A Note on Distance-Regular Graphs with Girth 3 *

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Abstract

We give some relationships among the intersection numbers of a distance-regular graph Γ which contains a circuit (u_1, u_2, u_3, u_4) with $\partial(u_1, u_3) = 1$ and $\partial(u_2, u_4) = 2$. As an application, we obtain an upper bound of the diameter of Γ when $k \geq 2b_1$.

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1 Introduction

Let $\Gamma = (X, E)$ denote a finite, connected, undirected graph, without loops or multiple edges, with *vertex set* X and *edge set* E. We often write $V\Gamma$ for X and $E\Gamma$ for E. Let r denote a nonnegative integer and let u and v denote vertices of Γ. By a *path* of length r from u to v we mean a finite sequence of vertices $(u = w_0, w_1, \cdots, w_r = v)$ such that $(w_{t-1}, w_t) \in E\Gamma$ for $t = 1, \cdots, r$. By a circuit of length r we mean a path $(w_0, w_1, \cdots, w_{r-1})$ such that $r \geq 3$ and $(w_{r-1}, w_0) \in E\Gamma$. A shortest circuit is called a *minimal circuit*. The *girth* g of Γ is the length of a minimal circuit. The number of edges traversed in a shortest path joining u and v is called the *distance* between u and v, denoted by $\partial(u, v)$. Let d denote the maximal value of the distance function. We call d the diameter of Γ.

For vertices $u, v \in V\Gamma$, let

$$\Gamma_i(u) = \{x \in V\Gamma \mid \partial(u, x) = i\}, \quad D^i_j(u, v) = \Gamma_i(u) \cap \Gamma_j(v).$$

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For any two subsets Y and Z of $V\Gamma$, let $e(Y,Z)$ denote the number of edges (u,v) with $u \in Y$ and $v \in Z$. If Y contains a single vertex y, i.e., $Y = \{y\}$, we write as $e(y,Z)$.

A connected graph Γ is said to be *distance-regular* if, for any two vertices u and v at distance h, the parameters $p^h_{i,j} = |D^i_j(u,v)|$ depend only on i, j and h. The parameters

$$c_i = p^i_{i-1,1},\ a_i = p^i_{i,1},\ b_i = p^i_{i+1,1}$$

are called the *intersection numbers* of Γ. It is clear that $c_i + a_i + b_i = b_0$ for all i with $0 \le i \le d$, and $k = b_0$ is the valency of Γ.

In [2], Terwilliger found some relationships among the intersection numbers of a distance-regular graph Γ when Γ contains a circuit (u_1,u_2,u_3,u_4) with $\partial(u_1,u_3) = \partial(u_2,u_4) = 2$, and gave an upper bound of the diameter of Γ. In this paper, we apply Terwilliger's method to a distance-regular graph containing a circuit (u_1,u_2,u_3,u_4) with $\partial(u_1,u_3) = 1$ and $\partial(u_2,u_4) = 2$, and obtain some relationships among the intersection numbers of Γ. As an application, we obtain an upper bound of the diameter of Γ when $k \ge 2b_1$. Namely, our main results are the following.

Theorem 1.1 *Let Γ be a distance-regular graph of girth 3. For any two adjacent vertices u and v, let Δ be an induced subgraph on a nonempty subset of $D^1_1(u,v)$ such that $e(p,\Delta) < |\Delta| - 1$ for all $p \in \Delta$. Let $r = |\Delta|$ and $m = \frac{2|E\Delta|}{r}$. If $m \le \frac{r}{2} - 1$, then for all integers i $(1 \le i \le d-1)$ the intersection numbers of Γ satisfy the following.*

(*i*) $b_1 - c_i - b_{i+1} \ge \frac{r-2m-2}{2r}(\sqrt{b_{i+1}} + \sqrt{c_i})^2 - \frac{1}{2}(\sqrt{b_{i+1}} - \sqrt{c_i})^2$,

(*ii*) $b_1 - c_i - b_{i+1} \ge \min\{\sqrt{c_i}(\sqrt{b_{i+1}} - \sqrt{c_i}),\ \frac{c_i(r-2m-2)}{m+1}\}$,

(*iii*) $b_1 - c_i - b_{i+1} \ge \min\{\sqrt{b_{i+1}}(\sqrt{c_i} - \sqrt{b_{i+1}}),\ \frac{b_{i+1}(r-2m-2)}{m+1}\}$.

