Lower bounds on the signed k-subdomination number of graphs [1]

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Abstract

Let $G = (V, E)$ be a simple graph. For any real valued function $f : V \to R$ and $S \subseteq V$, let $f(S) = \sum_{v \in S} f(u)$. A signed k-subdominating function is a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for at least k vertices $v \in V$. The signed k-subdomination number of a graph G is $\gamma_{ks}^{-11}(G) = \min\ \{f(V) | f$ is a signed k-subdominating function on $G\}$. In this paper, we obtain lower bounds on this parameter and extend some results in other papers.

Key words: majority domination number; signed k-subdomination number

1. Introduction

Let $G = (V, E)$ be a simple graph and v a vertex in V. The open neighborhood of v, denoted by $N(v)$, is the set of vertices adjacent to v, i.e. $N(v) = \{u \in V | uv \in E\}$. The closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. The degree of v is $d(v) = |N(v)|$. The minimum degree and maximum degree of the vertices of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. When no ambiguity can occur, we often simply write δ, Δ instead of $\delta(G)$ and $\Delta(G)$.

For any real valued function $f : V \to R$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The weight of f is defined as $f(V)$. We will denote $f(N[v])$ by $f[v]$ where $v \in V$. A function $f : V \to \{-1, 1\}$ is a majority dominating function on G if for at least half the vertices $v \in V$, $f[v] \geq 1$. The majority domination number is defined as $\gamma_{maj}(G) = \min\ \{f(V) | f$ is a

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majority dominating function on $G\}$. A signed k-subdominating function for G is defined as a function $f : V \to \{-1, 1\}$ such that $f[v] \geq 1$ for at least k vertices of G. The signed k-subdomination number of a graph G is $\gamma_{ks}^{-11}(G)$ =min$\{f(V)|f$ is a signed k-subdominating function on $G\}$. In this paper, we obtain lower bounds on the signed k-subdomination number of a graph G in terms of the order, minimum and maximum degree.

2. Main results

Theorem 1 *For any graph G with order n and $1 \leq k \leq n$,*

$$\gamma_{ks}^{-11}(G) \geq \frac{(\delta - 3\Delta - 2)n + 2(\Delta + 2)k}{\Delta + \delta + 2}.$$

Proof Let f be a minimum signed k-subdominating function on G. Partition the vertices of G based on their degree and function value as follows. Let

$$P_\Delta = \{v \in V | f(v) = 1 \text{ and } d(v) = \Delta\},$$

$$P_\delta = \{v \in V | f(v) = 1 \text{ and } d(v) = \delta\},$$

$$P_\Theta = \{v \in V | f(v) = 1 \text{ and } \delta < d(v) < \Delta\},$$

$$M_\Delta = \{v \in V | f(v) = -1 \text{ and } d(v) = \Delta\},$$

$$M_\delta = \{v \in V | f(v) = -1 \text{ and } d(v) = \delta\},$$

$$M_\Theta = \{v \in V | f(v) = -1 \text{ and } \delta < d(v) < \Delta\}.$$

Further, we let

$$V_\Delta = P_\Delta \cup M_\Delta, \quad V_\delta = P_\delta \cup M_\delta, \quad V_\Theta = P_\Theta \cup M_\Theta,$$

$$P = P_\Delta \cup P_\delta \cup P_\Theta, \quad M = M_\Delta \cup M_\delta \cup M_\Theta.$$

For at least k vertices $x \in V$, $f[x] \geq 1$. Hence we have

$$\sum_{x \in V} f[x] \geq k - (\Delta + 1)(n - k) = (\Delta + 2)k - (\Delta + 1)n. \tag{1}$$

In this sum $f(x)$ is added a total of $d(x) + 1$ times, for each vertex x. Therefore, we have

$$\sum_{x \in V} f(x)(d(x)+1) \geq (\Delta+2)k - (\Delta+1)n. \tag{2}$$

Breaking the sum up into the six summations and replacing $f(x)$ with the corresponding value of 1 or -1 yields

$$\begin{aligned}
&\sum_{x \in P_\Delta} (d(x)+1) + \sum_{x \in P_\delta} (d(x)+1) + \sum_{x \in P_\Theta} (d(x)+1) \\
&- \sum_{x \in M_\Delta} (d(x)+1) - \sum_{x \in M_\delta} (d(x)+1) - \sum_{x \in M_\Theta} (d(x)+1) \geq (\Delta+2)k \\
&- (\Delta+1)n.
\end{aligned}$$

