Orthogonal Labeling of Constant Weight Gray Codes by Partitions With Blocks Of Size At Most Two

Inessa Levi and Steve Seif
Department of Mathematics
University of Louisville
Louisville, KY 40292

Abstract

We extend results concerning orthogonal edge labeling of constant weight Gray codes. For positive integers n and r with $n > r$, let $G_{n,r}$ be the graph whose vertices are the r-sets of $\{1, \ldots, n\}$, with r-sets adjacent if they intersect in $r-1$ elements. The graph $G_{n,r}$ is Hamiltonian; Hamiltonian cycles of $G_{n,r}$ are early examples of error-correcting codes, where they came to be known as constant weight Gray codes.

An r-set A and a partition π of weight r said to be orthogonal if every block of π meets A in exactly one element. Given a class $\mathcal{P}$ of weight r partitions of X_n, one would like to know if there exists a $G_{n,r}$ Hamiltonian cycle $A_1 A_2 \ldots A_{\binom{n}{r}}$ whose edges admit a labeling $A_1 \pi_1 A_2 \ldots A_{\binom{n}{r}} \pi_{\binom{n}{r}}$ by distinct partitions from $\mathcal{P}$, such that a partition label of an edge is orthogonal to the vertices that comprise the edge. The answer provides non-trivial information about Hamiltonian cycles in $G_{n,r}$ and has application to questions pertaining to the efficient generation of finite semigroups.

Let τ be a partition of n as a sum of r positive integers. We let τ also refer to the set of all partitions of X_n whose block sizes comprise the partition τ. J. Lehel and the first author have conjectured that for $n \geq 6$ and partition type τ of $\{1, \ldots, n\}$ of weight r partitions, there exists a τ-labeled Hamiltonian cycle in $G_{n,r}$.

In the present paper for $n = s + r$, we prove that there exist Hamiltonian cycles in $G_{n,r}$ which admit orthogonal labelings by the partition types which have s blocks of size two and $r - s$ blocks of size one, thereby extending a result of J. Lehel and the first author and completing the work on the conjecture for all partition types with blocks of size at most two.

Key words: combinatorial Gray code, constant weight Gray code, Hamiltonian cycle, partition, Middle Levels Conjecture, transversal, semigroup, semigroup of transformations, rank of a semigroup, idempotent rank of a semigroup.

MOS Classification number: 05A18, 05A17, 05C20, 20M20

Orthogonal Labeling of Constant Weight Gray Codes by Partitions With Blocks Of Size At Most Two

1 Introduction

We solve an algebra motivated combinatorics problem, raised in [5], involving notions related to constant weight Gray codes. We prove results which present certain non-trivial properties of constant weight Gray codes; these have application to the growing literature on finite idempotent-generated semigroups.

Let n and r be positive integers with $n > r$, and let $G_{n,r}$ be the graph whose vertices are the r-sets of $X_n = \{1, \ldots, n\}$ such that two r-sets are adjacent if their intersection consists of $r-1$ elements. It is well-known that $G_{n,r}$ is Hamiltonian ([1], [2], [10], [12]); that is, there exists a cycle in $G_{n,r}$ passing once through each of its vertices. The Hamiltonian cycles of $G_{n,r}$ are early examples of error-correcting codes, where they are known as *constant weight Gray codes.* ([11]).

A partition π of X_n is said to be of *weight* r if π partitions X_n into r classes; the set of all weight r partitions will be denoted by $Part(n,r)$. Let $\mathcal{P}$ be a subset of $Part(n,r)$.

Definition 1.1. *An* orthogonally $\mathcal{P}$-labeled *list* $A_1, \alpha_1, \ldots, A_{\binom{n}{r}}, \alpha_{\binom{n}{r}}$ *is an alternating sequence of the* $\binom{n}{r}$ *distinct* r*-sets and distinct partitions in* $\mathcal{P}$*, such that for all* i *with* $1 \leq i \leq \binom{n}{r}$*, the partition* π_i *is orthogonal to* A_i *and* A_{i+1}*, and* $\pi_{\binom{n}{r}}$ *is orthogonal to* $A_{\binom{n}{r}}$ *and* A_1.

An orthogonally $\mathcal{P}$*-labeled list* $A_1, \alpha_1, \ldots, A_{\binom{n}{r}}, \alpha_{\binom{n}{r}}$ *such that* $A_1, \ldots, A_{\binom{n}{r}}$ *is a Hamiltonian cycle in* $G_{n,r}$ *is said to be an* orthogonally $\mathcal{P}$-labeled Hamiltonian cycle.

The sequence $A_1, \ldots, A_{\binom{n}{r}}$ (which consists of all the r-sets of X_n) is called the *set-sequence*; the sequence $\alpha_1, \ldots, \alpha_{\binom{n}{r}}$ is called the *partition-sequence.* In the sequel we omit commas between the elements of labeled lists, set-sequences and partition-sequences. We refer to α_i as the *label* for the edge A_iA_{i+1}. An orthogonally $\mathcal{P}$-labeled list is denoted by the ordered pair $(\mathcal{C}, \Pi)$, where $\mathcal{C}$ is the set-sequence and Π is the partition-sequence. We describe some earlier results concerning orthogonally labeled lists and orthogonally labeled Hamiltonian cycles. The following result was proved in [4].

Theorem 1.2. *[4] For positive integers* n *and* r *with* $n > r$*, there exist orthogonally* $Part(n,r)$*-labeled lists.*

The authors of [4], John M. Howie and Robert B. McFadden, use Theorem 1.2 to prove a result concerning the size of minimal generating sets of certain finite semigroups of transformations. The connection between orthogonal labeling and semigroups is described in Section 4. The next result of the authors and McFadden, cited in Theorem 1.3 below, extends Theorem 1.2. Let $Part(n)$ be the lattice of partitions of X_n and let $Cov(Part(n))$ be the graph with vertices $Part(n)$; two such vertices are adjacent if one covers the other in $Part(n)$. Note that in $Cov(Part(n))$, distinct elements of $Part(n,r)$ have distance at least two (since $Part(n,r)$ is an anti-chain in $Part(n)$). Let $\mathbf{Part(n,r)}$ be the graph with vertices $Part(n,r)$ such that two vertices are adjacent if they are distance-two in $Cov(Part(n))$.

Theorem 1.3. *[6] Let n and r be positive integers with $n > r$.*

1. *Every Hamiltonian cycle in $G_{n,r}$ admits an orthogonal $Part(n,r)$ labeling.*

2. *There exists a Hamiltonian cycle in $G_{n,r}$ which an orthogonal $Part(n,r)$ labeling $A_1\alpha_1 \dots A_{\binom{n}{r}}\alpha_{\binom{n}{r}}$ with the additional property that the list of partitions $\alpha_1 \dots \alpha_{\binom{n}{r}}$ is a cycle in* $\mathbf{Part(n,r)}$.

A *partition* τ of a positive integer n into r *parts* is a decomposition of n as a sum of r positive integers, denoted by $\tau = a_1^{m_1} \dots a_k^{m_k}$, where $\Sigma_{j=1}^k a_i m_i = n$ and $\Sigma_{j=1}^k m_j = r$. A partition $\pi \in Part(n,r)$ is said to be of *type* τ, if the sizes of its partition classes form a partition τ of the integer n. We also use τ to refer to the subset of $Part(n,r)$ of all the partitions of that type. For a partition type τ let $\mathcal{N}(\tau)$ be the number of distinct τ partitions. In [5], the authors make the following conjecture.

Conjecture 1.4. *For any partition type τ of weight r on X_n satisfying inequality $N(\tau) \geq \binom{n}{r}$, there exist τ-labeled lists of all r-sets of X_n.*

In [5] J. Lehel and the first author prove the next theorem. Note that the second statement of the theorem asserts existence of an orthogonally τ-labeled list, containing all the vertices of $G_{n,r}$, it is not stipulated that the list is a path in $G_{n,r}$.

Theorem 1.5. *[5] Let n and r be positive integers such that $2r \geq n > r$.*

1. *For positive integers r and s with $r > s$ and $\tau = 2^s 1^{r-s}$ there exists an orthogonally τ-labeled Hamiltonian cycle.*

2. *For $r \geq 4$ and $\tau = 2^r$ there exists an orthogonally τ-labeled list of all r-sets of X_n.*

For $r = 2$ and $r = 3$, observe that $\mathcal{N}(2^r)$ is smaller than $\binom{2r}{r}$, the number of r-subsets; hence, there are no orthogonal 2^r-labeled Hamiltonian cycles in $G_{2r,r}$ for $r = 2, 3$.

Theorem 1.5 leaves open the problem of determining the existence of orthogonally 2^r-labeled Hamiltonian cycles, and leads to the following conjecture generalizing Conjecture 1.4.

Conjecture 1.6. Partition Type Conjecture *For any partition type τ of weight r on X_n satisfying inequality $N(\tau) \geq \binom{n}{r}$, there exist τ-labeled Hamiltonian cycle in $G_{n,r}$.*

It is worth noting that for $\tau = r\,1^{r-1}$, the Partition Type Conjecture 1.6 is equivalent to the celebrated *Middle Levels Conjecture*. Indeed the mapping from the set of all $(r-1)$-sets to the partitions of type is a one-to-one correspondence between $(r-1)$-sets on X_n and partitions of type $r\,1^{r-1}$. See [11] for background on the Middle Levels Conjecture.

We confirm the Partition Type Conjecture 1.6 for $\tau = 2^r$, with $r \geq 4$ by proving the following theorem.

Theorem 1.7. *For $r \geq 4$ and $r \geq s \geq 1$, there there exists an orthogonally $2^s\,1^{r-s}$-labeled Hamiltonian cycle in $G_{r+s,r}$.*

The proof of Theorem 1.7 is intricate, purely combinatorial, and independent of the constructions used in [5].

For $r \geq 4$, not *every* Hamiltonian cycle in $G_{2r,r}$ admits an orthogonal 2^r-labeling; a counter-example for the $r = 4$ case is provided in Lemma 3.3. In a sequel to this paper ([8]) we show that for $s \geq 9$, every Hamiltonian cycle in $G_{r+s,r}$ can be orthogonally $2^s\,1^{r-s}$-labeled.