Corollary 1.2 *Let Γ be a distance-regular graph containing a circuit (u_1,u_2,u_3,u_4) with $\partial(u_1,u_3) = 1$ and $\partial(u_2,u_4) = 2$. If $k \ge 2b_1$, then $b_1 \ge c_i + b_{i+1}$ for $0 \le i \le d-1$. Moreover, we get*

$$d \le \frac{c_d + a_1 + 1}{a_1 - b_1 + 2}.$$

2 Proof of main results

In this section, we follow the notation in Theorem 1.1.

For each integer i with $1 \le i \le d$ and for each vertex $w \in D^i_i(u,v)$, set

$$U_i(w) = |\{y \mid y \in \Delta,\ \partial(y,w) = i+1\}|,$$

$$D_i(w) = |\{y \mid y \in \Delta,\ \partial(y,w) = i-1\}|.$$

Note that

$$U_i(w) + D_i(w) \le r,\ (1 \le i \le d).$$

For each i with $1 \le i \le d$, let

$$R_i = \{w \mid w \in D_i^i(u,v),\ U_i(w) \ge 1\}.$$

For each vertex $p \in \Delta$ and each i with $1 \le i \le d$, we define

$$u_i(p) = \{w \mid w \in R_i,\ \partial(w,p) = i+1\},$$

$$d_i(p) = \{w \mid w \in R_i,\ \partial(w,p) = i-1\}.$$

By computing the pairs (y,w) of vertices $y \in \Delta$ and $w \in R_i$ with $\partial(y,w) = i+1$, we get

$$\sum_{w \in R_i} U_i(w) = \sum_{p \in \Delta} |u_i(p)|. \tag{1}$$

Likewise, we have

$$\sum_{w \in R_i} D_i(w) = \sum_{p \in \Delta} |d_i(p)|. \tag{2}$$

Now we will follow Terwilliger's idea in [2] to prove Theorem 1.1. At first, We give a lemma.

Lemma 2.1 *Let $p \in \Delta$ and $m_i = \frac{k_i c_i}{k_2 c_2}$. Then the following inequalities hold.*

(a) $|u_{i-1}(p)| + |d_i(p)| \le m_i b_1,\ (2 \le i \le d)$,

(b) $|d_i(p)| \ge \frac{b_i |d_{i-1}(p)|}{c_{i-1}},\ (2 \le i \le d)$,

(c) $|d_i(p)| \ge m_i b_i,\ (1 \le i \le d)$,

(d) $|u_i(p)| \ge m_i b_i,\ (1 \le i \le d)$,

(e) $\frac{1}{(r-m-1)m_i b_i} \ge \frac{1}{\sum_{p \in \Delta} |d_i(p)|} + \frac{1}{\sum_{p \in \Delta} |u_i(p)|},\ (1 \le i \le d-1)$.

Proof. (a). For all positive integers r, s, and t, let

$$n(r,s,t) = |\{w \mid \partial(w,u) = r,\ \partial(w,p) = s,\ \partial(w,v) = t\}|.$$

Then the following equalities hold.

