PERMUTATIONS RESTRICTED BY PATTERNS OF TYPE (2, 1)

TOUFIK MANSOUR

LaBRI (UMR 5800), Université Bordeaux 1,
351 cours de la Libération, 33405 Talence Cedex, France
toufik@labri.fr

ABSTRACT

Recently, Babson and Steingrímsson (see [BS]) introduced generalized permutations patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation.

In this paper we study the generating functions for the number of permutations on n letters avoiding a generalized pattern *ab-c* where $(a, b, c) \in S_3$, and containing a prescribed number of occurrences of a generalized pattern *cd-e* where $(c, d, e) \in S_3$. As a consequence, we derive all the previously known results for this kind of problem, as well as many new results.

1. INTRODUCTION

Classical patterns. Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ*-avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids any $\tau \in T$; the corresponding subset of S_n is denoted $S_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SS]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]), and for $\tau_1, \tau_2 \in S_4$ (see [B, K] and references therein). Several recent papers deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2 (see [CW, Kr, MV2] and references therein). Another natural question is to study permutations avoiding τ_1 and containing τ_2 exactly t times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $t = 1$ was investigated in [R], and for certain $\tau_1 \in S_3$, $\tau_2 \in S_k$ in [RWZ, MV1, Kr, MV2, MV3, MV4]. The tools involved in these papers include continued fractions, Chebyshev polynomials, and Dyck paths.

Generalized patterns. In [BS] Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation.

Following [C], we define our *generalized patterns* as words with letters $1, 2, 3, \ldots$ where two adjacent letters may or may not be separated by a dash. The absence of a dash between two adjacent letters in a pattern indicates that the corresponding letters in the permutation must be adjacent, and in the order (order-isomorphic) given by the pattern. For example, the subword 23-1 of a permutation $\pi = (\pi_1, \pi_2, \cdots, \pi_n)$ is a subword $(\pi_i, \pi_{i+1}, \pi_j)$ where $i + 1 < j$ such that $\pi_j < \pi_i < \pi_{i+1}$. We say that τ a generalized pattern of type $(2, 1)$ if it has the form ab-c where $(a, b, c) \in S_3$.

Remark 1.1. *There exist six generalized patterns of type* $(2,1)$ *which are* 12-3, 13-2, 21-3, 23-1, 31-2, *and* 32-1. *By the complement symmetric operation (that is* $(\pi_1, \pi_2, \ldots, \pi_n) \mapsto (n+1-\pi_1, n+1-\pi_2, \ldots, n+1-\pi_n)$*) we get three different classes:* $\{12\text{-}3, 32\text{-}1\}$, $\{13\text{-}2, 31\text{-}2\}$, *and* $\{21\text{-}3, 23\text{-}1\}$.

While the case of permutations avoiding a single generalized pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of generalized patterns τ_1, τ_2. This problem was solved completely for τ_1, τ_2 two generalized patterns of length three with exactly one adjacent pair of letters (see [CM1]). Claesson and Mansour [CM2] showed (using a result [CSZ, Corollary 11] by Clarke, Steingrímsson and Zeng) that the distribution of the patterns 2-31 and 31-2 is given by Stieltjes continued fraction as follows.

Theorem 1.2. *We have*

$$\sum_{n\geq 0}\sum_{\pi\in S_n} x^{1+(12)\pi} y^{(21)\pi} p^{(2\text{-}31)\pi} q^{(31\text{-}2)\pi} t^{|\pi|} = \cfrac{1}{1-\cfrac{x[1]_{p,q}t}{1-\cfrac{y[1]_{p,q}t}{1-\cfrac{x[2]_{p,q}t}{1-\cfrac{y[2]_{p,q}t}{\ddots}}}}},$$

where $[n]_{p,q} = q^{n-1} + pq^{n-2} + \cdots + p^{n-2}q + p^{n-1}$, $(\tau)\pi$ *is the number of occurrences of* τ *in* π.

Several recent papers deal with the case where τ_1 is a generalized pattern of length 3 and τ_2 is a generalized pattern of length k, for various pairs τ_1, τ_2 (see [M1, M2, M3] and references therein). Another natural question is to study permutations avoiding τ_1 and containing τ_2 exactly t times. Such a problem for certain generalized patterns τ_1 and τ_2 of length 3 and $t = 0$ was investigated in [CM2], and for certain $\tau_1 \in S_3$, $\tau_2 \in S_k$ in [M1, M2, M3]. The tools involved in these papers include continued fractions and Chebyshev polynomials of the second kind.

In the present paper, as follow-up to [CM1] (see also [C, CM2]), we develop a general approach to study the number of permutations avoiding a generalized pattern of type $(2,1)$ and containing a prescribed number of occurrences of a generalized pattern τ of type $(2,1)$. As a consequence, we derive all the previously known results for this kind of problem, as well as many new results.

2. Avoiding 12-3

Let $f_{\tau;r}(n)$ be the number of all permutations in $S_n(12\text{-}3)$ containing τ exactly r times. We denote the corresponding exponential and ordinary generating function by $\mathcal{F}_{\tau;r}(x)$ and $F_{\tau;r}(x)$ respectively; that is, $\mathcal{F}_{\tau;r}(x) = \sum_{n\geq 0} \frac{f_{\tau;r}(n)}{n!} x^n$ and $F_{\tau;r}(x) = \sum_{n\geq 0} f_{\tau;r}(n)x^n$. We extend the above definitions by $f_{\tau;r}(n) = 0$ for any τ and $r < 0$.

Our aim is to count the number of permutations avoiding 12-3 and avoiding (or containing exactly r times) an arbitrary generalized pattern τ. Let $f_{\tau;r}(n; i_1, i_2, \dots, i_j)$ be the number of permutations $\pi \in S_n(12\text{-}3)$ containing τ exactly r times such that $\pi_1\pi_2\dots\pi_j = i_1 i_2 \dots i_j$.

The main body of this section is divided into 6 subsections corresponding to the cases: τ is a general generalized pattern, 13-2, 21-3, 23-1, 31-2, and 32-1.

2.1. τ is a generalized pattern of length k.