The work here is a part of a now completed program aimed at showing that the Partition Type Conjecture 1.6 is valid for all partition types, except for the types of the form $m\,1^{r-1}$, where the Middle Levels Conjecture is an obstruction. The proof is completed in [8] and [9].

We prove Theorem 1.7 in the next section. In the last section we explain the connection between the combinatorial results concerning orthogonal labeling and semigroup theory, and we pose problems in combinatorics and semigroup theory.

2 Proof of Theorem 1.7

For a cycle $\mathcal{B} = B_1 \ldots B_m$, for $i = 1, \ldots, m$, when we refer to an edge B_iB_{i+1}, we assume that B_mB_1 is one of the possibilities; in general, subscripts associated with a cycle of length m are interpreted "mod m".

Let $C = A_1 \ldots A_{\binom{n}{r}}$ be a Hamiltonian cycle in $G_{n,r}$. The $(r-1)$-set $A_i \cap A_{i+1}$ is termed the *core* of the edge A_iA_{i+1}. The two-element symmetric difference of A_i and A_{i+1} is denoted by $\Delta(A_i, A_{i+1})$. For a positive integer s and a non-negative integer k, with $s + k \leq n$, subsets of the form $\{s, s+1\ldots, s+k\}$ are denoted by $[s, s+k]$. If $k > 0$, then $[s+k, s]$ stands for the empty set. To prove Theorem 1.7, we will prove the following stronger result.

Proposition 2.1. *1. For $r \geq 4$ and $r \geq s \geq 1$, there exists an orthogonal $2^s\,1^{r-s}$-labeled Hamitonian cycle $A_1\pi_1A_2\pi_2\ldots A_{\binom{r+s}{r}}\pi_{\binom{r+s}{r}}$ satisfying the following conditions:*

(a) $A_1 = \{1,2,\ldots,r\}$ and $A_{\binom{r+s}{r}} = \{1,\ldots,r-1,r+s\}$;

(b) $\{1,\ldots,r-1\}$ is the core of exactly one edge of C, the edge $A_{\binom{r+s}{r}}A_1$;

(c) if $b \in X_{r+s}$, $b \leq s+2$, and $\{b\}$ is a singleton class of a partition π_i labeling the edge A_iA_{i+1} ($i = 1,\ldots,\binom{r+s}{r}$), then the core $A_i \cap A_{i+1}$ contains at least $s-1$ elements in the interval $[1, b-1]$.

2. For all $r \geq 4$, there exists an orthogonally 2^r-labeled Hamiltonian cycle of $G_{2r,r}$ satisfying the conditions (1a) and (1b) above.

Suppose π is a partition of type $2^s\,1^{r-s}$ orthogonal to an edge A_iA_{i+1} in $G_{r+s,r}$. Observe that if $\{b\}$ is a singleton class of π, then b is in the core of A_iA_{i+1}. Thus, statement (1c) of Proposition 2.1 is equivalent to the following: the core $A_i \cap A_{i+1}$ contains s elements in the interval $[1, b]$.

Suppose that $A_1 \ldots A_{\binom{r+s}{r}}$ is a Hamiltonian cycle in $G_{r+s,r}$. A partition π of type $2^s\,1^{r-s}$ is said to be *available for the edge* A_iA_{i+1} if π is orthogonal to both A_i and A_{i+1}, and π and A_iA_{i+1} satisfy Proposition 2.1(1c). In the subsections to follow, we provide the proof of Proposition 2.1, and thereby the proof of Theorem 1.7.

2.1 Proof of Proposition 2.1

With each partition type $2^s\,1^{r-s}$, we associate an ordered pair (r, s). The set of all such ordered pairs, restricted to $r \geq 4$ and $r \geq s \geq 1$, and ordered lexicographically, is a well-ordered set. We prove Proposition 2.1 by induction. The base steps correspond to the pairs $(4,1), (4,2), (4,3), (4,4)$ (which together comprise the $r = 4$ cases), along with $(r, 1)$, for all $r \geq 5$. The constructions for the base steps $(4,2), (4,3), (4,4)$ are ad-hoc and for that reason are presented explicitly in Lemma 2.8, at the end of this section.

Lemma 2.2. *For $r \geq 4$, Proposition 2.1 is valid for $2\,1^{r-1}$.*

Proof. Consider the following list of r-subsets of X_{r+1}: $A_1 = \{1, 2, \ldots, r\}$, $\{2, 3, \ldots, r+1\}, \ldots, \{1, \ldots, i-1, i+1, \ldots, r+1\}, \ldots, \{1, 2, \ldots, r-1, r+1\} = A_{r+1}$. (For $i = 1, \ldots, r+1$, we have $A_i = X_{r+1} - \{i-1\}$, where the computation of $i-1$ is "mod $r+1$".) For $i = 1, \ldots, r+1$, let π_i be the partition of type $2\,1^{r-1}$ with unique doubleton class $\Delta(A_i, A_{i+1})$. It is not difficult to check that $A_1\pi_1 A_2\pi_2 \ldots A_{r+1}\pi_{r+1}$ satisfies the conditions of Proposition 2.1. □

We assume that the lists In Lemma 2.8 are indeed orthogonally type labeled Hamiltonian cycles satisfying Proposition 2.1; thus, the base step of the proof of Proposition 1.7 is complete.

Assume there exist positive integers r and s, with $r \geq 4$ and $r \geq s \geq 1$, such that Proposition 2.1 holds for all (r', s') with $(r, s) \geq (r', s') > (4, 1)$. We show that Proposition 2.1 holds for (r, s) as well. By Lemma 2.2 and Lemma 2.8, we can assume $(r, s) \geq (5, 2)$. The proof involves two quite separate cases, the first involving types of the form (r, s), where $r > s$, and the second (more difficult) case where $r = s$, the 2^r case.

2.2 Proof of Proposition 2.1: $r > s$

Let $\tau = 2^s\,1^{r-s}$, with $r > s > 1$ and $r \geq 5$. We assume inductively that for $\tau_1 = 2^{s-1}\,1^{r-(s-1)}$ and $\tau_2 = 2^s\;1^{(r-1)-s}$ there exist orthogonally τ_1 and τ_2-labeled Hamiltonian cycles $(C_1, \Pi_1), (C_2, \Pi_2)$ respectively, satisfying Proposition 2.1. Let

$$(C_1, \Pi_1) = A_1\alpha_1 A_2\alpha_2 \ldots A_{\binom{r+s-1}{r}}\alpha_{\binom{r+s-1}{r}},$$

and

$$(C_2, \Pi_2) = B_1\beta_1 B_2\beta_2 \ldots B_{\binom{r+s-1}{r-1}}\beta_{\binom{r+s-1}{r-1}}.$$

For $j = 1, \ldots, \binom{r+s-1}{r-1})$, let $\tilde{B}_j = B_j \cup \{r+s\}$, and let $\tilde{C}_2 = \tilde{B}_1\tilde{B}_2 \ldots \tilde{B}_{\binom{r+s-1}{r-1}}$. We construct a Hamiltonian cycle C in $G_{r+s,r}$ by a concatenation letting C be

$$A_1 A_2 \ldots A_{\binom{r+s-1}{r}}\tilde{B}_{\binom{r+s-1}{r-1}} \ldots \tilde{B}_1.$$

Observe that C is indeed a Hamiltonian cycle. The first set of C is A_1 and the last set of C is $\tilde{B}_1$; these satisfy satisfy Proposition 2.1(1a). We show that C satisfies Proposition 2.1(1b). Observe that the edge $A_{\binom{r+s-1}{r}}A_1$ of C_1 is not an edge of C. Moreover, because C_1 satisfies Proposition 2.1(1b), we have $A_{\binom{r+s-1}{r}}A_1$ is the only edge of C_1 whose core is $\{1, \ldots, r-1\}$. Cores of edges of the form $\tilde{B}_j\tilde{B}_{j-1}$ each contain $r+s$. Moreover, the core of the edge $A_{\binom{r+s-1}{r}}\tilde{B}_{\binom{r+s-1}{r-1}}$ is $\{1, \ldots, r-2, r+s-1\}$. We have $r+s-1 > r-1$, since s is positive; hence, the core of the edge $A_{\binom{r+s-1}{r}}\tilde{B}_{\binom{r+s-1}{r-1}}$ is not $\{1, \ldots, r-1\}$.

It follows that $\{1,\ldots,r-1\}$ occurs as the core of exactly one edge of C, namely $\tilde{B}_1 A_1$.

The edges $\tilde{B}_1 A_1$ and $A_{\binom{r+s-1}{r}}\tilde{B}_{\binom{r+s-1}{r-1}}$, which we use to link C_1 and $\tilde{C}_2$ to form C, are called *connecting edges.*

We construct a partition-sequence Π in such a way that (C,Π) is an orthogonally τ-labeled Hamiltonian cycle satsifying Proposition 2.1. The partition-sequence Π is constructed in stages, beginning with a modification of the existing α_i partitions, for $i=1,\ldots,\binom{r+s-1}{r}-1$; then followed by a modification of the existing β_j partitions, for $j=1,\ldots,\binom{r+s-1}{r-1}-1$, and completed with the construction of two partitions δ and γ that label the two connecting edges.

