Detour Domination in Graphs

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ABSTRACT

For distinct vertices u and v of a nontrivial connected graph G, the detour distance $D(u,v)$ between u and v is the length of a longest u-v path in G. For a vertex $v \in V(G)$, define $D^-(v) = \min\{D(u,v) : u \in V(G) - \{v\}\}$. A vertex u ($\neq v$) is called a detour neighbor of v if $D(u,v) = D^-(v)$. A vertex v is said to detour dominate a vertex u if $u = v$ or u is a detour neighbor of v. A set S of vertices of G is called a detour dominating set if every vertex of G is detour dominated by some vertex in S. A detour dominating set of G of minimum cardinality is a minimum detour dominating set and this cardinality is the detour domination number $\gamma_D(G)$. We show that if G is a connected graph of order $n \geq 3$, then $\gamma_D(G) \leq n-2$. Moreover, for every pair k, n of integers with $1 \leq k \leq n-2$, there exists a connected graph G of order n such that $\gamma_D(G) = k$. It is also shown that for each pair a, b of positive integers, there is a connected graph G with domination number $\gamma(G) = a$ and $\gamma_D(G) = b$.

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1 Introduction

In the standard definition of domination in a graph, a vertex v dominates itself and each of its neighbors. By a neighbor of v, we mean, of course, any vertex that is adjacent to v. A neighbor of v can also be interpreted as a vertex distinct from v and whose distance from v is minimum. Certainly, if u is a vertex distinct from v whose distance from v is minimum, then this distance $d(u, v)$ must be 1. This assumes though, that we are using the standard definition of distance between two vertices (the length of a shortest path between the vertices). One major advantage of using this definition of distance is that it is a metric on the vertex set of every connected graph. There are other distances, however, defined on the vertex set of a connected graph that are also metrics. One of these is the length of a longest path between two vertices u and v in a connected graph G. This distance is called the detour distance between u and v and is denoted by $D(u, v)$. Detour distance has been studied in [3] and [4].

Let G be a nontrivial connected graph. For each vertex $v \in V(G)$, define

$$d^-(v) = \min\{d(u, v) : u \in V(G) - \{v\}\}.$$

A vertex u ($\neq v$) is called a *neighbor* of v if $d(u, v) = d^-(v)$. A vertex v is said to *dominate* a vertex u if $u = v$ or u is a neighbor of v. Since $d^-(v) = 1$ for all $v \in V(G)$, this is equivalent to the standard definition of neighbor and the standard definition of domination. For distinct vertices u and v of G, the *detour distance* $D(u, v)$ between u and v is the length of a longest u-v path in G. Thus $1 \leq D(u, v) \leq n - 1$, where $D(u, v) = 1$ if and only if uv is a bridge of G and $D(u, v) = n - 1$ if and only if G contains a hamiltonian u-v path. Also, $D(u, v) = d(u, v)$ for every two distinct vertices u and v of G if and only if G is a tree. We note that it is possible for $D(u, v) = d(u, v)$ for some pairs u, v of distinct vertices without G being a tree. For example, if $G = C_{2k}$, $k \geq 2$, and u and v are two antipodal vertices of G, then $D(u, v) = d(u, v) = k$.

For a vertex v in G, define

$$D^-(v) = \min\{D(u, v) : u \in V(G) - \{v\}\}.$$

A vertex u ($\neq v$) is called a *detour neighbor* of v if $D(u, v) = D^-(v)$. (Note that v is not necessarily a detour neighbor of u.) If every edge incident with v is a bridge, then the detour neighbors of v are precisely the neighbors of v. Consequently, if G is a tree, then a vertex u is a detour neighbor of a vertex v if and only if u is a neighbor of v. If $G = C_{2k}$, $k \geq 2$, then every vertex v of G has a unique detour neighbor, namely the vertex in G that is antipodal to v. The *detour neighborhood* $N_D(v)$ of a vertex v is the set of detour neighbors of v; and its *closed detour neighborhood* is $N_D[v] = N_D(v) \cup \{v\}$.

A vertex v is said to *detour dominate* a vertex u if $u = v$ or u is a detour neighbor of v. A set S of vertices of G is called a *detour dominating set* if every vertex of G is detour dominated by some vertex in S. A detour dominating set of G of minimum cardinality is a *minimum detour dominating set* and this cardinality is the *detour domination number* $\gamma_D(G)$. A minimum detour dominating of G is also called a γ_D-set for G.