$$n(i-1,i,i-1) + n(i-1,i,i) = p^1_{i-1,i}, \tag{3}$$

$$n(i-1,i-1,i)+n(i,i-1,i)=p^1_{i-1,i}, \tag{4}$$

$$n(i-1,i-1,i)+n(i-1,i,i)=p^1_{i-1,i}, \tag{5}$$

By adding (3) and (4), and subtracting (5), we get

$$n(i-1,i,i-1)+n(i,i-1,i)=p^1_{i-1,i}.$$

Since $n(i-1,i,i-1)=|u_{i-1}(p)|$ and $n(i,i-1,i)\geq|d_i(p)|$, we have

$$|u_{i-1}(p)|+|d_i(p)|\leq p^1_{i-1,i}=m_ib_1.$$

(b). For each $w\in d_{i-1}(p)$, let $\bar{w}\in\Delta$ with $\partial(w,\bar{w})=i$. Pick $y\in D^{i+1}_1(\bar{w},w)$. Then we get $\partial(y,u)=\partial(y,v)=i$ and $\partial(y,p)=i-1$, so that $y\in d_i(p)$. Therefore $e(d_{i-1}(p),d_i(p))\geq|d_{i-1}(p)|b_i$. On the other hand, each vertex $y\in d_i(p)$ is adjacent to at most c_{i-1} vertices in $d_{i-1}(p)$, so $e(d_{i-1}(p),d_i(p))\leq|d_i(p)|c_{i-1}$. Consequently, $|d_i(p)|c_{i-1}\geq|d_{i-1}(p)|b_i$.

(c) and (d). By assumption, there exists a vertex $q\in\Delta$ such that $\partial(p,q)=2$. For any vertex $w\in D^{i-1}_{i+1}(p,q)$, we have $w\in d_i(p)\cap u_i(q)$. Hence

$$|d_i(p)|\geq p^2_{i-1,i+1}=m_ib_i,\ |u_i(q)|\geq p^2_{i-1,i+1}=m_ib_i.$$

(e). For any integer i with $1\leq i\leq d$, let

$$Y_i=\{(w_1,w_2,w_3)\mid w_2,w_3\in\Delta,\ w_1\in d_i(w_2)\cap u_i(w_3)\}.$$

It is obvious that $|Y_i|=\sum_{w\in R_i}U_i(w)D_i(w)$. We may also write

$$Y_i=\{(y_1,y_2,y_3)\mid y_2,y_3\in\Delta,\ \partial(y_2,y_3)=2,\ y_1\in D^{i-1}_{i+1}(y_2,y_3)\},$$

so

$$\begin{aligned}|Y_i|&=|\{(y_2,y_3)\mid y_2,y_3\in\Delta,\ \partial(y_2,y_3)=2\}|p^2_{i-1,i+1}\\&=r(r-m-1)m_ib_i.\end{aligned}$$

Consequently, we obtain

$$\sum_{w\in R_i}U_i(w)D_i(w)=r(r-m-1)m_ib_i. \tag{6}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}&(\textstyle\sum_{w\in R_i}U_i(w)D_i(w))^2\\&\leq(\textstyle\sum_{w\in R_i}U_i(w)^2)(\sum_{w\in R_i}D_i(w)^2)\\&\leq(\textstyle\sum_{w\in R_i}U_i(w)(r-D_i(w)))(\sum_{w\in R_i}D_i(w)(r-U_i(w))).\end{aligned}$$

Solve for $r(\sum_{w\in R_i} U_i(w)D_i(w))^{-1}$ in above inequality to get

$$\frac{r}{\sum_{w\in R_i} U_i(w)D_i(w)} \geq \frac{1}{\sum_{w\in R_i} D_i(w)} + \frac{1}{\sum_{w\in R_i} U_i(w)}.$$

Applying (1), (2) and (6) to the above inequality, we get (e). ■

Proof of Theorem 1.1. Let

$$E_i = \sum_{p\in\Delta} |d_i(p)|, \; F_i = \sum_{p\in\Delta} |u_i(p)|.$$

Then (a) and (b) yield

$$\frac{b_{i+1}}{c_i}E_i + F_i \leq \frac{rm_ib_ib_1}{c_i}, \tag{7}$$

and (c), (d) and (e) can be rewritten as

$$E_i \geq rm_ib_i, \tag{8}$$

$$F_i \geq rm_ib_i, \tag{9}$$

$$\frac{1}{(r-m-1)m_ib_i} \geq \frac{1}{E_i} + \frac{1}{F_i}. \tag{10}$$

If $i = d-1$, by (7) and (9), we have

$$b_1 - c_{d-1} \geq 0,$$

which implies the theorem holds.