For a partition θ of X_n with singleton classes $\{k_1\},\{k_2\},\ldots,\{k_t\}$, by the *smallest singleton class* $\{k_i\}$ of θ, we mean the smallest integer k_i of the set of integers $\{k_1,k_2,\ldots,k_t\}$. For $i=1,\ldots,\binom{r+s-1}{r}-1$, let $\tilde{\alpha}_i$ be a partition of X_{r+s} of type τ obtained from α_i by adjoining the element $r+s$ to the smallest singleton class of α_i. Let $\tilde{\beta}_j$ be the partition of X_{r+s} of type τ obtained from β_j by adjoining the new singleton class $\{r+s\}$ to the partition β_j. When completed Π will have the following form:

$$\tilde{\alpha}_1\tilde{\alpha}_2\ldots\tilde{\alpha}_{\binom{r+s-1}{r}-1}\delta\tilde{\beta}_{\binom{r+s-1}{r-1}-1}\ldots\tilde{\beta}_1\gamma.$$

Observe that $\tilde{\alpha}_i$ is orthogonal to A_iA_{i+1} and $\tilde{\beta}_j$ is orthogonal to $\tilde{B}_{j+1}\tilde{B}_j$. Moreover, because the partitions of Π_1 are distinct, as are the partitions of Π_2, we have that $\tilde{\alpha}_i$ partitions are distinct and $\tilde{\beta}_j$ partitions are distinct. The doubleton set $\{t,r+s\}$ is a class of each $\tilde{\alpha}_i$ partition; on the other hand, $\{r+s\}$ is a singleton class of each $\tilde{\beta}_j$ partition. Thus, the $\binom{r+s}{r}-2$ partitions defined above are distinct.

We show that the labels $\tilde{\alpha}_i$ and their corresponding edges satisfy Proposition 2.1(1c). Assume that $\{a\}$ is a singleton class of $\tilde{\alpha}_i$ and that $a\le s+2$. By the definition of $\tilde{\alpha}_i$, there exists a singleton class $\{a'\}$ of α_i such that $a'<a$ and $\{a',r+s\}$ is a class of $\tilde{\alpha}_i$. Because $a'\le s+1$, we can apply Proposition 2.1(1c) inductively to α_i and a': the core $C_i=A_i\cap A_{i+1}$ contains at least $(s-1)-1=s-2$ elements of the set $[1,a'-1]$. Since a' is in the core, it follows that the core $C_i=A_i\cap A_{i+1}$ contains at least $s-1$ elements from the interval $[1,a-1]$, as required.

To see that the labels $\tilde{\beta}_j$ and their corresponding edges satisfy Proposition 2.1(1c), assume that $\{b\}$ is a singleton class of $\tilde{\beta}_j$ and that $b\le s+2$. Of course $\{b\}$ must also be a singleton class of β_j; hence, by the inductive assumption, the core $B_j\cap B_{j+1}$ has at least $s-1$ elements in the interval $[1,b-1]$, as does then the core $\tilde{B}_j\cap\tilde{B}_{j+1}$, as required.

We complete the construction of the partition-sequence Π by labeling the two connecting edges. Let γ be a type $\tau=2^s\,1^{r-s}$ partition which is available for the connecting edge $\tilde{B}_1 A_1 = \{1, \dots, r\}\{1, \dots, r-1, r+s\}$. Such γ clearly exist. Because γ is available for $\tilde{B}_1 A_1$, it follows readily that an element $a \in X_{r+s}$ is potentially a singleton class of γ if and only if $a \in [s, r-1]$. In particular, γ is not equal to $\tilde{\beta}_j$, for $j = 1, \dots, \binom{r+s-1}{r-1} - 1$, since $\tilde{\beta}_j$ contains the singleton class $\{r+s\}$. We claim $\gamma \neq \tilde{\alpha}_j$, for any $j = 1, \dots, \binom{r+s-1}{r-1} - 1$. Observe that γ contains the doubleton class $\{r, r+s\}$. By the definition of $\tilde{\alpha}_j$, if $\{r, r+s\}$ is a doubleton class of $\tilde{\alpha}_j$, then r is the smallest singleton class of α_j, and by the definition of $\tilde{\alpha}_j$ again, we have that such $\tilde{\alpha}_j$ contains no singleton class smaller than r. On the other hand, all singletons classes of γ are contained in $[s, r-1]$. It follows that $\gamma \neq \tilde{\alpha}_j$, for any $j = 1, \dots, \binom{r+s-1}{r-1} - 1$, as claimed. We can label $\tilde{B}_1 A_1$ with any such γ.

Let δ be a $2^s\,1^{r-s}$ partition available for $A_{\binom{r+s-1}{r}} \tilde{B}_{\binom{r+s-1}{r-1}} = \{1, \dots, r-1, r+s-1\}\{1, \dots, r-2, r+s-1, r+s\}$. Because δ is available for $A_{\binom{r+s-1}{r}} \tilde{B}_{\binom{r+s-1}{r-1}}$, it follows that an element $a \in X_{r+s}$ is a potentially a singleton class of δ if and only if $a \in [s, r-2] \cup \{r+s-1\}$. Moreover, δ contains the doubleton class $\{r-1, r+s\}$, and so $\delta \neq \tilde{\beta}_j$, for each j, since b_j contains the singleton class $\{r+s\}$.

Assume $\delta = \tilde{\alpha}_i$. Because $\{r-1, r+s\}$ is a doubleton class of $\tilde{\alpha}_i$, by the definition of $\tilde{\alpha}_i$, the smallest singleton class of α_i must be $r-1$. Thus, all singleton classes of $\tilde{\alpha}_i$ are contained in $[r, r+s]$. In view of the restriction described above for singleton classes of δ, it follows that if $\delta = \tilde{\alpha}_i$, then $[s, r-2]$ is an empty set, so $r-2 < s$ therefore $r-1 \leq s$, and hence $r-s = 1$. In particular, if $r-s > 1$, then we can label $A_{\binom{r+s-1}{r}} \tilde{B}_{\binom{r+s-1}{r-1}}$ with any $2^s\,1^{r-s}$ partition available for $A_{\binom{r+s-1}{r}} \tilde{B}_{\binom{r+s-1}{r-1}}$, since it would be distinct from any $\tilde{\alpha}_i$'s.

So we assume that $r-s = 1$ and thus that $\tau = 2^{r-1}\,1$ and $\tau_1 = 2^{r-2}\,1^2$. Since the smallest singleton class of α_i is $r-1$, we can apply the induction hypothesis (to α_i and the edge $A_i A_{i+1}$) and conclude that the core $A_i \cap A_{i+1}$ contains at least $r-3$ elements in the interval $[1, r-2]$. The core $A_i \cap A_{i+1}$ must also contain $r-1$, and because $r+s-1 = 2r-2$ is a singleton class of $\delta = \alpha_i$, the core must also contain $2r-2$. Thus $A_i \cap A_{i+1} = ([1, r-1] - \{u\}) \cup \{2r-2\}$, where $u \in [1, r-2]$. That δ is available for $A_{\binom{r+s-1}{r}} \tilde{B}_{\binom{r+s-1}{r-1}}$ implies that u is contained in a doubleton class $\{u, v\}$ of δ, where $v \in [r, 2r-3]$. Because $\tilde{\alpha}_i = \delta$, we have $\{u, v\}$ must also be a doubleton class of α_i. An examination of the core of $A_i A_{i+1}$ leads to the conclusion that $\{u, v\}$ is $\Delta(A_i, A_{i+1})$. Therefore, one of the two sets A_i, A_{i+1} must be equal to $[1, r-1] \cup \{2r-2\} = \{1, 2 \dots, r-1, 2r-2\} = A_{\binom{r+s-1}{r}}$. In particular, at most one of the $\binom{r+s}{r} - 1$ partitions of Π defined thus far is an available label for $A_{\binom{r+s-1}{r}} \tilde{B}_{\binom{r+s-1}{r-1}}$. But there are

$(r-2)!$ available partitions for $A_{\binom{r+s-1}{r}}\tilde{B}_{\binom{r+s-1}{r-1}}$. Because $r \geq 5$, it follows that there exists an unused available partition for $A_{\binom{r+s-1}{r}}\tilde{B}_{\binom{r+s-1}{r-1}}$ with which to label that edge. This completes the construction of (C, Π), an orthogonally τ-labeled Hamiltonian cycle.

We have shown that for all (r, s), with $r \geq 4$ and $r - s \geq 1$, if Proposition 2.1 holds for all partition types corresponding to (r', s') satisfying $(r, s) > (r', s') \geq (4, 1)$, then the proposition holds for the partition type corresponding to (r, s), namely $2^s\,1^{r-s}$.

2.3 Completion of proof of Proposition 2.1: $r = s$ case

Take $r \geq 5$ and assume that Proposition 2.1 holds for all (r', s') such that $(r, r) > (r', s') \geq (4, 1)$. We assume inductively that for some $r \geq 5$, there exists an orthogonally $2^{r-1}\,1$-labeled Hamiltonian cycle

$$(C_0, \Pi_0) = A_1\pi_1 \ldots A_{\binom{2r-1}{r}}\pi_{\binom{2r-1}{r}},$$

satisfying Proposition 2.1. To complete the proof of Proposition 2.1, we show that it holds for the type associated with (r, r), namely 2^r, for $r \geq 5$.

We begin by modifying the orthogonally $(2^{r-1}1)$-labeled the Hamiltonian cycle (C_0, Π_0) to construct a cycle in $G_{2r,r}$ of length $\binom{2r-1}{r}$. We then construct a 2^r-labeled cycle of length $\binom{2r-1}{r}$ in $G_{2r,r}$, disjoint from the previously constructed one, and use these two cycles to construct a 2^r-labeled Hamiltonian cycle in $G_{2r,r}$.

For $i = 1, \ldots, \binom{2r-1}{r}$, let $\tilde{\pi}_i$ be the 2^r partition obtained by adjoining $2r$ to the unique singleton class of π_i. Note that partitions $\tilde{\pi}_i$ are distinct and each is orthogonal to its associated edge. Later we use an observation that Proposition 2.1(1c) applied to (C_0, Π_0), implies that for $j = 1, \ldots, r-2$, the doubleton set $\{2r, j\}$ is not a class of $\tilde{\pi}_i$. Let $\tilde{C}_0 = C_0$, and let $\tilde{\Pi}_0$ be obtained from Π_0 by replacing each π_i with $\tilde{\pi}_i$. Then $(\tilde{C}_0, \tilde{\Pi}_0)$ is an orthogonally 2^r-labeled cycle in $G_{2r,r}$. We refer to $(\tilde{C}_0, \tilde{\Pi}_0)$ as the *initial list* and to $\tilde{C}_0$ as the *initial cycle.*

Next, we build a cycle in $G_{2r,r}$, denoted by $\mathcal{E}$, out of collections of so-called "exclusion sets" which are defined below. This cycle contains as vertices all the r-sets of X_{2r} containing the element $2r$. As a means to keep subsequent notation uniform, for $i = r-2$, let $\{i+1, \ldots, r-2\}$ denote the empty set.