Here we study certain cases of τ, where τ is a generalized pattern of length k without dashes, or with exactly one dash.

Theorem 2.1. *Let $k \geq 2$ and $P_k(x) = \sum_{j=0}^{k-2} \frac{x^j}{j!}$. We define $G_{-1}(x) = 0$, $G_0(x) = e^x - P_k(x)$, and $G_s(x) = G_0(x) \int_0^x G_{s-1}(t)dt$ for all $s \geq 1$. Then*

$$\mathcal{F}_{(k-1)\dots 21k;r}(x) = e^{\int_0^x P_k(t)dt} \int_0^x G_{r-1}(t)dt.$$

Proof. Let $\alpha \in S_n(12\text{-}3)$ such that $\alpha_j = n$; so $\alpha_1 > \alpha_2 > \cdots > \alpha_{j-1}$. Therefore, α contains $\tau = (k-1)\dots 21k$ exactly r times if and only if

$(\alpha_{j+1},\ldots,\alpha_n)$ contains τ exactly r times if $j \le k-1$, and contains τ exactly $r-1$ times if $j \ge k$. Thus

$$f_{\tau;r}(n) = \sum_{j=1}^{k-1} \binom{n-1}{j-1} f_{\tau;r}(n-j) + \sum_{j=k}^{n} \binom{n-1}{j-1} f_{\tau;r-1}(n-j).$$

Multiplying by $x^n/(n-1)!$ and summing over all $n \ge 1$ we get

$$\tfrac{d}{dx}\mathcal{F}_{(k-1)\ldots 21k;r}(x) = P_k(x)(\mathcal{F}_{(k-1)\ldots 21k;r}(x) - \mathcal{F}_{(k-1)\ldots 21k;r-1}(x)) + \\ + e^x \mathcal{F}_{(k-1)\ldots 21k;r-1}(x).$$

The rest is easy to check. □

Example 2.2. (see Claesson [C]) *Theorem 2.1 yields for $r=0$ that*

$$\mathcal{F}_{(k-1)\ldots 21k;0}(x) = e^{\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^{k-1}}{(k-1)!}}.$$

If $k \to \infty$, then we get that the exponential generating function for the number of 12-3*-avoiding permutations in S_n is given by e^{e^x-1}. Besides, Theorem 2.1 yields for given $k \ge 2$ and $r \to \infty$ that $G_s(x) \to e^{e^x - \int_0^x P_k(t)dt}$, so $\mathcal{F}_{(k-1)\ldots 21k;r}(x) \to e^{e^x-1}$.*

Claesson [C] (see also [CM1, Proposition 28]) proved that the number of permutations in S_n(12-3, 21-3) is the same number of involutions in S_n. The case of varying k is more interesting. As an extension of this result.

Theorem 2.3. *For $k \ge 2$,*

$$\mathcal{F}_{(k-1)\ldots 21\text{-}k;0}(x) = e^{\frac{x^1}{1!}+\cdots+\frac{x^{k-1}}{(k-1)!}}.$$

Proof. Let $\tau = (k-1)\ldots 21$-k, by definitions we get

$$f_{\tau;0}(n) = \sum_{j=1}^{n} f_{\tau;0}(n;j), \quad f_{\tau;0}(n;n) = f_{\tau;0}(n-1), \tag{1}$$

and

$$f_{\tau;0}(n;i_1,\ldots,i_j) = \sum_{i_{j+1}=1}^{i_j-1} f_{\tau;0}(n;i_1,\ldots,i_j,i_{j+1}) + f_{\tau;0}(n;i_1,\ldots,i_j,n) \tag{2}$$

for all $n-1 \ge i_1 > i_2 > \cdots > i_j \ge 1$. Therefore, since $f_{\tau;0}(n;i_1,\ldots,i_j) = 0$ for all $n-1 > i_1 > \cdots > i_j \ge 1$ where $j \ge k-1$ and since

$$f_{\tau;0}(n;i_1,\ldots,i_j,n) = f_{\tau;0}(n-1-j)$$

for all $n-1 > i_1 > \cdots > i_j \ge 1$ where $0 \le j \le k-2$ we have for all $n \ge 1$

$$f_{\tau;0}(n) = \sum_{j=0}^{k-2} \binom{n-1}{j} f_{\tau;0}(n-1-j).$$

The rest is easy to see as proof of Theorem 2.1. □

In view of Example 2.2 and Theorem 2.3 we get that the number of permutations in $S_n(12\text{-}3,(k-1)\dots 21\text{-}k)$ is the same number of permutations in $S_n(12\text{-}3,(k-1)\dots 21k)$. In addition,

$$S_n(12\text{-}3,(k-1)\dots 21\text{-}k) = S_n(12\text{-}3,(k-1)\dots 21k), \tag{3}$$

which can prove as follows. Let $\alpha = (\alpha', n, \alpha'')$; since α avoids 12-3 we get α' decreasing, so by the principle of induction on length of α it is easy to see that α avoids $(k-1)\dots 21k$ if and only if avoids $(k-1)\dots 21\text{-}k$.
In [CM1] showed that the number of permutations in $S_n(12\text{-}3, 13\text{-}2)$ is given by 2^{n-1}. We generalize this result in the theorem below.