While a vertex v dominates a vertex u if and only if u dominates v, this is not the case for detour domination. That is, detour domination is not symmetric in general. For example, in the graph $G = K_{2,3}$ of Figure 1, the vertex v detour dominates the vertex u, but u does not detour dominate v.

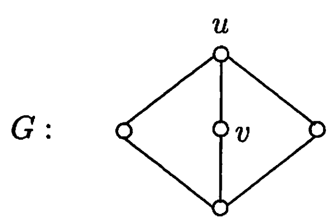

Figure 1: Detour domination is not symmetric

We define the *detour dominator set* of a vertex v, denoted by $D_D(v)$, to be the set of vertices different from v that detour dominate v, that is, $D_D(v) = \{u \mid v \in N_D(u)\}$. The closed detour dominator set is $D_D[v] = D_D(v) \cup \{v\}$.

2 Detour Domination Numbers of Some Familiar Graphs

As we have already mentioned, a vertex u dominates a vertex v in a tree G if and only if u detour dominates v in G. From this, we have the following.

Observation 2.1 *If G is a tree, then $\gamma_D(G) = \gamma(G)$.*

If G is a hamiltonian-connected graph of order $n \geq 3$, then $D(u,v) = n-1$ for every pair u, v of distinct vertices of G. Hence $N_D[v] = V(G)$ for every vertex v of G, which gives the following.

Observation 2.2 *If G is hamiltonian-connected, then $\gamma_D(G) = 1$.*

Among the many hamiltonian-connected graphs are the complete graphs.

Observation 2.3 *If G is a complete graph, then $\gamma_D(G) = 1$.*

The converse of Observation 2.2 is not true. A graph G of order $n \geq 2$ is called *hamiltonian-connected from a vertex* v (see [1]) if $D(v,w) = n-1$ for all $w \in V(G)-\{v\}$. The graph G of Figure 2 is hamiltonian-connected from v but from no other vertices. Therefore, G is not hamiltonian-connected, but $\gamma_D(G) = 1$. This illustrates the following.

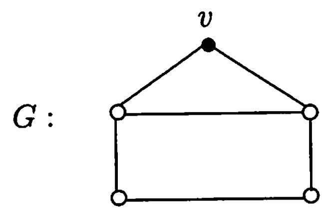

Figure 2: A graph G with $\gamma_D(G) = 1$ that is hamiltonian-connected from a unique vertex

Observation 2.4 *Let G be a nontrivial connected graph of order n. Then $\gamma_D(G) = 1$ if and only if there exists a constant k $(1 \leq k \leq n-1)$ and a vertex v such that $D(u,v) = k$ for all $u \in V(G)-\{v\}$. In particular, if G contains a vertex v such that G is hamiltonian-connected from v, then $\gamma_D(G) = 1$.*

The graph G of Figure 3 is hamiltonian-connected from none of its vertices, but $\gamma_D(G) = 1$. This follows since $D(v,w) = 2$ for all $w \in V(G) - \{v\}$.

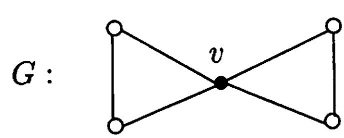

Figure 3: A graph G with $\gamma_D(G) = 1$ that is hamiltonian-connected from none of its vertices

The example shown in Figure 3 illustrates the following result.

Proposition 2.5 *For positive integers k and n such that $k \mid (n-1)$, there exists a connected graph G of order n and a vertex v such that $D(v, w) = k$ for all $w \in V(G) - \{v\}$.*

Proof. Let $n - 1 = kr$ and define G to be the join $K_1 + rK_k$ (see Figure 4), where $V(K_1) = \{v\}$. Then G has order $n = kr + 1$ and $D(v, w) = k$ for all $w \in V(G) - \{v\}$, as desired. ■

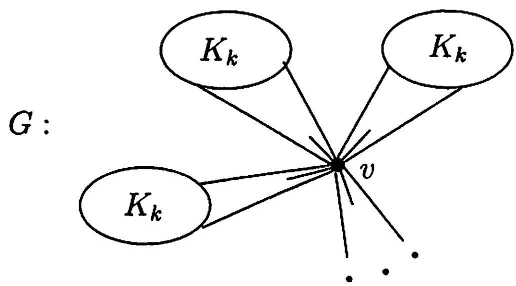

Figure 4: The graph G in Proposition 2.5

The following is an immediate consequence of Proposition 2.5.