Definition 2.3. *The collections* $\mathbf{E_i}$ *and* $\mathbf{M}$ *of sets defined below are referred to as collections of exclusion sets.*

Let $\mathbf{E_i}$ *consist of all* r*-sets of* X_{2r} *containing* $\{i+1, \ldots, r-2, 2r\}$ *but not containing* i*, for* $i = 1, \ldots, r-3$.

Let $\mathbf{E}_{r-2}$ *consist of all* r*-sets of* X_{2r} *containing* $2r$ *but not containing* $r-2$.

Let M *be the set of all* r*-sets of* X_{2r} *which contain* $\{1,\ldots,r-2,2r\}$.

Observe that for $i = 1,\ldots,r-2$, $\mathbf{E}_i$ consists of $\binom{r+i}{1+i}$ sets, and M consists of $r+1$ sets. The union of the collection of exclusion sets is the set of all r-sets of X_{2r} containing $2r$. (A well-known formula states that the sum of $\binom{r+j}{1+j}$, as j ranges from 0 to $r-2$ is $\binom{2r-1}{r-1}-1$; the formula and our statement in the previous sentence are in agreement.)

For $i = 1,\ldots,r-2$, we show that there exists a listing of $\mathbf{E}_i$ as a cycle in $G_{2r,r}$ as follows. Let $X = X_{2r} - \{i, i+1,\ldots,r-2,2r\}$. Note that $|X| = r+i$. Let $G_{X,i+1}$ be the graph whose vertices are the $(i+1)$-sets of X, two such sets adjacent if their intersection contains i elements. Using the Hamiltonicity of $G_{X,i+1}$, we select a Hamiltonian cycle $\mathcal{H}_i$ of $G_{X,i+1}$; we may specify any pair of sets in X whose intersection has i elements as the first and last sets of $\mathcal{H}_i$. Later we will make use of that freedom. Add $\{i+1,\ldots,r-2,2r\}$ to each set of $\mathcal{H}_i$; the result is a cycle in $G_{2r,r}$ whose vertices consist of all members of $\mathbf{E}_i$. Henceforth, $\mathcal{E}_i$ will denote an unspecified cycle in $G_{2r,r}$, constructed in the manner above. In Listing 2.4 below, we specify the first and last sets of $\mathcal{E}_i$; these are chosen so that we can link the exclusion sets to form a cycle (which we denote by $\mathcal{E}$) in $G_{2r,r}$. For the purposes of the proof, we will not need to know further explicit information about $\mathcal{E}_i$.

Listing 2.4. *1. The first set of* $\mathcal{E}_1$ *is* $\{2,\ldots,r-2,2r,2r-2,2r-3\}$*; the last set of* $\mathcal{E}_1$ *is* $\{2,\ldots,r-2,2r,2r-1,2r-2\}$.

2. For $i = 2,\ldots r-2$,

the first set of $\mathcal{E}_i$ *is* $\{i+1,\ldots,r-2,2r,2r-1,\ldots,2r-i-1\}$*;*

the last set of $\mathcal{E}_i$ *is* $\{i-1,i+1,\ldots,r-2,2r,2r-1,\ldots,2r-i\}$.

3. M *is listed as a cycle* $\mathcal{M}$ *in* $G_{2r,r}$ *as follows:*

$\{1,\ldots,r-2,2r,2r-1\},\{1,\ldots,r-2,2r,r-1\},\{1,\ldots,r-2,2r,r\},$
$\{1,\ldots,r-2,2r,r+1\},\ldots, \{1,\ldots,r-2,2r,2r-2\}.$

Given a list $\mathcal{L} = l_1,\ldots,l_k$, its *reverse* $l_k,\ldots,l_1$ is denoted by $\mathcal{L}^{rev}$.

Listing 2.5. *1. If* r *is odd, then* $\mathcal{E}$ *is the cycle in* $G_{2r,r}$ *determined by the following listing of exclusion sets:*

$$\mathcal{E}_{r-3},\ldots,\mathcal{E}_{r-2m-1},\ldots,\mathcal{E}_2,\mathcal{M},\mathcal{E}_1^{rev},\ldots,\mathcal{E}_{1+2j}^{rev},\ldots,\mathcal{E}_{r-2}^{rev},$$

where $m = 1,\ldots,(r-3)/2$ *and* $j = 1,\ldots,(r-3)/2$.

2. *If r is even, then $\mathcal{E}$ is the cycle in $G_{2r,r}$ determined by the the following listing of exclusion sets:*

$$\mathcal{E}_{r-3},\ldots,\mathcal{E}_{r-2m-1},\ldots,\ldots,\mathcal{E}_1,\mathcal{M},\mathcal{E}_2^{rev},\ldots,\mathcal{E}_{2j}^{rev},\ldots,\mathcal{E}_{r-2}^{rev},$$

where $m = 1,\ldots,(r-2)/2$ and $j = 2,\ldots,(r-2)/2$.

For example, if $r = 6$, then $\mathcal{E}$ is $\mathcal{E}_3,\mathcal{E}_1,\mathcal{M},\mathcal{E}_2^{rev},\mathcal{E}_4^{rev}$. If $r = 7$, then $\mathcal{E}$ is $\mathcal{E}_4,\mathcal{E}_2,\mathcal{M},\mathcal{E}_1^{rev},\mathcal{E}_3^{rev},\mathcal{E}_5^{rev}$.

It is not difficult to check that $\mathcal{E}$ defined above is indeed a cycle; Listing 2.6 below may help in checking. The vertices of $G_{2r,r}$ have been partioned into two disjoint cycles, the initial cycle $\tilde{C}_0$ and the exclusion cycle $\mathcal{E}$. Adjoining these two cycles, we will form a Hamiltonian cycle in $G_{2r,r}$.

Edges which connect a pair of distinct exclusion sets will be referred to as *bridges*. The ordered pair in parentheses indicates the exclusion sets which are connected by the given bridge, where the number i refers to $\mathcal{E}_3$. The bridges for odd r are: $B(r-2,r-3), B(r-3,r-5),\ldots,B(4,2),B(2,\mathcal{M}).B(\mathcal{M},1$ $B(r-4,r-2)$. The symmetric difference of the vertices of each edge is also provided.

Listing 2.6. *Let r be an odd integer. Then*

1. $B(r-2,r-3) = \{2r,\ldots,r+1\}\{r-2,2r,\ldots,r+2\}$, *with symmetric difference* $\{r+1,r-2\}$.

2. *For i even and $4 \le i \le r-3$, we have that* $B(i,i-2) =$

   $\{i-1,i+1,\ldots,r-2,2r,\ldots,2r-i\}\{i-1,i,\ldots,r-2,2r,\ldots,2r-i-1\}$, *with symmetric difference* $\{i-2,2r-i\}$.

3. $B(2,\mathcal{M}) = \{1,3,\ldots,r-2,2r,2r-1,2r-2\}\{1,\ldots,r-2,2r,2r-1\}$, *with symmetric difference* $\{2r-2,2\}$.

4. $B(\mathcal{M},1) = \{1,\ldots,r-2,2r,2r-2\}\{2,\ldots,r-2,2r,2r-1,2r-2\}$, *with symmetric difference* $\{1,2r-1\}$.

5. $B(1,3) = \{2,\ldots,r-2,2r,2r-2,2r-3\}\{2,4,\ldots,r-2,2r,2r-1,$

   $2r-2,2r-3\}$, *with symmetric difference* $\{3,2r-1\}$.

6. *For an odd i with $3 \le i \le r-4$, we have that* $B(i,i+2) = \{i+1,\ldots,r-2,2r,$

   $2r-1,\ldots,2r-i-1\}\{i+1,i+3,\ldots,r-2,2r,2r-1,\ldots,2r-i-2\}$, *with symmetric difference* $\{i+2,2r-i-2\}$.

For even r, the bridges are similar to those for the odd r: the bridges are $B(r-2,r-3), B(r-3,r-5), \dots, B(3,1), B(1,\mathcal{M}), B(\mathcal{M},2), \dots, B(r-4,r-2)$. The symmetric difference of the two sets comprising each bridge is a two element set, as can be easily checked. It will turn out that for $r \geq 6$, a quick counting argument allows us to label the bridges without a detailed analysis. But for $r = 5$, the labeling requires a careful examination of the bridges.

We link the initial and exclusion cycles to produce a Hamiltonian cycle $\mathcal{D}$ in $G_{2r,r}$. To do so, we define a pair of *connecting edges*. Let

$$B(ini,1) = \{1,\dots,r\}\{1,\dots,r-2,2r,r-1\}, \tag{1}$$

with symmetric difference $\{r,2r\}$. Let

$$B(ini,2) = \{1,\dots,r-1,2r-1\}\{1,\dots,r-2,2r,2r-1\}, \tag{2}$$

with symmetric difference $\{r-1,2r\}$. Note that we have formed a Hamiltonian cycle by linking two (adjacent) vertices of the initial cycle to two (adjacent) vertices of $\mathcal{M}$ of the exclusion cycle. It is straight-forward to verify that $\mathcal{D}$ is a Hamiltonian cycle in $G_{2r,r}$, one that satisfies Proposition 2.1(1a), (1b). Notice that $\{1,\dots,r\}\{1,\dots,r-1,2r-1\}$ and $\{1,\dots,r-2,2r,2r-1\}\{1,\dots,r-2,2r,$
$r-1\}$ are edges of $\tilde{\mathcal{C}}_0$ and $\mathcal{E}$ respectively, but they are not edges of $\mathcal{D}$.