Theorem 2.4. *Let $k \geq 3$, then for all $n \geq 1$*

$$\begin{aligned} &f_{(k-2)\dots 21k\text{-}(k-1);0}(n) = \\ &= \sum_{j=0}^{k-3} \binom{n-1}{j} f_{(k-2)\dots 21k\text{-}(k-1);0}(n-1-j) + \\ &\qquad + \sum_{j=k-2}^{n-1} \binom{n-j+k-4}{k-3} f_{(k-2)\dots 21k\text{-}(k-1);0}(n-1-j). \end{aligned}$$

Proof. Let $\tau = (k-2)\dots 21k\text{-}(k-1)$ and let $n-1 \geq i_1 > \cdots > i_j \geq 1$, so for $j \leq k-3$

$$f_{\tau;0}(n; i_1, \dots, i_j, n) = f_{\tau;0}(n-1-j)$$

and for $j \geq k-2$

$$\begin{aligned} &f_{\tau;0}(n; i_1, \dots, i_j, n) = \\ &\quad = f_{\tau;0}(n; n-1, \dots, n-(j-k+3), i_{j-k+4}, \dots, i_j, n) = \\ &\quad = f_{\tau;0}(n-j). \end{aligned}$$

Therefore, Equation (1) and Equation (2) yield the desired result. □

Example 2.5. (see Claesson and Mansour [CM2]) *Theorem 2.4 yields for $k=3$ that the number of permutations in $S_n(12\text{-}3, 13\text{-}2)$ is given by 2^{n-1}. As another example, for $k=4$ we get*

$$\mathcal{F}_{214\text{-}3;0}(x) = 1 + \int_0^x e^{2t+t^2/2}dt.$$

As a remark, similarly as proof of Equation (3), we have for all $k \geq 3$

$$\mathcal{F}_{(k-2)\dots 21k\text{-}(k-1);0}(x) = \mathcal{F}_{(k-2)\dots 21\text{-}k\text{-}(k-1);0}(x).$$

2.2. $\tau = 13\text{-}2$.

Theorem 2.6. *Let r be a nonnegative integer. Then*

$$F_{13\text{-}2;r}(x) = \frac{1-x}{1-2x}\delta_{r,0} + \frac{x^2}{1-2x}\sum_{d=1}^{r} \frac{F_{13\text{-}2;r-d}(x) - \sum_{j=0}^{d-1} f_{13\text{-}2;r-d}(j)x^j}{(1-x)^d}.$$

Proof. Let $r \geq 0$, $b_r(n) = f_{13\text{-}2;r}(n)$, and let $1 \leq i \neq j \leq n$. If $i < j$, then since the permutations avoiding 12-3 we have $b_r(n; i, j) = 0$ for $j \leq n-1$ and $b_r(n; i, n) = b_{r-(n-1-i)}(n-2)$, hence

$$\sum_{j=i+1}^{n} b_r(n; i, j) = b_{r-(n-1-i)}(n-1; n-1) = b_{r-(n-1-i)}(n-2).$$

If $i > j$ then by definitions we have

$$b_r(n; i, j) = b_r(n-1; j).$$

Owing to Equation (2) we have showed that, for all $1 \leq i \leq n-2$,

$$b_r(n; i) = b_{r-(n-1-i)}(n-2) + \sum_{j=1}^{i-1} b_r(n-1; j). \tag{4}$$

Moreover, it is plain that

$$b_r(n; n) = b_r(n; n-1) = b_r(n-1), \tag{5}$$

and by means of induction we shall show that Equation (4) implies: If $2 \leq m \leq n-1$ then

$$\begin{aligned} b_r(n; n-m) = &\sum_j (-1)^j \left[\binom{m-1}{j} + \binom{m-2}{j-1}\right] b_r(n-1-j) + \\ &+ b_{r-(m-1)}(n-2) - \sum_{d\geq 1}\sum_j (-1)^j \binom{m-2-d}{j} b_{r-d}(n-3-j). \end{aligned} \tag{6}$$

First we verify the statement for $m = 2$; in this case Equation (6) becomes

$$b_r(n; n-2) = b_r(n-1) - 2b_r(n-2) + b_{r-1}(n-2).$$

Indeed,

$$\begin{aligned} b_r(n; n-m) &= \\ &= \sum_{j=1}^{n-m-1} b_r(n; n-m, j) + \sum_{j=n-m+1}^{m} b_r(n; n-m, j) \\ &= \sum_{j=1}^{n-m-1} b_r(n-1; j) + b_{r-(m-1)}(n-1; n-1) \\ &= b_r(n-1) - 2b_r(n-2) + b_{r-(m-1)}(n-2) \\ &\qquad - \sum_{k=2}^{m-1} b_r(n-1; n-1-k), \end{aligned} \tag{7}$$

where the three equalities follow from Equation (2), and Equation (1) together with (4) and (5), respectively. Now simply put $m = 2$ to obtain Equation (6).

Assume that Equation (7) holds for all k such that $2 \leq k \leq m-1$. Then, employing the familiar identity $\binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$, the trailing sum in Equation (7) expands as follows. Since

$$\sum_{k=2}^{m-1}\sum_j(-1)^j\left[\binom{k-1}{j}+\binom{k-2}{j-1}\right]b_r(n-2-j)=$$
$$=\sum_j(-1)^j\left[\binom{m-1}{j+1}+\binom{k-2}{j}\right]b_r(n-2-j)-b_r(n-2)=$$
$$=-\sum_j(-1)^j\left[\binom{k-1}{j}+\binom{k-2}{j-1}\right]b_r(n-1-j)+b_r(n-1)-2b_r(n-2),$$

$$\sum_{k=2}^{m-1} b_{r-(k-1)}(n-3)=\sum_{d\geq 1} b_{r-d}(n-3),$$

and

$$\sum_{k=2}^{m-1}\sum_{d\geq1}\sum_j(-1)^j\binom{k-2-d}{j}b_{r-d}(n-4-j)=$$
$$=-\sum_{d\geq1}\sum_j(-1)^j\binom{m-2-d}{j}b_{r-d}(n-3-j)+\sum_{d\geq1} b_{r-d}(n-3)$$

with using Equation (7) we get that Equation (6) holds for $k=m$, by the principle of induction the universal validity of Equation (6) follows.

Now, if summing $b_r(n;n-m)$ over all $0\leq m\leq n-1$, then by using Equation (1), (5), and (6) we get

$$\sum_j(-1)^j\left[\binom{n-1}{j}+\binom{n-2}{j-1}\right]b_r(n-j)=$$
$$=\sum_{d\geq1}\sum_j(-1)^j\binom{n-2-d}{j}b_{r-d}(n-2-j). \tag{8}$$

Using [CM2, Lemma 7] to transfer the above equation in terms of ordinary generating functions

$$(1-u)F_{13\text{-}2;r}\left(\tfrac{u}{1+u}\right)=\delta_{r,0}+$$
$$+u^2\sum_{d=1}^{r}(1+u)^{d-1}\left[F_{13\text{-}2;r-d}\left(\tfrac{u}{1+u}\right)-\sum_{j=0}^{d-1}f_{13\text{-}2;r-d}(j)\left(\tfrac{u}{1+u}\right)^j\right].$$

Putting $x=u/(1+u)$ $(u=x/(1-x))$ we get the desired result. □

As application of Theorem 2.6 we get the exact formula for $f_{13\text{-}2;r}(n)$ for $r=0,1,2,3,4$.