We know that $d(x) = \Delta$ for all x in V_Δ and $d(x) = \delta$ in V_δ and $\delta + 1 \leq d(x) \leq \Delta - 1$ in V_Θ. Therefore, we have

$$\begin{aligned}
&\sum_{x \in P_\Delta} (\Delta+1) + \sum_{x \in P_\delta} (\delta+1) + \sum_{x \in P_\Theta} (\Delta) \\
&- \sum_{x \in M_\Delta} (\Delta+1) - \sum_{x \in M_\delta} (\delta+1) - \sum_{x \in M_\Theta} (\delta+2) \geq (\Delta+2)k - (\Delta+1)n.
\end{aligned}$$

Thus

$$\begin{aligned}
&(\Delta+1)|P_\Delta| + (\delta+1)|P_\delta| + \Delta|P_\Theta| - (\Delta+1)|M_\Delta| - (\delta+1)|M_\delta| \\
&- (\delta+2)|M_\Theta| \ \geq (\Delta+2)k - (\Delta+1)n.
\end{aligned}$$

For $i \in \{\Delta, \delta, \Theta\}$, we replace $|P_i|$ with $|V_i| - |M_i|$ in the above inequality. Therefore, we have

$$\begin{aligned}
&(\Delta+1)|V_\Delta| + (\delta+1)|V_\delta| + \Delta|V_\Theta| - 2(\Delta+1)|M_\Delta| - 2(\delta+1)|M_\delta| \\
&- (\Delta+\delta+2)|M_\Theta| \ \geq (\Delta+2)k - (\Delta+1)n.
\end{aligned}$$

Thus we have

$$\begin{aligned} 2(\Delta+1)n-(\Delta+2)k &\geq (\Delta-\delta)|V_\delta|+|V_\Theta|+2(\Delta+1)|M_\Delta|+2(\delta+1)|M_\delta| \\ &\quad +(\Delta+\delta+2)|M_\Theta|. \end{aligned}$$

Hence

$$\begin{aligned} 2(\Delta+1)n-(\Delta+2)k &\geq 2(\Delta+1)|M_\Delta|+(\Delta+\delta+2)|M_\delta|+(\Delta+\delta+3)|M \\ &\geq (\Delta+\delta+2)|M|. \end{aligned}$$

Therefore we have that

$$|M| \leq \frac{2(\Delta+1)n-(\Delta+2)k}{\Delta+\delta+2}. \tag{3}$$

Since $\gamma_{ks}^{-11}(G)=n-2|M|$, we have that

$$\gamma_{ks}^{-11}(G) \geq \frac{(\delta-3\Delta-2)n+2(\Delta+2)k}{\Delta+\delta+2}.$$

This completes the proof. ∎

Corollary 1 *([2]) If G is a k−regular graph, k is even, of order n, then*

$$\gamma_{maj}(G) \geq \frac{-k}{2(k+1)}n.$$

Corollary 2 *([1]) If $n \geq 3$ and $1 \leq k \leq n$, then for every r-regular ($r \geq 2$ is even) graph G of order n,*

$$\gamma_{ks}^{-11}(G) \geq \frac{r+2}{r+1}k-n.$$

If v is a vertex of odd degree in a graph G, and if f is a signed k-subdomination function of G, then $f[v]$ is even. Hence, using an almost identical proof to that of Theorem 1 we have the following result.

Theorem 2 *For any graph G of order n with all vertices odd degree and $1 \leq k \leq n$, then*

$$\gamma_{ks}^{-11}(G) \geq \frac{(\delta - 3\Delta - 2)n + 2(\Delta + 3)k}{\Delta + \delta + 2}.$$

Corollary 3 *([2]) If G is a $k-$regular graph, k is odd, of order n, then*

$$\gamma_{maj}(G) \geq \frac{1-k}{2(k+1)}n.$$

Corollary 4 *([1]) If $n \geq 3$ and $1 \leq k \leq n$, then for every r-regular ($r \geq 2$) graph G of order n, r is odd, then*

$$\gamma_{ks}^{-11}(G) \geq \frac{r+3}{r+1}k - n.$$

References

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