2.4 Labeling $\mathcal{E}_i$

We continue the construction of the partition-sequence Π by providing partition labels for $\mathcal{E}_i$, $i = 1,\dots,r-2$. We make further use of the induction hypothesis at this stage of the proof. Recall that for $i = 1,\dots,r-2$, the vertices of the cycle $\mathcal{E}_i$ consist of $\binom{r+i}{i+1}$ $(= \binom{r+i}{r-1})$ r-sets. As before, we let $X = X_{2r}$ - $\{i,i+1,\dots,r-2,2r\}$. Let $G_{X,r-1}$ be the graph whose vertices are the $(r-1)$-subsets of X; two $(r-1)$-sets are adjacent in $G_{X,r-1}$ if they intersect in $r-2$ elements.

We have $r-1 \geq i+1$ because $r-2 \geq i$. Because $r \geq 5$, we can apply the induction hypothesis to the partition type represented by the pair $(r-1,i+1)$: in $G_{X,r-1}$, there exists an orthogonally $2^{i+1}\,1^{r-i-2}$ -labeled Hamiltonian cycle

$$(\mathcal{D}_i,\Theta_i) = D_{i,1}\alpha_{i,1}\dots D_{i,\binom{r+i}{r-1}}\alpha_{i,\binom{r+i}{r-1}}.$$

By applying a permutation to the set X, we can assume that for i, $2 \leq i \leq r-2$, $D_{i,1} = [1,i-1]\cup[r-1,2r-i-2]$ and $D_{i,\binom{r+i}{r-1}} = [1,i-2]\cup[r-1,2r-i-1]$, and for $i = 1$, $D_{1,1} = [r-1,2r-4]\cup\{2r-2\}$ and $D_{\binom{r+1}{r-1}} = [r-1,2r-3]$.

Motivation on the choices above for the first and the last sets will come soon.

For a subset T of X, the complement of T in X will be denoted by T^C. Ignoring for the moment the partition-sequence Θ_i, consider the following list of $i+1$ sets formed from $\mathcal{D}_i$ by taking complements within X as follows: $(D_{i,1})^C \dots (D_{i,\binom{r+i}{r-1}})^C$. The list is a Hamiltonian cycle in $G_{X,i+1}$. We now form an exclusion cycle $\mathcal{E}_i$ in $G_{2r,r}$ as follows:

$$\mathcal{E}_i = ((D_1)^C \cup \{i+1,\dots,r-2,2r\}) \dots ((D_{\binom{r+i}{r-1}})^C \cup \{i+1,\dots,r-2,2r\}).$$

It is not difficult to verify that the first and last sets of each exclusion cycle are as stated in Listing 2.4.

Consider the $2^{i+1}\,1^{r-i-2}$-partitions $\alpha_{i,j}$, for $1 \le j \le \binom{r+i}{r-1} = \binom{r+i}{i+1}$. Observe that every doubleton class of α_j intersects $(D_{i,j})^C$ in one element and intersects $(D_{i,j+1})^C$ in one element. We modify $\alpha_{i,j}$ in two steps, in order to produce a 2^r partition in X_{2r}. First we choose a one-to-one assignment of the the singleton classes of $\alpha_{i,j}$ to the set $\{i+1,\dots,r-2\}$ thereby creating $r-i-2$ new doubleton classes. Next add the doubleton class $\{i,2r\}$, forming a 2^r partition of X_{2r}, denoted by $\tilde{\alpha}_{i,j}$. The reader can now verify that the following list is an orthogonal 2^r-labeled path whose set-sequence is $\mathcal{E}_i$:

$$((D_{i,1})^C \cup \{i+1,\dots,r-2,2r\})\tilde{\alpha}_{i,1} \dots ((D_{i,\binom{r+i}{r-1}})^C \cup \{i+1,\dots,r-2,2r\})\tilde{\alpha}_{i,\binom{r+i}{r-1}}.$$

Notice that each partition label on the exclusion cycle $\mathcal{E}_i$ contains the doubleton class $\{2r,i\}$; thus, the set of partition labels used for distinct exclusion cycles are disjoint. Moreover, by Proposition 2.1(1c), it follows that no partition label of an edge of an exclusion cycle is also used to label an edge of the initial cycle $\tilde{\mathcal{C}}_0$.

2.5 Labeling bridges and labeling edges of $\mathcal{M}$

To complete an orthogonal 2^r labeling of the constructed Hamiltonian cycle, we need to label the bridges, the edges of $\mathcal{M}$, and the two connecting edges. We specialize the previously defined "available label for an edge". Let UV be an edge of $G_{2r,r}$ and let $i \in [1,2r-1]$. An *i-available label for UV* is a partition π of type 2^r which is orthogonal to both U and V and for which $\{i,2r\}$ is a doubleton class. Observe that if $\{i,2r\}$ is not the difference set of UV, then $(r-2)!$ is the number of i-available labels for UV; if $\{i,2r\}$ is the difference set, then $(r-1)!$ is the number of i-available labels for UV.

We have described the bridges and their symmetric differences; here is a list of the edges of $\mathcal{M}$, along with their symmetric differences.

Listing 2.7. *1.* $\mathcal{M}_{2r-1} = \{1,\ldots,r-2,2r,2r-1\}\{1,\ldots,r-2,2r,r-1\}$, *with symmetric difference* $\{2r-1,r-1\}$.

2. $\mathcal{M}_r = \{1,\ldots,r-2,2r,r-1\}\{1,\ldots,r-2,2r,r\}$, *with symmetric difference* $\{r-1,r\}$.

3. More generally, for $r \leq u \leq 2r-3$, *we have that*

$\mathcal{M}_{u+1} = \{1,\ldots,r-2,2r,u\}\{1,\ldots,r-2,2r,u+1\}$, *with symmetric difference* $\{u,u+1\}$.

Notice that the linking of the initial cycle $\tilde{C}_0$ and $\mathcal{E}$, the connecting edges $B(ini,1)$ and $B(ini,2)$ replace the edges $\mathcal{M}_{2r-1}$ in the Hamiltonian cycle $\mathcal{D}$ and $C_1C_{\binom{n}{r-1}}$ in $\tilde{C}_0$; in particular, it is unnecessary to label $\mathcal{M}_{2r-1}$.

2.5.1 Labeling $\mathcal{MB}$ edges

With the exception of $\mathcal{M}_r$, the element $r-1$ is not an element of a vertex of any bridge, or of a vertex of any still to be labeled edges of $\mathcal{M}$. We restrict our attention to labeling the bridges and the edges of $\mathcal{M}$ that remain to be labeled, with the exception of $\mathcal{M}_r$. Call this collection of edges $\mathcal{MB}$. As can be easily checked, there are $2r-3$ edges to be labeled in $\mathcal{MB}$ and that each such edge has an $(r-1)$-available partition for that edge.

No partition label up to now contains $\{r-1,2r\}$ as a doubleton class: in the initial cycle, if $\{r-1,2r\}$ is a doubleton set of a partition label, then by Proposition 2.1(1c), the core of the edge must be $\{1,\ldots,r-1\}$. By Proposition 2.1(1b), the only edge with core $\{1,\ldots,r-1\}$ has its vertices the first and last set of the initial cycle; however, that edge is not an edge of $\mathcal{D}$. In the appended cycle $\mathcal{E}$, the edges of each exclusion cycle $\mathcal{E}_i$ are labeled with i-available partitions, $i = 1,\ldots,r-2$.

Of course, $2r$ is in the core of each edge in $\mathcal{MB}$. In particular, the number of $(r-1)$-available labels for a given edge of $\mathcal{MB}$ is $(r-2)!$

Fix one of these $2r-3$ edges of $\mathcal{MB}$. If $r > 5$, then as is easily verified, $(r-2)! > 2r-3$. Hence, for $r > 5$ we can label the $\mathcal{MB}$ edges "greedily", labeling with an $(r-1)$-available partition for the given edge, proceeding to the next $\mathcal{MB}$ edge, guaranteed that there exist unused $(r-1)$-available partitions to label that edge.

The case r =5

For $r = 5$, we claim that the union of the set of 4-available partitions for the edges $B(2,\mathcal{M})$ and $B(3,2)$ is disjoint from the union of the set of 4-available partitions for the rest of $\mathcal{MB}$ edges.

Indeed, no 4-available partition with doubleton class $\{u,u+1\}$ $(u = 5,6,7)$ is a 4-available partition for $B(2,\mathcal{M})$ and $B(3,2)$ (see Listing 2.7).

However, the $\mathcal{MB}$ edges with vertices in $\mathcal{M}$ have distinct symmetric differences of the form $\{u, u+1\}$, where $u = 5, 6, 7$. Moreover, $B(\mathcal{M}, 1)$ has 4-available partitions with a class $\{7, 8\}$ but not $\{6, 7\}$, and $B(1, 3)$ has 4-available partitions with a class $\{6, 7\}$ but not $\{7, 8\}$.

Now, orthogonally 2^5-label the edges $B(2, \mathcal{M})$ and $B(3, 2)$ with distinct 4-available partitions; this is possible, since $(5-2)! = 6 > 2$. Neither of these two partitions have a class of the form $\{u, u+1\}$ $(u = 5, 6, 7)$. There are at least two 4-available partitions with a class $\{u, u+1\}$ for a fixed $u = 5, 6, 7$. Thus we can label the remaining edges "greedily" with 4-available partitions with classes $\{u, u+1\}$ for appropriate i in the range $\{5, 6, 7\}$. This completes the $r = 5$ case.

We have shown that for any $r \geq 5$, we can orthogonally 2^r-label the $2r-3$ edges of $\mathcal{MB}$ sequentially, with distinct partitions, each containing $\{2r, r-1\}$ as a doubleton class.

2.6 Labeling M_r

We continue, labeling the edge $\mathcal{M}_r$ (which has a difference set of $\{r-1, r\}$) with one of the $(r-2)!$ labels which is $(r+1)$-available for $\mathcal{M}_r$. Since for each $i = 1, \ldots, r-2$, each partition label of $\mathcal{E}_i$ contains the doubleton class $\{2r, i\}$, the labeling of $\mathcal{E}$ has not involved a 2^r partition with $\{r+1, 2r\}$ as a doubleton class.