Corollary 2.7. *For all $n \geq 1$,*

$$f_{13\text{-}2;0}(n) = 2^{n-1};$$

$$f_{13\text{-}2;1}(n) = (n-3)2^{n-2} + 1;$$

$$f_{13\text{-}2;2}(n) = (n^2 - 3n - 6)2^{n-4} + n;$$

$$f_{13\text{-}2;3}(n) = \tfrac{1}{3}(n^3 - 31n - 18)2^{n-5} + n^2 - n + 1;$$

$$f_{13\text{-}2;4}(n) = \tfrac{1}{3}(n-1)(n^3 + 7n^2 - 546n - 312)2^{n-8} + \\ + \tfrac{2}{3}(n-1)(n^2 - 2n + 3).$$

2.3. $\tau = 21\text{-}3$. The following lemma is the base of all the other results in this subsection, which holds immediately from definitions.

Lemma 2.8. *Let $n \geq 1$; then*

$$f_{21\text{-}3;r}(n) = f_{21\text{-}3;r} + \sum_{i=1}^{n-1} f_{21\text{-}3;r}(n;i),$$
$$f_{21\text{-}3;r}(n;i) = f_{21\text{-}3;r}(n-2) + \sum_{j=1}^{i-1} f_{21\text{-}3;r-(n-i)}(n-2;j), \;\; 1 \leq i \leq n-1.$$

Using the above lemma for given r we obtain the exact formula for $f_{21\text{-}3;r}$. Here we present the first three cases $r = 0, 1, 2$.

Theorem 2.9. *For all $n \geq 1$*

$$f_{21\text{-}3;0}(n) = f_{21\text{-}3;0}(n-1) + (n-1)f_{21\text{-}3;0}(n-2);$$

$$f_{21\text{-}3;1}(n) = f_{21\text{-}3;1}(n-1) + (n-1)f_{21\text{-}3;1}(n-2) + \\ + f_{21\text{-}3;0}(n-1) - f_{21\text{-}3;0}(n-2);$$

$$f_{21\text{-}3;2}(n) = f_{21\text{-}3;2}(n-1) + (n-1)f_{21\text{-}3;2}(n-2) + f_{21\text{-}3;1}(n-1) \\ - f_{21\text{-}3;1}(n-2) + f_{21\text{-}3;0}(n-1) - 2f_{21\text{-}3;0}(n-2).$$

Proof. Case $r = 0$: Lemma 2.8 yields $f_{21\text{-}3;0}(n;i) = f_{21\text{-}3;0}(n-1)$ where $1 \leq i \leq n-1$, so for all $n \geq 1$

$$f_{21\text{-}3;0}(n) = f_{21\text{-}3;0}(n-1) + (n-1)f_{21\text{-}3;0}(n-2).$$

Case $r = 1$: Lemma 2.8 yields $f_{21\text{-}3;1}(n;i) = f_{21\text{-}3;1}(n-2)$ where $1 \leq i \leq n-2$, and

$$f_{21\text{-}3;1}(n;n-1) = \sum_{j=1}^{n-2} f_{21\text{-}3;0}(n-1;j) + f_{21\text{-}3;1}(n-2)$$

which equivalent to (by use the case $r = 0$)

$$f_{21\text{-}3;1}(n; n-1) = f_{21\text{-}3;1}(n-2) + f_{21\text{-}3;0}(n-1) - f_{21\text{-}3;0}(n-2).$$

Therefore, for all $n \geq 1$

$$f_{21\text{-}3;1}(n) = f_{21\text{-}3;1}(n-1)+(n-1)f_{21\text{-}3;1}(n-2)+f_{21\text{-}3;0}(n-1)-f_{21\text{-}3;0}(n-2).$$

Case $r = 2$: Similarly as the cases $r = 0, 1$. □

2.4. $\tau = 23\text{-}1$. **The following lemma is the base of the main result in this subsection, which holds immediately from definitions.**

Lemma 2.10. *Let $n \geq 1$; then*

$$f_{23\text{-}1;r}(n) = f_{23\text{-}1;r}(n-1) + \sum_{i=1}^{n-1} f_{23\text{-}1;r}(n; i),$$

$$f_{23\text{-}1;r}(n; i) = f_{23\text{-}1;r-(i-1)}(n-2) + \sum_{j=1}^{i-1} f_{23\text{-}1;r}(n-1; j), \ 1 \leq i \leq n-1.$$

Theorem 2.11. *For any nonnegative integer r, the generating function $F_{23\text{-}1;r}(x)$ is given by*

$$\frac{\delta_{r,0}}{1-x} + x^2 \sum_{d=0}^{r} (1-x)^{j-2} \left[F_{23\text{-}1;r-d}\left(\frac{x}{1-x}\right) - \sum_{j=0}^{d-1} f_{23\text{-}1;r-d}(j) \left(\frac{x}{1-x}\right)^j \right].$$

Proof. Using the same argument proof of Equation (6) together with Lemma 2.10 we obtain for all $1 \leq m \leq n-1$,

$$\begin{aligned} f_{23\text{-}1;r}(n; m) = \\ = f_{23\text{-}1;r+1-m}(n-2) + \sum_{d \geq 1} \sum_{j} \binom{m-1-d}{j} f_{23\text{-}1;r+1-d}(n-3-j). \end{aligned}$$

Therefore, by summing $f_{23\text{-}1;r}(n; m)$ over all $1 \leq m \leq n$ we shall show that Lemma 2.10 implies for all $n \geq 1$,

$$f_{23\text{-}1;r}(n) = f_{23\text{-}1;r}(n-1) + \sum_{d=0}^{r} \sum_{j=0}^{n-2-d} \binom{n-2-d}{j} f_{23\text{-}1;r-d}(n-2-j).$$