Corollary 2.6 *For every positive integer n, there is a connected graph G of order n with $\gamma_D(G) = 1$.*

Whether the converse of Proposition 2.5 is true is not known, however.

We now turn to a class of graphs having detour domination number 2. It is known (see [2], for example) that for every integer $n \geq 2$, the hypercube Q_n contains a hamiltonian u-v path if and only if $d(u, v)$ is odd.

Proposition 2.7 *For $n \geq 2$, $\gamma_D(Q_n) = 2$.*

Proof. Let $x, y \in V(Q_n)$. We know that $D(x, y) = 2^n - 1$ if x and y belong to different paritite sets of Q_n. We show that $D(x, y) = 2^n - 2$ if x and y belong to the same paritite set of Q_n. If $n = 2$, then $Q_2 = C_4$ and the result is true; so we may assume that $n \geq 3$. Let Q and Q^* be the two copies of Q_{n-1} in $Q_n = Q_{n-1} \times K_2$. Let $x, y \in V(Q_n)$ such that x and y belong to the same paritite set of Q_n. Since $D(x, y)$ is even, $D(x, y) \leq 2^n - 2$. Thus, it suffices to show that there is an x-y path of length $2^n - 2$ in Q_n. We consider two cases.

Case 1. *x and y both belong to Q or both belong to Q^*, say the former.* Let $x', y' \in V(Q^*)$ that correspond to x and y, respectively. Let $z \in V(Q)$ such that z lies in the partite set different from that of x. Let $z' \in V(Q^*)$

correspond to z. Since x and y are in the same partite set of Q, it follows that z and y are in different partite sets of Q. Let $P : z = u_0, u_1, \ldots, u_i = x, u_{i+1} \ldots u_{2^{n-1}-1} = y$ be a z-y hamiltonian path in Q and $P^* : z' = u_0^*, u_1^*, \ldots, u_i^* = x', u_{i+1}^*, \ldots, u_{2^{n-1}-1}^* = y'$ be the hamiltonian path in Q^* corresponding to P. Since z and x belong to different partite sets of Q, it follows that $D_Q(z, x)$ is odd. Similarly, $D_{Q^*}(z', x')$ is odd. Thus $D_Q(x, y)$ and $D_{Q^*}(x', y')$ are even. Then the path $x = u_i, u_{i-1}, \ldots, u_0 = z, z' = u_0^*, u_1^*, \ldots, u_i^* = x', u_{i+1}^*, u_{i+1}, u_{i+2}, u_{i+2}^*, \ldots, u_{2^{n-1}-2}^*, u_{2^{n-1}-2}, u_{2^{n-1}-1} = y$ is an x-y path of length $2^n - 2$ in Q_n.

Case 2. *One of x and y belongs to Q and the other belongs to Q^*, say x belongs to Q and y belongs to Q^*.* Let $x' \in V(Q^*)$ and $y' \in V(Q)$ that correspond to x and y, respectively. Since x and y are in the same partite set of Q_n, it follows that x and y' are in different partite sets of Q and there is an x-y' hamiltonian path P in Q. Let $P : x = u_0, u_1, \ldots, u_{2^{n-1}-1} = y'$ and let $P^* : x' = u_0^*, u_1^*, \ldots, u_{2^{n-1}-1}^* = y$ be the path corresponding to P in Q^*. Let $z = u_{2^{n-1}-2}$ and $z' = u_{2^{n-1}-2}^*$. Then x and z belong to the same partite set of Q and so $D_Q(x, z)$ is even. Similarly, $D_{Q^*}(x', z')$ is even. Then the path $x = u_0, u_0^*, u_1^*, u_1, u_2, u_2^*, u_3^*, \ldots, u_{2^{n-1}-2} = z, u_{2^{n-1}-2}^* = z', y$ is an x-y path of length $2^n - 2$ in Q_n.