Now we consider the case $1 \leq i \leq d-2$. Combining (7)–(10), we obtain

$$\frac{1}{(r-m-1)m_ib_i} \geq \frac{1}{E_i} + \frac{1}{rm_ib_1b_ic_i^{-1} - b_{i+1}E_ic_i^{-1}}, \tag{11}$$

where

$$rm_ib_i \leq E_i \leq \frac{rm_ib_i(b_1-c_i)}{b_{i+1}}. \tag{12}$$

Let $s = \frac{rm_ib_1b_i}{c_i}$ and $w = \frac{rm_ib_i(b_1-c_i)}{b_{i+1}}$. By inequalities (11) and (12), it is clear that

$$f(y) = \frac{1}{(r-m-1)m_ib_i} - \frac{1}{y} - \frac{1}{s - b_{i+1}c_i^{-1}y}.$$

is nonnegative somewhere in the range $[rm_ib_i, w]$. Since

$$\lim_{y\to 0^+} f(y) = \lim_{y\to \frac{sc_i}{b_{i+1}}^-} f(y) = -\infty,$$

in $(0, \frac{sc_i}{b_{i+1}})$, $f(y)$ has a maximum at

$$y_0 = s(\sqrt{\frac{b_{i+1}}{c_i}} + \frac{b_{i+1}}{c_i})^{-1} \geq 0.$$

Therefore, we have

$$\frac{1}{(r-m-1)m_ib_i} \geq \frac{1}{y_0} + \frac{1}{s - b_{i+1}c_i^{-1}y_0},$$

i.e.,

$$\frac{rb_1}{r-m-1} \geq (\sqrt{c_i} + \sqrt{b_{i+1}})^2,$$

which reduces to

$$\begin{aligned} & b_1 - c_i - b_{i+1} \\ & \geq \tfrac{-m-1}{r}(\sqrt{c_i} + \sqrt{b_{i+1}})^2 + 2\sqrt{b_{i+1}c_i} \\ & \geq \tfrac{r-2m-2}{2r}(\sqrt{c_i} + \sqrt{b_{i+1}})^2 - \tfrac{1}{2}(\sqrt{b_{i+1}} - \sqrt{c_i})^2, \end{aligned}$$

which is (i). Next suppose $b_1 - c_i - b_{i+1} \leq \sqrt{c_i}(\sqrt{b_{i+1}} - \sqrt{c_i})$, then $y_0 \leq m_ib_ir$. In this case, $f(y)$ is decreasing in $[m_ib_ir, w]$, and so $f(m_ib_ir) \geq 0$, i.e.,

$$b_1 - c_i - b_{i+1} \geq c_i\frac{r-2m-2}{m+1}.$$

Consequently, (ii) is valid. In a similar way, we can prove (iii).

Hence, we complete the proof of Theorem 1.1. ■

Proof of Corollary 1.2. Let Δ be the induced subgraph on $\{u_2, u_4\}$. Then r and m in Theorem 1.1 are 2 and 0, respectively. Applying (ii) if $b_{i+1} \geq c_i$, or applying (iii) if $b_{i+1} < c_i$, we get

$$b_1 \geq c_i + b_{i+1},\ (1 \leq i \leq d-1),$$

so

$$(d-1)b_1 \geq \sum_{i=1}^{d} c_i + \sum_{i=1}^{d} b_i - c_d - b_1. \tag{13}$$

Proposition 5.5.1 in [1] tells us that

$$b_i + c_{i+1} \geq a_1 + 2,\ (1 \leq i \leq d-1),$$

which implies that

$$\sum_{i=1}^{d} c_i + \sum_{i=1}^{d} b_i - 1 \geq (d-1)(a_1 + 2). \tag{14}$$

Combining (13) and (14), we find that

$$(a_1 + 2 - b_1)d \leq c_d + a_1 + 1.$$

Since $k \geq 2b_1$, $a_1 - b_1 + 2$ is a positive integer. We divide the both sides of the above inequality by $a_1 - b_1 + 2$ to obtain

$$d \leq \frac{c_d + a_1 + 1}{a_1 - b_1 + 2},$$

as desired. ■

References

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