Hence, using [CM2, Lemma 7] we get the desired result. □

Example 2.12. (see Claesson and Mansour [CM1, Proposition 7]) *Theorem 2.11 for $r = 0$ yields*

$$F_{23\text{-}1;0}(x) = \frac{1}{1-x} + \frac{x^2}{(1-x)^2} F_{23\text{-}1;0}\left(\frac{x}{1-x}\right).$$

An infinite number of application of this identity we have

$$F_{23\text{-}1;0}(x) = \sum_{k \geq 0} \frac{x^{2k}}{p_{k-1}(x) p_{k+1}(x)},$$

where $p_m(x) = \prod_{j=0}^{m}(1-jx)$. *An another example, Theorem 2.11 for* $r=1$ *yields (similarly)*

$$F_{23\text{-}1;1}(x) = \sum_{d\geq 0}\left[\frac{x^{2d+2}}{p_{d+1}(x)}\left(\sum_{k\geq 0}\frac{x^{2k}}{p_{k+d-1}(x)p_{k+d+1}(x)} - 1\right)\right].$$

2.5. $\tau = 31\text{-}2$. The following lemma is the base of the main result in this subsection, which holds immediately from definitions.

Lemma 2.13. *Let* $n \geq 1$; *then*

$$f_{31\text{-}2;r}(n;n) = \sum_{j=1}^{n-1} f_{31\text{-}2;r+1-j}(n-1;n-j),$$
$$f_{31\text{-}2;r}(n;i) = f_{31\text{-}2;r}(n-1;n-1) + \sum_{j=1}^{i-1} f_{31\text{-}2;r-(i-1-j)}(n-1;j),$$

where $1 \leq i \leq n-1$.

Theorem 2.14. *Let* r *be a nonnegative integer, then* $f_{31\text{-}2;r}(n)$ *is a polynomial of degree at most* $2r+2$ *with coefficients in* $\mathcal{Q}$, *where* $n \geq 0$.

Proof. Using Lemma 2.13 for $r=0$ we obtain that, first $f_{31\text{-}2;0}(n;n) = 1$ and $f_{31\text{-}2;0}(n;1) = 1$, second $f_{31\text{-}2;0}(n;j) = j$ for all $1 \leq j \leq n-1$. Hence, for all $n \geq 0$,

$$f_{31\text{-}2;0}(n) = \binom{n}{2} + 1.$$

Now, assume that $f_{31\text{-}2;d}(n;j)$ is a polynomial of degree at most $2d+1$ with coefficient in $\mathcal{Q}$ for all $1 \leq j \leq n$ where $d = 0,1,2,\ldots,r-1$. Therefore, Lemma 2.13 with induction hypothesis imply, first $f_{31\text{-}2;r}(n;n)$ and $f_{31\text{-}2;r}(n;1)$ are polynomials of degree at most $2r$, and then $f_{31\text{-}2;r}(n;j)$ is a polynomial of degree at most $2r+1$. So, by use the principle of induction on r we get that $f_{31\text{-}2;r}(n;j)$ is a polynomial of degree at most $2r+1$ with coefficient in $\mathcal{Q}$ for all $r \geq 0$. Hence, since $f_{31\text{-}2;r}(n) = \sum_{j=1}^{n} f_{31\text{-}2;r}(n;j)$ we get the desired result. □

Aa application of Theorem 2.14 with the initial values of the sequence $f_{31\text{-}2;r}(n)$ we have the exact formula for $f_{31\text{-}2;r}(n)$ where $r = 0,1,2,3$.

Corollary 2.15. *For all $n \geq 0$,*

$$f_{31\text{-}2;0}(n) = 1 + \tfrac{n(n-1)}{2};$$

$$f_{31\text{-}2;1}(n) = \tfrac{n(n-1)(n-2)}{24}(3n-5);$$

$$f_{31\text{-}2;2}(n) = \tfrac{n(n-1)(n-2)(n-3)}{720}(5n^2-3n-38);$$

$$f_{31\text{-}2;3}(n) = \tfrac{n(n-1)(n-2)(n-3)(n-4)}{40320}(7n^3+10n^2+205n-1142).$$

2.6. $\tau = $ 32-1. The following lemma is the base of the main result in this subsection, which holds immediately from definitions.

Lemma 2.16. *Let $n \geq 1$; then*

$$f_{32\text{-}1;r}(n;n) = \sum_{j=1}^{n-1} f_{31\text{-}2;r+1-j}(n-1;j),$$
$$f_{32\text{-}1;r}(n;i) = f_{32\text{-}1;r}(n-1;n-1) + \sum_{j=1}^{i-1} f_{31\text{-}2;r+1-j)}(n-1;j),$$

where $1 \leq i \leq n-1$.

Theorem 2.17. *Let r be a nonnegative integer, then $f_{32\text{-}1;r}(n)$ is a polynomial of degree at most $r+1$ with coefficients in $\mathcal{Q}$, where $n \geq r+2$.*

Proof. Using Lemma 2.16 for $r = 0$ we obtain that, first $f_{32\text{-}1;0}(n;n) = 1$ and $f_{32\text{-}1;0}(n;1) = 1$, second $f_{31\text{-}2;0}(n;j) = 2$ for all $2 \leq j \leq n-1$. Hence, for all $n \geq 2$,

$$f_{31\text{-}2;0}(n) = 2n-2.$$

Let $n \geq r+2$ and let us assume that $f_{32\text{-}1;d}(n;j)$ is a polynomial of degree at most d with coefficients in $\mathcal{Q}$ for all $1 \leq j \leq n$ where $d = 0, 1, 2, \ldots, r-1$. Lemma 2.16 with induction hypothesis imply, first $f_{32\text{-}1;r}(n;n)$ and $f_{32\text{-}1;r}(n;1)$ are polynomials of degree at most r with coefficients in $\mathcal{Q}$, and then $f_{31\text{-}2;r}(n;j)$ where $2 \leq j \leq n-1$ is a polynomial of degree at most r with coefficients in $\mathcal{Q}$. So, by use the principle of induction on r we get that $f_{32\text{-}1;r}(n;j)$ is a polynomial of degree at most r with coefficients in $\mathcal{Q}$ for all $r \geq 0$. Hence, since $f_{32\text{-}1;r}(n) = \sum_{j=1}^{n} f_{32\text{-}1;r}(n;j)$ we get the desired result. □

As application of Theorem 2.17 with the initial values of the sequence $f_{32\text{-}1;r}(n)$ we get the exact formula for $f_{32\text{-}1;r}(n)$ for $r = 0, 1, 2, 3$.