By Observation 2.4, $\gamma_D(Q_n) \geq 2$. Let V_1 and V_2 be the two partite sets of Q_n and let $v_i \in V_i$ for $i = 1, 2$. Then $N_D[v_i] = V_i$ for $i = 1, 2$. Hence $\{v_1, v_2\}$ is a γ_D-set of Q_n, and so $\gamma_D(Q_n) = 2$ for $n \geq 2$. ■

We now compute the detour domination number of all complete bipartite graphs.

Proposition 2.8 *If $G = K_{r,s}$, where $2 \leq r \leq s$, then*

$$\gamma_D(G) = \begin{cases} 2 & \text{if } r = s \\ s & \text{if } r < s. \end{cases}$$

Proof. Let the partite sets of G be V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$. We consider two cases.

Case 1. $r = s$. For distinct vertices u and v of $K_{r,r}$,

$$D(u, v) = \begin{cases} 2r - 1 & \text{if } uv \in E(K_{r,r}) \\ 2r - 2 & \text{if } uv \notin E(K_{r,r}). \end{cases}$$

By Observation 2.4, $\gamma_D(K_{r,r}) \geq 2$. It remains to show that $\gamma_D(K_{r,r}) \leq 2$. Let $S = \{v_1, v_2\}$, where $v_i \in V_i$, $1 \leq i \leq 2$. Then v_1 detour dominates every vertex in V_1 and v_2 detour dominates every vertex in V_2. Thus S is a detour dominating set and so $\gamma_D(K_{r,r}) = 2$.

Case 2. $r < s$. For distinct vertices u and v of $K_{r,s}$,

$$D(u,v) = \begin{cases} 2r-2 = n-s+r-2 & \text{if } u,v \in V_1 \\ 2r-1 = n-s+r-1 & \text{if } uv \in E(K_{r,s}) \\ 2r = n-s+r & \text{if } u,v \in V_2. \end{cases}$$

This implies that every vertex in V_1 detour dominates V_1; while each vertex in V_2 detour dominates itself and V_1. Thus V_2 is contained in every detour dominating set of $K_{r,s}$. Since V_2 is a detour dominating set for $K_{r,s}$, it follows that $\gamma_D(K_{r,s}) = |V_2| = s$. ∎

All regular graphs that we have seen so far have detour domination number 1 or 2 regardless of their diameter. This is not always true, however.

Proposition 2.9 *For* $n \geq 3$,

$$\gamma_D(C_n) = \begin{cases} \lceil n/3 \rceil = \gamma(C_n) & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

3 Upper Bound

Our next result establishes an upper bound on the detour domination number of connected graphs. First, we make an observation.

Observation 3.1 *Let G be a connected graph and v a vertex of G. Then $V(G) - N_D(v)$ is a detour dominating set of G.*

Theorem 3.2 *If G is a connected graph of order $n \geq 3$, then $\gamma_D(G) \leq n-2$.*

Proof. If $n = 3$, then $G = K_3$ or $G = P_3$ and $\gamma_D(G) = 1 = n-2$. Hence we assume that $n \geq 4$.

Since G is connected, $|N_D[v]| \geq 2$ for all $v \in V(G)$. Thus, $\gamma_D(G) \leq n-1$. Suppose that $\gamma_D(G) = n-1$. Since $V(G) - N_D(v)$ is a detour dominating set of G by Observation 3.1, it follows that $|N_D(v)| = 1$ for every vertex v of G.

Claim 1: There exists a vertex v^* in G such that every vertex in $V(G) - \{v^*\}$ detour dominates v^*.

Proof of Claim 1: Let S be a γ_D-set for G, and so $|S| = n-1$. Let $V(G) - S = \{u\}$ and let $x \in S$ such that $N_D(x) = \{u\}$. Let $z \in S - \{x\}$ and $N_D(z) = \{w\}$. If $w \notin \{u, x\}$, then $V(G) - \{u, w\}$ is a detour dominating set of G, a contradiction. Hence, $N_D(z) = \{x\}$ or $N_D(z) = \{u\}$. Suppose that there exist $z_1, z_2 \in S - \{x\}$ such that $N_D(z_1) = \{x\}$ and $N_D(z_2) = \{u\}$. In this case, $V(G) - \{u, x\}$ is a detour dominating set of G, again a contradiction. Hence, either $N_D(z) = \{x\}$ for all $z \in S - \{x\}$ or $N_D(z) = \{u\}$ for all $z \in S$. We consider these two cases.

Case 1. $N_D(z) = \{u\}$ *for all* $z \in S$. In this case, $v^* = u$ has the desired property.