We investigate edges of the initial cycle $\tilde{C}_0$ to determine how many such edges admit an $(r+1)$-available label for $\mathcal{M}_r$; the idea is to show that there are fewer than $(r-2)!$ such edges. Call a $\tilde{C}_0$ edge a *competing edge* if it can be orthogonally labeled with a $(r+1)$-available partition for $\mathcal{M}_r$. So a competing edge admits a partition which has $\{r+1, 2r\}$ as a doubleton class; by Proposition 2.1(1c), the core of the competing edge contains at least $r-2$ elements from the interval $[1, r]$. If the competing edge has symmetric difference $\{r-1, r\}$, then the core of the competing edge must be $\{1, \ldots, r-2, r+1\}$ and the edge is completely determined. So there is at most one competing edge with symmetric difference $\{r-1, r\}$. If $\{r-1, r\}$ is not the symmetric difference, then it must be possible to find a partition orthogonal to the vertices of the competing edge which has $\{r-1, r\}$ as a doubleton class (or else the edge would not be a competing edge). Upon inspection, it follows that if $\{r-1, r\}$ is not the symmetric difference of the competing edge, then exactly one of the elements of the set $\{r-1, r\}$ is in the core of the competing edge. Thus the core of the competing edge is as follows: $([1, r-2] - \{u\}) \cup \{r+1, z\}$, where $u \in [1, r-2]$ and $z \in \{r-1, r\}$.

We claim that u is in the symmetric difference set of the competing edge. If not, then using the information above about the core of the competing edge, it is not difficult to see that any partition orthogonal to the competing edge must contain a doubleton set $\{u, w\}$, where $w \in [1, r-2]$. However, each of the $(r+1)$-available partitions for $\mathcal{M}_r$ have $\{u, v\}$ as a doubleton class, for some $v \in [r+2, 2r-1]$. Thus, u is in the symmetric difference, as claimed.

Since u is in the symmetric difference, one of the two vertices of the competing edge is of the form $[1, r-2] \cup \{r+1, z\}$, where z is either $r-1$ or r. Thus there are at most four competing edges for which $\{r-1, r\}$ is not the symmetric difference.

It follows that there exist at most five competing edges in the initial cycle labeled with one of the $(r+1)$-available partitions of $\mathcal{M}_r$, of which there are $(r-2)!$. For $r > 4$, $(r-1)! > 5$; hence, there exists an unused $(r+1)$-available partition for $\mathcal{M}_r$. Note that whichever label for $\mathcal{M}_r$ is used, it has not been used in a prior stage of the labeling (since $\{2r, r+1\}$ is not a doubleton class of any label used in a prior stage.)

2.7 Labeling the two connecting edges

To complete the proof of the theorem, we label the connecting edges (see Equations (1) and (2)). Recall that $B(ini, 2) = \{1, \ldots, r-1, 2r-1\}\{1, \ldots, r-2, 2r, 2r-1\}$, with symmetric difference $D(ini, 2) = \{r-1, 2r\}$. The doubleton class containing $2r$ has been determined - it must be the symmetric difference $\{r-1, 2r\}$. Thus, the number of $(r-1)$-available partitions for $B(ini, 2)$ is $(r-1)!$ To this point, we have used $2r-3$ labels containing $\{r-1, 2r\}$ as a doubleton class. For $r \geq 5$, $(r-1)! > 2r-3$; hence, there exists an unused $(r-1)$-available partition for $B(ini, 2)$.

We label $B(ini, 1) = \{1, \ldots, r\}\{1, \ldots, r-2, 2r, r-1\}$, with symmetric difference $\{r, 2r\}$. No edge in the appended cycle $\mathcal{E}$ has been labeled with a partition containing $\{r, 2r\}$ as a doubleton set. We use the notation "competing edge" again to denote an edge in $\tilde{\mathcal{C}}_0$ which can be orthogonally 2^r-labeled with one of the $(r-1)!$ partitions that are r-available for the edge $B(ini, 1)$. By Proposition 2.1(1c), any competing edge in the initial cycle has r in its core along with $r-2$ elements from the set $[1, r-1]$. In particular, there exists a unique $v \in [1, r-1]$ which is not in the core of the competing edge. If v is not in the symmetric difference of the competing edge, then any orthogonal label for the competing edge must identify v with an element from $[1, r-1]$; by inspection, such a partition is not an r-available partition for $B(ini, 1)$. If v is in the symmetric difference, then one of the vertices for the competing edge is $\{1, \ldots, r\}$. Thus there is at most one edge which is labeled with one of the $(r-1)! > 1$ partitions r-available for $B(ini, 1)$. This completes the proof of Proposition 2.1, and the proof of Theorem 1.7. □

2.8 The base steps $2^2\,1^2, 2^3\,1, 2^4$

The next lemma describes the requirements of Propostion 2.1 for each of the base steps $2^2\,1^2, 2^3\,1$, and 2^4. The corresponding labeled Hamiltonian cycles are presented below.

Lemma 2.8. *1. There exists an orthogonally $2^2\,1^2$-labeled Hamiltonian cycle in $G_{6,4}$ such that*

(a) $\{1,2,3\}$ is the core of exactly one edge, namely the edge formed from the first and last sets, $\{1,2,3,4\}\{1,2,3,6\}$;

(b) $\{1\}$ is not a singleton class of a partition label of any edge;

(c) for $b = 2,3,4$, if $\{b\}$ is a singleton class of a partition label of an edge, then the core of that edge contains b and at least one element of $[1, b-1]$;

2. There exists an orthogonally $2^3\,1$-labeled Hamiltonian cycle in $G_{7,4}$ satisifying the following:

(a) $\{1,2,3\}$ is the core of exactly one edge, namely the edge formed from the first and last sets, $\{1,2,3,4\}\{1,2,3,7\}$;

(b) neither $\{1\}$ nor $\{2\}$ is a singleton class of a partition label of any edge;

(c) for $b = 3,4,5$, if $\{b\}$ is a singleton class of a partition label of an edge, then the core of that edge contains b and at least two elements of $[1, b-1]$;

3. There exists an orthogonally 2^4-labeled Hamiltonian cycle in $G_{8,4}$ such that $\{1,2,3\}$ is the core of exactly one edge, namely the edge formed from the first and last sets, $\{1,2,3,4\}\{1,2,3,7\}$.

2.8.1 Orthogonally $2^2\,1^2$-labeled Hamiltonian cycle

In Figure 1 below, we provide the orthogonally $2^2\,1^2$-labeling of a Hamiltonian cycle in $G_{6,4}$ satisfying all the conditions of Lemma 2.8(1a).

For $i, \ldots, 15$, the left hand column provides the set A_i; the second column provides the core of the edge A_iA_{i+1}; the last column provides the orthogonal $2^2\,1^2$-label for A_iA_{i+1}. The symmetric difference of A_iA_{i+1} is the left-most doubleton class of the partition α_i. It is not difficult to verify that the list satisfies the conditions of Lemma 2.8 for $\tau = 2^2\,1^2$.

Vertex	Core	Partition
A_i	$A_i \cap A_{i+1}$	π_i
1234	124	$3,5\|1,6\|2\|4$
1245	245	$1,3\|2,6\|4\|5$
2345	245	$3,6\|1,2\|4\|5$
2456	456	$2,3\|1,4\|5\|6$
3456	456	$1,3\|2,4\|5\|6$
1456	156	$3,4\|1,2\|5\|6$
1356	135	$4,6\|1,2\|3\|5$
1345	134	$5,6\|1,2\|3\|4$
1346	146	$2,3\|1,5\|4\|6$
1246	246	$1,3\|2,5\|4\|6$
2346	236	$4,5\|1,2\|3\|6$
2356	235	$1,6\|2,4\|3\|5$
1235	125	$3,6\|1,4\|2\|5$
1256	126	$3,5\|1,4\|2\|6$
1236	123	$4,6\|1,5\|2\|3$

Figure 1: Orthogonally $2^2\,1^2$-labeled Hamiltonian cycle

2.8.2 An orthogonally $2^3\,1$-labeled list

Figure 2 presents an orthogonal $2^3\,1$-labeled a Hamiltonian cycle in $G_{7,4}$ satisfying the conditions of Lemma 2.8(2). To help the reader verify that the partitions are distinct, we have counted 13 partitions with 7 as the singleton class; of these, three partitions have $\{1,2\}$ as a doubleton class, three partitions have $\{1,3\}$ as a doubleton class, three have $\{1,4\}$ as a doubleton class, two have $\{1,5\}$ as a doubleton class, and two have $\{1,6\}$ as a doubleton class. There are 13 partitions which have 6 as the singleton class. Of these, there are three that have $\{1,2\}$ as a doubleton class, three that have $\{1,3\}$ as a doubleton class, two that have $\{1,4\}$ as a doubleton class, two that have $\{1,5\}$ as a doubleton class, and three that have $\{1,7\}$ as a doubleton class.