Corollary 2.18.

(i) *For all* $n \geq 2$, $f_{32\text{-}1;0}(n) = 2n - 2$;

(ii) *For all* $n \geq 3$, $f_{32\text{-}1;1}(n) = (n-3)(2n-1)$;

(iii) *For all* $n \geq 4$, $f_{32\text{-}1;2}(n) = (n-4)(n^2-3n+1)$;

(iv) *For all* $n \geq 5$, $f_{32\text{-}1;3}(n) = \frac{1}{6}(n-5)(2n^3-13n^2+47n-6)$.

3. Avoiding 13-2

Let $g_{\tau;r}(n)$ be the number of all permutations in $S_n(13\text{-}2)$ containing τ exactly r times. We denote the corresponding ordinary generating function by $G_{\tau;r}(x)$; that is, $G_{\tau;r}(x) = \sum_{n\geq 0} g_{\tau;r}(n)x^n$. We extend the above definitions by $g_{\tau;r}(n) = 0$ for any τ and $r < 0$.

In the current section, our aim is to count the number of permutations avoiding 13-2 and containing τ exactly r times where τ a generalized pattern of type $(2,1)$. Let $g_{\tau;r}(n; i_1, i_2, \ldots, i_j)$ be the number of permutations $\pi \in S_n(13\text{-}2)$ containing τ exactly r times such that $\pi_1\pi_2\ldots\pi_j = i_1i_2\ldots i_j$.

The main body of this section is divided to three subsections corresponding to the cases τ is 12-3; 21-3, 23-1; 31-2; or 32-1.

3.1. $\tau = 12\text{-}3$.

Theorem 3.1. *Let r be any nonnegative integer; then there exist polynomials $p_r(n)$ and $q_{r-1}(n)$ of degree at most r and $r-1$ respectively, with coefficients in Q such that for all $n \geq 1$,*

$$g_{12\text{-}3;r}(n) = p_r(n) \cdot 2^n + q_{r-1}(n).$$

Proof. Let $r \geq 0$, and let $1 \leq i \neq j \leq n$. If $i < j$, then since the permutations avoiding 13-2 we have $g_{12\text{-}3;r}(n; i, j) = 0$ for $i+2 \leq j \leq n$ and $g_{12\text{-}3;r}(n; i, i+1) = g_{12\text{-}3;r-(n-1-i)}(n-1; i)$, hence

$$\sum_{j=i+1}^{n} g_{12\text{-}3;r}(n; i, j) = g_{12\text{-}3;r-(n-1-i)}(n-1; i).$$

If $i > j$ then by definitions we have

$$g_{12\text{-}3;r}(n; i, j) = g_{12\text{-}3;r}(n-1; j).$$

Owing to the definitions we have showed that, for all $1 \leq i \leq n-2$,

$$g_{12\text{-}3;r}(n; i) = g_{12\text{-}3;r-(n-1-i)}(n-1; i) + \sum_{j=1}^{i-1} g_{12\text{-}3;r}(n-1; j). \qquad (1')$$

Moreover, it is plain that

$$g_{12\text{-}3;r}(n;n) = g_{12\text{-}3;r}(n;n-1) = g_{12\text{-}3;r}(n-1), \tag{2'}$$

and for all $1 \leq j \leq n-r-2$

$$g_{12\text{-}3;r}(n;j) = 0. \tag{3'}$$

Now we ready to prove the theorem. Let $r = 0$; by Equation (3') we get $g_{12\text{-}3;0}(n;j) = 0$ for all $j \leq n-2$ and by Equation (2') we have $g_{12\text{-}3;0}(n;n-1) = g_{12\text{-}3;0}(n;n) = d_0(n-1)$, so $d_r(n) = 2^{n-1}$. Therefore, the theorem holds for $r = 0$.

Let $r \geq 1$, and let us assume that for all $0 \leq m \leq s-1$ and all $0 \leq s \leq r-1$ there exist polynomials $p_m(n)$ and $q_{m-1}(n)$ of degree at most m and $m-1$ respectively with coefficients in $\mathcal{Q}$ such that $g_{12\text{-}3;s}(n;n-s-1+m) = p_m(n)2^n + q_{m-1}(n)$, and there exist polynomials $v_s(n)$ and $u_{s-1}(n)$ of degree at most s and $s-1$ respectively with coefficients in $\mathcal{Q}$ such that $g_{12\text{-}3;s}(n-m) = v_s(n)2^n + u_{s-1}(n)$ where $m = 0, 1$.

So, using Equation (1') for $m = 0, 1, \ldots, r-1$ and the induction hypothesis imply that there exist polynomials $a_m(n)$ and $b_{m-1}(n)$ of degree at most m and $m-1$ respectively with coefficients in $\mathcal{Q}$ such that

$$g_{12\text{-}3;r}(n;n-r-1+m) = a_m(n)2^n + b_{m-1}(n).$$

Besides, Owing to Equations (1'), (2'), and (3') we have showed that

$$g_{12\text{-}3;r}(n) = 2g_{12\text{-}3;r}(n-1) + \sum_{j=2}^{r+1} g_{12\text{-}3;r}(n;n-j),$$

which means that $g_{12\text{-}3;r}(n)$ is given by $v_r(n)2^n + u_{r-1}(n)$ and $g_{12\text{-}3;r}(n;n) = g_{12\text{-}3;r}(n;n-1) = g_{12\text{-}3;r}(n-1)$. Therefore, the statement holds for $s = r$. Hence, by the principle of induction on r the theorem holds. □

As application of Theorem 3.1 with the initial values of the sequence $g_{12\text{-}3;r}(n)$ we obtain the exact formula for $g_{12\text{-}3;r}(n)$ for $r = 0, 1, 2, 3$.