Case 2. $N_D(z) = \{x\}$ *for all* $z \in S - \{x\}$. Let $N_D(u) = \{w\}$ where $w \in S$. If $w \neq x$, then $V(G) - \{x, w\}$ is a detour dominating set of G, a contradiction. Hence, we may assume that $N_D(u) = \{x\}$. This implies that $N_D(z) = \{x\}$ for all $z \in (S - \{x\}) \cup \{u\} = V(G) - \{x\}$. In this case, $v^* = x$ has the desired property.

Therefore, as claimed, there exists a vertex v^* in G such that every vertex in $V(G) - \{v^*\}$ detour dominates v^*.

Claim 2: The vertex v^* of Claim 1 lies on a cycle of G.

Proof of Claim 2: Suppose that this is not the case. Then every edge incident with v^* is a bridge of G. If v^* is incident with two bridges, say rv^* and sv^*, then $D(v^*, r) = D(v^*, s) = 1$, implying that $|N_D(v^*)| \geq 2$, a contradiction. So $\deg v^* = 1$. Let x^* be the vertex incident with v^* in G and so v^*x^* is an bridge of G. Let y ($\neq v^*$) be a vertex adjacent to x^* and let $D(y, v^*) = k$. However, then $D(y, x^*) = k - 1$, contradicting the fact that y detour dominates v^*. Therefore, as claimed, the vertex v^* lies on a cycle of G.

Let $C : v^* = u_0, u_1, u_2, \ldots, u_j, u_0$ be a longest cycle of G (of length $j+1$, say) containing v^*. Since there is a u_0-u_1 path of length j along C and $D(u_1, u_0) < D(u_1, u_2)$, it follows that $D(u_1, u_2) \geq j + 1$. A longest u_1-u_2 path must include u_0; otherwise, if we proceed from u_0 to u_1 along the edge u_0u_1 and then along a longest u_1-u_2 path, we obtain a u_0-u_2 path that is longer than $D(u_1, u_2)$, contradicting the fact that $D(u_2, u_0) < D(u_1, u_2)$. But now we have a cycle (formed from a longest u_1-u_2 path by adding the edge u_1u_2) of length at least $j + 2$ containing u_0, contradicting our choice of C. ■

It is straightforward to show that for all n and k where $1 \leq k \leq n - 2$, there exists a connected graph G of order n with $\gamma_D(G) = k$.

Proposition 3.3 *For every pair k, n of integers with $1 \leq k \leq n - 2$, there exists a connected graph G of order n such that $\gamma_D(G) = k$.*

Proof. Assume that $1 \leq k \leq n/2$. Then the caterpillar G obtained from a path P_k by adding at least one end-vertex adjacent to each vertex of the

P_k for a total of $n-k$ end-vertices has order n and $\gamma_D(G) = k$.

Assume that $n/2 < k \le n-2$. Since $k > n/2$, it follows that $k > n-k$. By Proposition 2.8, $G = K_{n-k,k}$ has the desired property. ∎

4 Graphs With Prescribed Domination Number and Detour Domination Number

We now show that there is no relationship between the detour domination number and the domination number of a connected graph; that is, knowing the value of one of these parameters for a connected graph supplies no information about the value of the other parameter.

Theorem 4.1 *For each pair a, b of positive integers, there is a connected graph G such that $\gamma(G) = a$ and $\gamma_D(G) = b$.*

Proof. Since $\gamma(P_{3a}) = \gamma_D(P_{3a}) = a$, the result is true if $a = b$. Thus we may assume that $a \ne b$. There are two cases, according to whether $a < b$ or $b < a$.

Case 1. $a < b$. We consider three subcases.