Vertex A_i	Core $A_i \cap A_{i+1}$	Partition π_i
1234	234	1,5|2,6|3,7|4
2345	245	3,6|1,2|4,7|5
2456	256	3,4|1,2|5,7|6
2356	235	6,7|1,2|3,4|5
2357	257	3,4|1,2|5,6|7
2457	247	3,5|2,6|1,4|7
2347	237	4,6|2,5|1,3|7
2367	236	4,7|1,2|3,5|6
2346	246	3,7|1,2|4,5|6
2467	267	4,5|1,2|3,6|7
2567	567	1,2|3,5|4,6|7
1567	167	2,5|1,3|4,7|6
1267	126	5,7|1,4|2,3|6
1256	156	2,4|1,3|5,7|6
1456	456	1,7|2,4|3,5|6
4567	457	1,6|2,4|3,5|7
1457	157	2,4|1,3|5,6|7
1257	127	4,5|1,3|2,6|7
1247	147	2,6|1,5|3,4|7
1467	467	1,3|4,5|2,7|6
3467	367	1,4|2,3|5,6|7
1367	136	2,7|1,4|3,5|6
1236	126	3,4|1,7|2,5|6
1246	146	2,3|1,5|4,7|6
1346	346	1,5|3,7|2,4|6
3456	345	6,7|1,3|2,4|5
3457	347	1,5|2,3|4,6|7
1347	134	5,7|1,2|3,6|4
1345	145	2,3|1,6|4,7|5
1245	125	3,4|1,7|2,6|5
1235	135	2,6|1,4|3,7|5
1356	356	1,7|2,3|4,5|6
3567	357	1,6|2,3|4,5|7
1357	137	2,5|1,4|3,6|7
1237	123	4,7|1,5|2,6|3

Figure 2: Orthogonally 2^3 1-labeled Hamiltonian cycle

2.8.3 An orthogonally 2^4-labeled Hamiltonian cycle

The construction of the cycle follows along the lines of construction of the Hamiltonian cycle $\mathcal{D}$ in the $r = s$ case in the proof of Proposition 2.1. We begin by modifying the 2^3 1-labeled cycle provided in Figure 2, by adding 8 to the unique singleton class of each partition label in Figure 2, with the exception of the edge A_1A_{35} which plays no role in the Hamiltonian cycle in $G_{8,4}$. Thus, we have 35 sets so far, with 34 partititions. Because Figure 2 is an orthogonally 2^3 1-labeled list, the 34 partitions are distinct. Observe each of these 34 partition labels of type 2^4 does not contain a doubleton class of the form $\{1,8\}$, $\{2,8\}$, or $\{3,8\}$, because neither 1, 2, or 3 are singleton classes of the 34 partitions of Figure 2 for which we add $\{8\}$ to the singleton class.

In Figure 3, the other "half" of the list is provided; there are 36 partitions of type 2^4 in the list, 37 sets. The first set of Figure 3 is listed for convenience; it is the last set of Figure 2. The first partition of Figure 3 is the partition label for the edge $\{1,2,3,7\}\{1,2,7,8\}$. The last partition of Figure 3 is the label of $\{1,2,3,8,\}\{1,2,3,4\}$.

The first and last sets of the resulting orthogonally 2^4-labeled Hamiltonian cycle are as Lemma 2.8 stipulate; a quick scan of the "core" column shows that $\{1,2,3\}$ occurs as a core exactly once, as the core of the edge connecting the first and last set.

3 Counterexample, connections with semigroup theory, open problems

After showing that not every Hamiltonian cycle of $G_{8,4}$ can be orthogonally 2^4-labeled, we present a brief description of the connection between the combinatorics of orthogonal labeling and questions related to minimal generating sets of certain finite semigroups of transformations. We also pose some problems.

3.1 Counterexample

We show that there exist Hamiltonian cycles in $G_{n,r}$ which do not admit a single orthogonal τ-labeling, even though the number $\mathcal{N}(\tau)$ of distinct partitions of type τ is greater or equal to $\binom{n}{r}$. We begin by presenting a class of Hamiltonian cycles in $G_{n,r}$ known as *constant weight reflected Gray codes* ([1],[6],[11],[12]).

Definition 3.1. *Let n, r be positive integers with $n \geq r$.*

1. *Let $H_{n,n} = \{X_n\}$.*
2. *Let $H_{n,1} = \{1\}, \ldots, \{n\}$.*
3. *Suppose $H_{n-1,r-1} = A_1 \ldots A_{\binom{n-1}{r-1}}$. Let $H^{rev}_{n-1,r-1} \oplus n$ be the list*

$$(\{n\} \cup A_{\binom{n-1}{r-1}}) \ldots (\{n\} \cup A_i)(\{n\} \cup A_{i-1}) \ldots (\{n\} \cup A_1).$$

So $H^{rev}_{n-1,r-1} \oplus n$ is the list that results by joining n to each set of $H_{n-1,r-1}$ and then reversing the order of the resulting listing.

4. *For $n > r > 1$, let $H_{n,r} = H_{n-1,r}(H^{rev}_{n-1,r-1} \oplus n)$, the list that results by concatenating $H_{n-1,r}$ and $H^{rev}_{n-1,r-1} \oplus n$.*

Vertex	Core	Partition
1237	127	1,4|2,6|5,7|3,8
1278	278	1,6|2,5|4,7|3,8
2678	278	3,6|2,4|5,7|1,8
2378	238	6,7|2,5|3,4|1,8
2368	238	4,6|2,5|3,7|1,8
2348	248	3,7|2,6|4,5|1,8
2478	248	6,7|2,3|4,5|1,8
2468	248	5,6|2,3|4,7|1,8
2458	258	3,4|2,7|5,6|1,8
2358	258	3,7|2,4|5,6|1,8
2578	258	6,7|2,4|3,5|1,8
2568	568	2,7|4,5|1,6|3,8
5678	568	3,7|4,6|1,5|2,8
3568	358	4,6|1,3|5,7|2,8
3458	458	3,6|1,4|5,7|2,8
4568	468	5,7|1,6|3,4|2,8
4678	478	5,6|1,7|3,4|2,8
4578	578	3,4|1,5|6,7|2,8
3578	378	4,5|1,7|3,6|2,8
3478	348	6,7|1,4|3,5|2,8
3468	368	4,7|1,6|3,5|2,8
3678	368	4,6|3,5|1,7|2,8
1368	138	5,6|3,7|1,4|2,8
1358	138	5,7|2,3|1,6|4,8
1378	138	4,7|3,6|1,5|2,8
1348	148	3,7|4,5|1,6|2,8
1478	148	6,7|4,5|1,3|2,8
1468	148	5,6|4,7|1,3|2,8
1458	158	4,6|1,2|5,7|3,8
1568	158	6,7|1,2|3,5|4,8
1578	178	5,6|1,3|2,7|4,8
1678	168	2,7|1,5|4,6|3,8
1268	128	5,6|1,7|2,4|3,8
1258	128	4,5|1,7|2,6|3,8
1248	128	3,4|1,6|2,7|5,8
1238	123	1,5|2,7|3,6|4,8

Figure 3: Second part of orthogonally 2^4-labeled Hamiltonian cycle

The next lemma is a series of basic observations whose proof is left to the reader. For a positive integer k satisfying $\binom{n}{r} \geq k \geq 1$, let $H_{n,r}(k)$ be the kth set in the Hamiltonian cycle $H_{n,r}$.

Lemma 3.2. *Let n and r be positive integers such that $n > r$. Then*

1. *$H_{n,r}$ is a Hamiltonian cycle.*
2. *$H_{n,r}(1) = \{1,\ldots,r\}$ and $H_{n,r}(\binom{n}{r}) = \{1,\ldots,r-1,n\}$.*
3. *$H_{n,r}(\binom{n-1}{r}) = \{1,\ldots,r-2,n-1\}$ and $H_{n,r}(\binom{n-1}{r}+1) = \{1,\ldots,r-1,n-1,n\}$*

Lemma 3.3. *There exists a Hamiltonian cycle in $G_{8,4}$ which admits no orthogonal 2^4 labeling.*

Proof. Consider $H_{8,4}(1)\ldots H_{8,4}(\binom{8}{4})$, the Hamiltonian cycle $H_{8,4}$ and the associated sequence $\Delta(H_{8,4}(1), H_{8,4}(2)), \ldots, \Delta(H_{8,4}(\binom{8}{4}), H_{8,4}(1))$, the sequence of two element symmetric differences. We begin with a claim that we prove by induction on n.

Claim. For $n > r \geq 1$, the set $\{1,2\}$ occurs $\binom{n-2}{r-1}$ times in the sequence of symmetric differences associated with $H_{n,r}$.

If $r = 1$, then $H_{n,1}$ is defined to be $\{1\},\{2\},\ldots,\{n\}$; the symmetric differences of the edges in the order in which they occur is given by $\{1,2\},\{2,3\},\ldots,\{n,1\}$. By inspection, $\{1,2\}$ occurs 1 ($= \binom{n-2}{0}$) time.

Assume $r > 1$. We have $H_{n,r} = H_{n-1,r}(H^{rev}_{n-1,r-1} \oplus n)$. Observe that $\{1,2\}$ is not the symmetric difference of the two "connecting edges", $\{1,\ldots,r\}\{1,\ldots,r-1,n\}$ and $\{1,\ldots,r-1,n-1\}\{1,\ldots,r-2,n-1,n\}$. Consider $H^{rev}_{n-1,r-1} \oplus n$. Reversing the order of the occurrences of vertices has no effect on the set of symmetric differences associated to this part of $H_{n,r}$; the same can be said adding n to each set of $H_{n-1,r-1}$. In summary, the number of times $\{1,2\}$ occurs as a symmetric difference in $H_{n,r}$ is the sum of the number of times it occurs in $H_{n-1,r}$ with the number of times it occurs in $H_{n-1,r-1}$. By the inductive assumption, we have that the total number of occurrences of $\{1,2\}$ as a symmetric difference is $\binom{n-3}{r-1} + \binom{n-3}{r-2} = \binom{n-2}{r-1}$, completing the proof of the Claim.

For $G_{8,4}$, the Claim shows that there are $\binom{6}{3} = 20$ edges of $H_{8,4}$ with symmetric difference $\{1,2\}$ in $G_{8,4}$. Thus, a 2^4-labeling of $H_{8,4}$ would require that at least 20 distinct 2^4 partition labels have $\{1,2\}$ as a doubleton set, which in turn would require the existence of 20 distinct partitions of type 2^3 of the set $\{3,4,5,6,7,8\}$. But as can be verified quickly, there exist only 15 distinct partitions of type 2^3 of the set $\{3,4,5,6,7,8\}$. This completes the proof of the lemma. □

3.2 Semigroup connections, conclusion

All semigroups here are assumed to be finite. For a semigroup S with a subset $U \subseteq S$, let $< U >$ denote the subsemigroup of S generated by U. We say S is generated by U if $S =< U >$. The *rank* of S is the cardinality of a smallest generating set of S.

An element $e \in S$ is said to be idempotent if $e^2 = e$. If S has a generating set consisting of idempotents, then S is said to be *idempotent-generated.* The *idempotent rank* of an idempotent-generated semigroup S is the cardinality of the smallest set of idempotent elements which generates S.