Corollary 3.2. *For all $n \geq 1$;*

$$g_{12\text{-}3;0}(n) = 2^{n-1};$$

$$g_{12\text{-}3;2}(n) = (n-3)2^{n-2} + 1;$$

$$g_{12\text{-}3;2}(n) = (n^2 - 11n + 34)2^{n-4} - n - 2;$$

$$g_{12\text{-}3;3}(n) = \tfrac{1}{3}(n^3 - 24n^2 + 257n - 954)2^{n-5} + n^2 + 4n + 10.$$

3.2. $\tau = 21\text{-}3$. The following lemma is the base of the main result in this subsection, which holds immediately from definitions.

Lemma 3.3. *Let $n \geq 1$; then*

$$g_{21\text{-}3;r}(n) = g_{21\text{-}3;r}(n-1) + \sum_{i=1}^{n-1} g_{23\text{-}1;r}(n;i),$$
$$g_{21\text{-}3;r}(n;i) = g_{21\text{-}3;r}(n-1;i) + \sum_{j=1}^{i-1} g_{21\text{-}3;r-(n-i)}(n-1;j),$$

where $1 \leq i \leq n-1$

Theorem 3.4. *Let r be any nonnegative integer, then there exists a polynomial $p_r(n)$ of degree at most r with coefficients in $\mathcal{Q}$, such that for all $n \geq r$*

$$g_{21\text{-}3;r}(n) = p_r(n) \cdot 2^n.$$

Proof. Lemma 3.3 implies for $r = 0$ as follows. First $g_{21\text{-}3;0}(n;m) = 2^{m-2}$ for all $m \geq 2$, and second $g_{21\text{-}3;0}(n;1) = 1$. Hence $g_{21\text{-}3;0}(n) = 2^{n-1}$, so the theorem holds for $r = 0$.

Let $r \geq 1$. Assume that for $2 \leq m \leq n-1$ the expression $\sum_{j=1}^{m} g_{21\text{-}3;d}(n;j)$ is given by $q_d^m(n)2^m$ where $q_d^m(n)$ is a polynomial of degree at most d with coefficients in $\mathcal{Q}$ for all $0 \leq d \leq r-1$. Lemma 3.3 yields

$$\begin{aligned}
g_{21\text{-}3;r}(n;1) &= g_{21\text{-}3;r}(n-1;1),\\
g_{21\text{-}3;r}(n;2) &= g_{21\text{-}3;r}(n-1;2),\\
&\vdots\\
g_{21\text{-}3;r}(n;n-r-1) &= g_{21\text{-}3;r}(n-1;n-r-1),\\
g_{21\text{-}3;r}(n;n-r+1) &= g_{21\text{-}3;r}(n-1;n-r+1) + \sum_{j=1}^{n-r} g_{21\text{-}3;1}(n-1;j),\\
&\vdots\\
g_{21\text{-}3;r}(n;n-1) &= g_{21\text{-}3;r}(n-1;n-1) + \sum_{j=1}^{n-2} g_{21\text{-}3;r-1}(n-1;j),\\
g_{21\text{-}3;r}(n;n) &= g_{21\text{-}3;r}(n-1),
\end{aligned}$$

with induction hypothesis imply for $2 \leq m \leq n-1$

$$\sum_{j=1}^{m} g_{21\text{-}3;r}(n;j) = \sum_{j=1}^{m} g_{21\text{-}3;r}(n-1;j) + q_{r-1}^m(n)2^m,$$

where $q_{r-1}^m(n)$ is a polynomial of degree at most $r-1$ with coefficients in $\mathcal{Q}$. Therefore, for $2 \leq m \leq n-1$ $\sum_{j=1}^{m} g_{12\text{-}3;r}(n;j)$ can be expressed by $q_r^m(n)2^m$ where $q_r^m(n)$ is a polynomial of degree at most r with coefficients in Q. Hence, with using Lemma 3.3 we get there exists a polynomial $a_r(n)$ of degree at most r with coefficients in Q such that

$$g_{12\text{-}3;r}(n) = g_{12\text{-}3;r}(n-1) + a_r(n)2^n,$$

so the theorem holds. □

Using Theorem 3.4 with the initial values of the sequence $g_{21\text{-}3;r}(n)$ for $r = 0, 1, 2, 3$ we get

Corollary 3.5.

(i) *For all* $n \geq 1$, $g_{21\text{-}3;0}(n) = 2^{n-1}$;

(ii) *For all* $n \geq 2$, $g_{21\text{-}3;1}(n) = (n-2)2^{n-3}$;

(iii) *For all* $n \geq 3$, $g_{21\text{-}3;2}(n) = (n^2 + n - 12)2^{n-6}$;

(iv) *For all* $n \geq 4$, $g_{21\text{-}3;3}(n) = \frac{1}{3}(n-4)(n^2 + 13n + 6)2^{n-8}$.

3.3. $\tau = 23\text{-}1$, $\tau = 31\text{-}2$, **or** $\tau = 32\text{-}1$. Similarly, using the argument proof of Theorem 3.4 with the principle of induction yield

Theorem 3.6. *Let r be any nonnegative integer. Then*

(i) *there exists a polynomial $p_{r-1}(n)$ of degree at most $r-1$ with coefficients in Q and a constant c, such that for all $n \geq r$*

$$g_{23\text{-}1;r}(n) = c \cdot 2^n + p_{r-1}(n).$$

(ii) *there exist polynomials $p_r(n)$ and $q_{2r-2}(n)$ of degree at most r and $2r-2$ respectively; with coefficients in Q such that for all $n \geq 1$*

$$g_{31\text{-}2;r}(n) = p_r(n)2^n + q_{2r-2}(n).$$

(iii) *there exists a polynomial $p_{r+2}(n)$ of degree at most $r+2$ with coefficients in Q such that for all $n \geq r$*

$$g_{32\text{-}1;r}(n) = p_{r+2}(n).$$

Using Theorem 3.6 with the initial values of the sequences $g_{23\text{-}1;r}(n)$, $g_{31\text{-}2;r}(n)$ and $g_{32\text{-}1;r}(n)$ for $r = 0, 1, 2, 3, 4$ we get that the following:

Corollary 3.7.

