quad f(v_{2i+1}) = 2, \quad \frac{t}{2} \le i \le t;
\end{aligned}
$$

$$
\begin{aligned}
f(u_{2i}) &= 4, \quad f(v_{2i}) = 1, &\frac{t+2}{2} \leq i \leq t;\\
f(u_i) &= 3, \quad f(v_i) = 4, &2t+2 \leq i \leq \frac{5t+4}{2};\\
f(u_i) &= 3, \quad f(v_i) = 5, &\frac{5t+6}{2} \leq i \leq 3t+2.
\end{aligned}
$$

It can be directly verified that two labels and two edge weights occur 't' times each and four labels and four edge weights occur '$t+1$' times each.

We give below a suitable labeling for $t = 2$ which corresponds to $n = 8$.

Cahit - 6 - equitable labeling of $C_8 \circ K_1$

Here $p = q = 16, t = 2, n = 8$.

(v_1)	1	5	2	1	2	4	4	5	(v_8)
(u_1)	0	5	0	4	0	3	3	3	(u_8)

Illustration

We apply the labeling function f given above in Sub-Case 1.1, for $t = 9$ which corresponds to $n = 27$.

Cahit - 6 - equitable labeling of $C_{27} \circ K_1$

(v_1)	2	1	2	1	$\cdots$	1	(v_8)
(u_1)	5	0	5	0	$\cdots$	0	(u_8)

(v_9)	5
(u_9)	5

(v_{10})	2	1	2	1	$\cdots$	1	(v_{17})
(u_{10})	0	4	0	4	$\cdots$	4	(u_{17})

(v_{18})	3	4	1	2	(v_{21})
(u_{18})	0	4	3	3	(u_{21})

(v_{22})	4	4	4	(v_{24})
(u_{22})	3	3	3	(u_{24})

(v_{25})	5	5	5	(v_{27})
(u_{25})	3	3	3	(u_{27})

Illustration

We apply the labeling function f given above in Sub-Case 1.2, for $t = 8$ which corresponds to $n = 24$.

Cahit - 6 - equitable labeling of $C_{24} \circ K_1$

(v_1)	1	2	1	2	$\cdots$	2	(v_6)
(u_1)	0	5	0	5	$\cdots$	5	(u_6)

(v_7)	2	5	2	(v_9)
(u_7)	0	5	0	(u_9)

(v_{10})	1	2	1	2	$\cdots$	1	(v_{14})
(u_{10})	4	0	4	0	$\cdots$	4	(u_{14})

(v_{15})	4	1	1	2	(v_{18})
(u_{15})	0	4	3	3	(u_{18})

(v_{19})	4	4	4	(v_{21})
(u_{19})	3	3	3	(u_{21})

(v_{22})	5	5	5	(v_{24})
(u_{22})	3	3	3	(u_{24})

Illustration

We apply the labeling function f given above in Sub-Case 2.1, for $t = 12$ which corresponds to $n = 37$.

Cahit - 6 - equitable labeling of $C_{37} \circ K_1$

Here $p = q = 74, t = 12, n = 37$.

(v_1)	1	2	1	2	$\cdots$	1	(v_{11})
(u_1)	0	5	0	5	$\cdots$	0	(u_{11})

(v_{12})	5	1	2	2	2	(v_{16})	
(u_{12})	5	0	4	0	4	(u_{16})	

(v_{17})	2	1	2	1	2	1	(v_{22})
(u_{17})	0	4	0	4	0	4	(u_{22})

(v_{23})	3	1	3	4	2	1	(v_{28})
(u_{23})	0	4	0	4	3	3	(u_{28})

(v_{29})	4	4	4	4	(v_{32})
(u_{29})	3	3	3	3	(u_{32})

(v_{33})	5	5	5	5	5	(v_{37})
(u_{33})	3	3	3	3	3	(u_{37})

Illustration

We apply the labeling function f given above in Sub-Case 2.2, for $t = 11$ which corresponds to $n = 34$.

Cahit - 6 - equitable labeling of $C_{34} \circ K_1$

(v_1)	1	2	1	2	$\cdots$	1	(v_{13})
(u_1)	0	5	0	5	$\cdots$	0	(u_{13})

(v_{14})	2
(u_{14})	4

(v_{15})	2	1	2	1	$\cdots$	2	(v_{21})
(u_{15})	0	4	0	4	$\cdots$	0	(u_{21}).

(v_{22})	4	3	4	2	1	(v_{26})
(u_{22})	4	0	4	3	3	(u_{26}).

(v_{27})	4	4	4	(v_{29})
(u_{27})	3	3	3	(u_{29}).

(v_{30})	5	5	5	5	5	(v_{34})
(u_{30})	3	3	3	3	3	(u_{34}).

Illustration

We apply the labeling function f given above in Sub-Case 3.1, for $t = 9$ which corresponds to $n = 29$.

Cahit - 6 - equitable labeling of $C_{29} \circ K_1$

(v_1)	1	2	1	2	$\cdots$	1	(v_9)
(u_1)	0	5	0	5	$\cdots$	0	(u_9)

(v_{10})	5
(u_{10})	5

(v_{11})	2	1	2	1	$\cdots$	2	(v_{19})
(u_{11})	0	4	0	4	$\cdots$	0	(u_{19})

(v_{20})	4	1	(v_{21})
(u_{20})	4	3	(u_{21})

(v_{22})	4	4	4	4	(v_{25})
(u_{22})	3	3	3	3	(u_{25})

(v_{26})	5	5	5	5	(v_{29})
(u_{26})	3	3	3	3	(u_{29})

Illustration

We apply the labeling function f given above in Sub-Case 3.2, for $t = 10$ which corresponds to $n = 32$.

Cahit - 6 - equitable labeling of $C_{32} \circ K_1$

(v_1)	1	2	1	2	$\cdots$	1	(v_9)
(u_1)	0	5	0	5	$\cdots$	0	(u_9)

(v_{10})	5
(u_{10})	5

(v_{11})	2	1	2	1	$\cdots$	2	(v_{21})
(u_{11})	0	4	0	4	$\cdots$	0	(u_{21})

(v_{22})	4	4	$\cdots$	4	(v_{27})
(u_{22})	3	3	$\cdots$	3	(u_{27})

(v_{28})	5	5	$\cdots$	5	(v_{32})
(u_{28})	3	3	$\cdots$	3	(u_{32})

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