For a transformation f of X_n, let $im(f)$ denote the image of f, let $ker(f) = \{(a,b) \in X_n^2 : af = bf\}$ denote the kernel of f, and let $h(f) = |im(f)|$, the height of f. Let $K(n,r)$ be the semigroup consisting of transformations of X_n of height at most r. In [3], it was proved that for $n > r$, the semigroup $K(n,r)$ is generated by its idempotents with height r. Recall that the Stirling number of the second kind $S(n,r)$ is the number of partitions of $\{1,\ldots,n\}$ of weight r. We describe the Howie and McFadden argument that is used in [4] to prove the following theorem.

Theorem 3.4. *[4] The idempotent-rank of $K(n,r)$ is $S(n,r)$.*

It is not difficult to see that the rank of $K(n,r)$ is at least $S(n,r)$. Suppose U is a generating set of $K(n,r)$ consisting of transformations with height r. Let $f \in K(n,r)$ be an arbitrary element of $K(n,r)$ with height r. If f is a product $s_1 s_2 \ldots s_m$, such that for $i = 1,\ldots,m$ we have $s_i \in U$, then because $ker(f)$ has weight r, it follows that $ker(f) = ker(s_1)$. Thus $|U|$ is at least $S(n,r)$ and so the rank (and the idempotent rank) of $K(n,r)$ is at least $S(n,r)$.

We demonstrate how the existence of orthogonally labeled lists (Theorem 1.2) leads to Theorem 3.4. Since an idempotent transformation is the identity on its image, an r-set A along with a weight r partition π orthogonal to A uniquely determines an idempotent transformation $e_{A,\pi} : X_n \to X_n$ such that $im(e_{A,\pi}) = A$ and $ker(e_{A,\pi}) = \pi$.

Let $A_1\pi_1 \ldots A_{\binom{n}{r}}\pi_{\binom{n}{r}}$ be an orthogonally labeled list of r-sets of $\{1,\ldots,n\}$. Consider the set $\{e_{A_i,\pi_i} : i = 1,\ldots,\binom{n}{r}\}$. We can extend $\{e_{A_i,\pi_i} : i = 1,\ldots,\binom{n}{r}\}$ to a set of $S(n,r)$ idempotents U, selecting $S(n,r) - \binom{n}{r}$ idempotents in such a way that each of the partitions of weight r is represented exactly once as the kernel of an idempotent in U. Now take an idempotent e of height r with $ker(e) = \pi$ and $im(e) = A_i$, for some $i = 1,\ldots,\binom{n}{r}$. By the choice of U, there exists a unique idempotent $f = e_{A_j,\pi}$ in U, for some $j = 1,\ldots,\binom{n}{r}$. By proceeding "clockwise" along the orthogonally labeled list, from $\pi_j A_{j+1}\pi_{j+1}A_{j+2}\ldots$ to $\pi_i A_i$, sequentially construct the idempotents $e_{A_{j+1},\pi_j},\ldots,e_{A_{j+2},\pi_{j+1}},\ldots,e_{A_i,\pi_{i-1}}$ associated with the orthogonal label affixed to each successive pair of r-sets by the orthogonal labeling. Observe that the orthogonality of the labeling guarantees that $g = fe_{A_{j+1},\pi_j} \cdots e_{A_i,\pi_{i-1}}$ has image A_i and kernel π. Since e is idempotent, we have A_i and π are orthogonal. By the finiteness of X_n, there exists a positive integer k such that g^k is idempotent. Since $ker(g^k) = ker(e)$ and $im(g^k) = im(e)$, it follows that $g^k = e$, thus proving that every idempotent with height r is contained in $< U >$. Since $K(n,r)$ is generated by its idempotent with height r, we have that $K(n,r) =< U >$ and $S(n,r)$ is a lower bound for the rank of $K(n,r)$, it follows that the rank of $K(n,r)$ is $S(n,r)$. That U consists of idempotents implies that the idempotent rank of $K(n,r)$ is also $S(n,r)$.

More generally, let $\mathcal{P}$ be a subset of the $Part(n,r)$, the set of all weight r partitions in $Part(n)$. Let $S(\mathcal{P})$ be the semigroup generated by all transformations with kernel in $\mathcal{P}$. Assume that $S(\mathcal{P})$ is idempotent-generated and there exists an orthogonally $\mathcal{P}$-labeled Hamiltonian cycle. In that case, the Howie and McFadden arguments above can be used to show that the rank and the idempotent rank of $S(\mathcal{P})$ are both $|\mathcal{P}|$. For example, let n, s, r be positive integers such that $n = s + r \geq 4$. Theorem 1.7 guarantees the existence of orthogonally 2^s1^{r-s}-labeled Hamiltonian cycles. It is not difficult to see that $S(2^s1^{r-s})$ is idempotent-generated. Thus, we have that $S(2^s1^{r-s})$ has rank and idempotent rank both equal to $(r+s)!/(s!(r-s)!2^s)$, the number of distinct partitions of type 2^s1^{r-s}.

Let S be a finite subsemigroup of $K(n,r)$ generated by its members of height r and let S_r be the elements of S of height r, let $k_S = |\{ker(f) : f \in S, |im(f)| = r\}|$, and let $i_S = |\{im(f) : f \in S, |im(f) = r\}|$. Using an argument of the type above that shows that the rank of $K(n,r)$ is at least $S(n,r)$, one can show that a generating set for S must have at least $max\{k_S, i_S\}$ elements.

Definition 3.5. *Let S be a finite subsemigroup of $K(n,r)$ generated by S_r. We say that S is* **extremally-generated** *if S has a generating set U such that $|U| = max\{k_S, i_S\}$.*

Using that $S(n,r) \geq \binom{n}{r}$, Theorem 3.4 shows that $K(n,r)$ is extremally-generated. The discussion above shows that for $r \geq s$, the semigroup $S(2^s1^{r-s})$ is extremally-generated. The authors have proved the non-trivial result that for any partition type τ, the semigroup $S(\tau)$ is extremally-generated ([7], [9]).

Problem 3.6. *Let $\mathcal{P}$ be a subset of $Part(n,r)$.*

1. *Determine conditions on $\mathcal{P}$ which guarantee that there exist orthogonally $\mathcal{P}$-labeled lists.*

2. *Determine conditions on $\mathcal{P}$ which guarantee that there exist orthogonally $\mathcal{P}$-labeled Hamiltonian cycles.*

3. *Determine conditions on $\mathcal{P}$ which guarantee that $S(\mathcal{P})$ is extremally-generated.*

The second part of Theorem 1.3 asserts the existence of an orthogonally $Part(n,r)$-labeled list $A_1\alpha_1 \ldots A_{\binom{n}{r}}\alpha_{\binom{n}{r}}$ which satisfies a "two-fold Gray condition", namely that $A_1 \ldots A_{\binom{n}{r}}$ is a Hamiltonian cycle in $G_{n,r}$ and that $\pi_1 \ldots \pi_{\binom{n}{r}}$ is a cycle in $\mathbf{Part(n,r)}$.

We ask if there exist orthogonally $2^s\,1^{r-s}$- labeled lists which satisfy the same "two-fold Gray condition".

Problem 3.7. *For which $r \geq 4$, $r \geq s$, and $s > 1$, does there exist an orthogonally labeled 2^s1^{r-s}-labeled Hamiltonian cycle of $G_{s+r,r}$ such that the associated partition-sequence $\pi_1 \ldots \pi_{\binom{n}{r}}$ is a cycle in* $\mathbf{Part(n,r)}$*?*

In Lemma 3.3 we provided an example of a Hamiltonian cycle in $G_{8,4}$ which does not admit any orthogonal 2^r-labeling.

Problem 3.8. *Which weight r partition types τ have the property that every Hamiltonian cycle of $G_{n,r}$ can be orthogonally τ-labeled?*

In [8], the authors prove that if τ is a type of weight r with no singleton classes and $n \geq 18$, then every Hamiltonian cycle in $G_{n,r}$ can be orthogonally τ-labeled.

References

[1] James R. Bitner, Gideon Ehrlich, Edward M. Reingold, Efficient Generation of the Binary Reflected Gray Code and Its Applications, *Communications of the ACM* , Vol. 9, Number 6, (1976), 517-521.

[2] Phillip J. Chase, Combinations of m out of n objects, *Communications of the ACM* , Vol. 13, Number 6, (1970), 368.

[3] A. E. Evseev and N.E. Podran, Transformation semigroups that are generated by idempotents with a given defect. (Russian) Izv. Vysš. Učebn. Zaved. Matematika 1972, no. 2(117), 44–50, translated in American Math. Soc. Transl. (2), 139, (1988) 77-84.

[4] J.M. Howie and R.B. McFadden, Idempotent rank in finite full transformation semigroups, *Proc. Royal Soc. Edinburgh,* 114A (1990), 161-167.

[5] Jenő Lehel and Inessa Levi, Loops with partitions and matchings, *Ars Combinatoria*, 54(2000), 237-253.

[6] Inessa Levi, Robert B. McFadden and Steve Seif, Efficient algorithms for producing minimal idempotent generating sets of finite transformation semigroups, submitted.

[7] Inessa Levi and Steve Seif, Constant Weight Gray codes labeled by partitions, with application to minimal generating sets of finite semigroups, *Proc. Edinburgh Math. Soc.*, to appear.

[8] Inessa Levi and Steve Seif, Counting Techniques to Label Constant Weight Gray Codes, in preparation.

[9] Inessa Levi and Steve Seif, Constructive Techniques for Labeling Constant Weight Gray Codes, in preparation.

[10] Carla Savage, A SURVEY OF COMBINATORIAL GRAY CODES, *SIAM Rev.*, Vol. **39**, No. 4, 605-629, December 1997.

[11] Carla D. Savage and Peter Winkler, Monotone Gray codes and the middle levels problem, *J. Combin. Theory Ser. A*, 70 (1995), no. 2, 230–248

[12] A.J. van Zanten, Index system and Separability of constant Weight Gray Codes, *IEEE Trans. Inform. Theory*, Vol. 91, Number 6, (1991), 1229-1223.