Subcase 1.1. $a = 1$. Let $n \ge 2b$ be an integer and let

$$G = [(b-1)K_2 \cup (n-2b+1)K_1] + K_1.$$

Since G contains a vertex adjacent to all other vertices, $\gamma(G) = 1$. We show that $\gamma_D(G) = b$. Let $u_1, v_1, u_2, v_2, \ldots, u_{b-1}, v_{b-1}$ be the vertices of degree 2 in G, where $u_iv_i \in E(G)$ for $1 \le i \le b-1$, let $w_1, w_2, \ldots, w_{n-2b+1}$ be the end-vertices of G, and let x be the cut vertex of G. Note that $D^-(u_i) = D^-(v_i) = D(u_i, v_i) = 2$ for $1 \le i \le b-1$ and $D^-(w_j) = D^-(x) = D(w_j, x) = 1$ for $1 \le j \le n-2b+1$. This implies that each vertex u_i $(1 \le i \le b-1)$ is only detour dominated by itself or by v_i. Similarly, each vertex v_i $(1 \le i \le b-1)$ is only detour dominated by itself or by u_i. Thus every $\gamma_D(G)$-set must contain at least one vertex from each set $\{u_i, v_i\}$ for all $1 \le i \le b-1$. However, no vertex w_j, $1 \le j \le n-2b+1$, is detour dominated by u_i or v_i for any i with $1 \le i \le b-1$. Thus $\gamma_D(G) \ge b$. On the other hand, since $S_0 = \{u_1, u_2, \ldots, u_{b-1}, x\}$ is a detour dominating set of G, it follows that $\gamma_D(G) \le |S_0| = b$. Therefore, $\gamma_D(G) = b$.

Subcase 1.2. $a = 2$. Let $G = K_{2,b}$, where $b \ge 3$. Then $\gamma(G) = 2$ and $\gamma_D(G) = b$ by Proposition 2.8.

Subcase 1.3. $a \ge 3$. Let H be the graph obtained from $C_a : u_1, u_2, \ldots, u_a, u_1$ by adding the $a-1$ new vertices $v_1, v_2, \ldots, v_{a-1}$ and the $a-1$ pendant edges u_iv_i for $1 \leq i \leq a-1$, and let $F = K_{1,b-a+1}$ be the star with $V(F) = \{w, w_1, w_2, \ldots, w_{b-a+1}\}$, where w is the central vertex of F. The graph G is obtained from H and F by joining each vertex w_i $(1 \leq i \leq b-a+1)$ to u_a in F. Let $U = \{u_1, u_2, \ldots, u_{a-1}\}$, $V = \{v_1, v_2, \ldots, v_{a-1}\}$, and $W = \{w_1, w_2, \ldots, w_{b-a+1}\}$.

First we show that $\gamma(G) = a$. Since $S_0 = U \cup \{w\}$ is a dominating set, $\gamma(G) \leq |S_0| = a$. On the other hand, every dominating set of G must contains at least one vertex from each set $\{u_i, v_i\}$ for $1 \leq i \leq a-1$. Since no vertex in W is dominated by any vertex in $U \cup V$, it follows that every dominating set of G must contains at least one vertex from $\{u_a, w\} \cup W$, implying that $\gamma(G) \geq a$. Therefore, $\gamma(G) = a$.

Next, we show that $\gamma_D(G) = b$. Since $S_1 = U \cup W$ is a detour dominating set of G, it follows that $\gamma_D(G) \leq |S_1| = (a-1) + (b-a+1) = b$. Every detour dominating set of G must contains at least one vertex from each set $\{u_i, v_i\}$ for $1 \leq i \leq a-1$ and no vertex in $W \cup \{w\}$ is detour dominated by any vertex in $U \cup V$. Next we show that every vertex in W belongs to each $\gamma_D(G)$-set. Without loss of generality, we only consider w_1. Since $D(w_1, w_i) = 4$ and $D(w_i, u_a) = D(w_i, w) = 3$ for $2 \leq i \leq b-a+1$ it follows that w_1 is not detour dominated by w_i. Since $D(w_1, u_a) = 3$ and $D(u_a, w) = 2$, it follows that w_1 is not detour dominated by u_a or by w. Furthermore, w_1 is not detour dominated by any vertex in $U \cup V$. Thus $D_D[w_1] = \{w_1\}$ and so w_1 belongs to every detour dominating set of G. Thus $\gamma_D(G) \geq a-1+|W| = (a-1)+(b-a+1) = b$. Therefore, $\gamma_D(G) = b$.

Case 2. $b < a$. There are three subcases.

Subcase 2.1. $b = 1$. We first construct a graph F_a for each integer $a \geq 2$. Let C' and C'' be two copies of the cycle C_{3a} of order $3a$, where $C' : x_1, x_2, \ldots, x_{3a}, x_1$ and $C'' : y_1, y_2, \ldots, y_{3a}, y_1$. Then the graph F_a is obtained from C' and C'' by adding the edges x_iy_i, x_iy_{i+1}, and $x_{i+1}y_i$ for $1 \leq i \leq 3a$ (addition performed modulo a). Since F_a is hamiltonian-connected, $\gamma_D(F_a) = 1$. It is straightforward to verify that $\gamma(F_a) = a$.

Subcase 2.2. $b = 2$. Then $a \geq 3$. Let G be the graph obtained from F_a described in Subcase 2.1 by adding a pendant edge xx_1. Then $\gamma(G) = a$. Since $D^-(x_1) = D^-(x) = D(x_1, x) = 1$, it follows that x is only detour dominated by itself or by x_1. Thus every $\gamma_D(G)$-set must contain at least one vertex from $\{x, x_1\}$. Moreover, no vertex in $V(F_a) - \{x_1\}$ can be detour dominated by x or x_1. Thus neither $\{x\}$ nor $\{x_1\}$ is a detour dominating set, implying that $\gamma_D(G) \geq 2$. On the other hand, $\{x_1, y_1\}$ is a detour dominating set of G and so $\gamma_D(G) = 2$.

Subcase 2.3. $b \geq 3$. Let H be the graph obtained from the cycle C_b : $u_1, u_2, \ldots, u_b, u_1$ by adding $b-1$ new vertices $v_1, v_2, \ldots, v_{b-1}$ and the $b-1$

pendant edges u_iv_i for $1 \le i \le b-1$. Let G be the graph obtained from H and the graph F_{a-b+1} in Subcase 2.1 by identifying the vertex u_b in H and the vertex x_1 in F_{a-b+1}. Then $\gamma(G) = a$. An argument similar to that given above shows that the set $\{u_1, u_2, \ldots, u_{b-1}, y_1\}$ is a minimum detour dominating set and so $\gamma_D(G) = b$. ∎

If G is a connected graph of order $n \ge 2$, then it is well known that $\gamma(G) \le n/2$. While Theorem 4.1 states that there exists a connected graph G with $\gamma(G) = 2$ and $\gamma_D(G) = 1$, there is no such graph of order 4 however; that is, we cannot simultaneously stipulate positive integers a, b, and n with $a \le n/2$ and $b \le n-2$ and be assured that there is a connected graph G of order n with $\gamma(G) = a$ and $\gamma_D(G) = b$. If $a = 1$, however, then both b and n can be specified.

Proposition 4.2 *For each pair b, n of integers with $1 \le b \le n-2$, there is a connected graph G of order $n \ge 3$ such that $\gamma(G) = 1$ and $\gamma_D(G) = b$.*

Proof. For $b = 1$, the graph $G = K_{1,n-1}$ has the desired properties. For $b \ge 2$, we consider two cases.

Case 1. $1 \le b \le n/2$. Let $G = [(b-1)K_2 \cup (n-2b+1)K_1] + K_1$. Then G is a connected graph of order n and $\gamma(G) = 1$. By the argument used in Subcase 1.1 in the proof of Theorem 4.1, we have $\gamma_D(G) = b$.

Case 2. $n/2 < b \le n-2$. Let $G = K_{n-b} + bK_1$, where $V(K_{n-b}) = \{u_1, u_2, \ldots, u_{n-b}\}$ and $V(bK_1) = \{v_1, v_2, \ldots, v_b\}$. Then G is a connected graph of order n and $\gamma(G) = 1$. Next we show that $\gamma_D(G) = b$. Since $D^-(v_i) = 2(n-b) - 1 = D(v_i, u_j)$ for $1 \le i \le b$ and $1 \le j \le n-b$, it follows that $N_D(v_i) = V(K_{n-b})$ for $1 \le i \le b$. Also, since $D^-(u_j) = 2(n-b)-2 = D(u_j, u_{j'})$, where $1 \le j \ne j' \le n-b$, it follows that $N_D(u_j) = V(K_{n-b}) - \{u_j\}$ for $1 \le j \le n-b$. This implies that $D_D[v_i] = \{v_i\}$ for each vertex v_i ($1 \le i \le b$). Therefore, $V(bK_1)$ belongs to every detour dominating set of G and so $\gamma_D(G) \ge |V(bK_1)| = b$. On the other hand, $V(bK_1)$ is a detour dominating set, and so $\gamma_D(G) = b$. ∎

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