(i) *For all* $n \geq 1$, $g_{23\text{-}1;0}(n) = 2^{n-1}$;

(ii) *For all* $n \geq 2$, $g_{23\text{-}1;1}(n) = 2^{n-2} - 1$;

(iii) *For all* $n \geq 3$, $g_{23\text{-}1;2}(n) = 2^{n-1} - n - 1$;

(iv) *For all* $n \geq 4$, $g_{23\text{-}1;3}(n) = 5 \cdot 2^{n-3} - \frac{1}{2}(n^2 - n + 8)$.

Corollary 3.8. *For all $n \geq 1$;*

(i) $g_{31\text{-}2;0}(n) = 2^{n-1}$;

(ii) $g_{31\text{-}2;1}(n) = (n-3)2^{n-2} + 1$;

(iii) $g_{31\text{-}2;2}(n) = (n^2 - 3n - 14)2^{n-4} + \frac{1}{2}(n^2 + n + 12)$;

(iv) $g_{31\text{-}2;3}(n) = \frac{1}{3}(n^3 - 55n - 90)2^{n-5} + \frac{1}{12}(n^4 + 11n^2 + 12n + 12)$.

Corollary 3.9. *For all $n \geq 1$;*

(i) $g_{32\text{-}1;0}(n) = \frac{1}{2}n(n-1) + 1$;

(ii) $g_{32\text{-}1;1}(n) = \frac{1}{6}(n-1)(n-2)(2n-3)$;

(iii) $g_{32\text{-}1;2}(n) = \frac{1}{6}(n-2)(n-3)(2n-5)$;

(iv) $g_{32\text{-}1;3}(n) = \frac{1}{8}(n-3)(n^3 - 3n^2 - 10n + 32)$;

(v) $g_{32\text{-}1;4}(n) = \frac{1}{24}(n-4)(3n^3 - 10n^2 - 55n + 198)$.

4. Avoiding 21-3

Let $h_{\tau;r}(n)$ be the number of all permutations in $S_n(21\text{-}3)$ containing τ exactly r times. We denote the corresponding exponential and ordinary generating function by $\mathcal{H}_{\tau;r}(x)$ and $H_{\tau;r}(x)$ respectively; that is, $\mathcal{H}_{\tau;r}(x) = \sum_{n\geq 0} \frac{h_{\tau;r}(n)}{n!}x^n$ and $H_{\tau;r}(x) = \sum_{n\geq 0} h_{\tau;r}(n)x^n$. We extend the above definitions by $h_{\tau;r}(n) = 0$ for any τ and $r < 0$.

In the current section, our aim is to count the number of permutations avoiding 21-3 and containing τ exactly r times where τ a generalized pattern of type (2,1). Let $h_{\tau;r}(n; i_1, i_2, \dots, i_j)$ be the number of permutations $\pi \in S_n(21\text{-}3)$ containing τ exactly r times such that $\pi_1\pi_2\dots\pi_j = i_1i_2\dots i_j$.

The main body of the current section is divided to five subsections corresponding to the cases τ is a general generalized pattern; 12-3; 13-2, 31-2; 23-1; and 32-1.

4.1. τ **is a general generalized pattern.** Here we study certain cases of τ, where τ is a generalized pattern of length k without dashes, or with exactly one dash.

First of all let us define a bijection Φ between the set $S_n(12\text{-}3)$ and the set $S_n(21\text{-}3)$ as follows. Let $\pi = (\pi', n, \pi'')$, where n the maximal element of

π, be any 12-3-avoiding permutation of s elements; we define by induction

$$\Phi(\pi) = (R(\pi'), n, \Phi(\pi'')),$$

where $R(\pi')$ is the reversal of π' (that is $R : (\pi_1, \dots, \pi_n) \mapsto (\pi_n, \dots, \pi_1)$). Since π is 12-3-avoiding permutation we have $\pi_1 > \cdots > \pi_{j-1}$ so by using the principle of induction on length π we get $\Phi(\pi)$ is 21-3-avoiding permutation. Also, it is easy to see by using the principle of induction that $\Phi^{-1} = \Phi$, hence Φ is a bijection.

Theorem 4.1. *For all $k \geq 1$;*

$$\mathcal{H}_{12\dots(k-1)k;0}(x) = \mathcal{F}_{(k-1)\dots21k;0}(x), \quad \mathcal{H}_{12\dots(k-1)k;1}(x) = \mathcal{F}_{(k-1)\dots21k;1}(x).$$

Proof. Using the bijection $\Phi : S_n(12\text{-}3) \to S_n(21\text{-}3)$ we get the desired result: the permutation $\pi \in S_n(12\text{-}3)$ contains $(k-1)\dots1k$ exactly r ($r = 0, 1$) times if and only if the permutations $\Phi(\pi)$ contains $12\dots(k-1)k$ exactly r. □

Example 4.2. (see Claesson [C]) *Theorem 2.1 and Theorem 4.1 yield for $r = 0$ that*

$$\mathcal{H}_{12\dots k;0}(x) = e^{\frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!}}.$$

If $k \to \infty$, then we get that the exponential generating function for the number of 21-3-*avoiding permutations in S_n is given by $e^{e^x - 1}$.*

In [C, CM1] proved that the number of permutations in S_n(21-3) avoiding 12-3 is the same number of involutions in S_n. The case of varying k is more interesting. As an extension of these results (the proofs are immediately holds by using the bijection Φ).

Theorem 4.3. *For $k \geq 1$,*

$$\mathcal{H}_{12\dots(k-1)\text{-}k;0}(x) = e^{\frac{x^1}{1!} + \cdots + \frac{x^{k-1}}{(k-1)!}};$$
$$\mathcal{H}_{12\dots(k-2)k\text{-}(k-1);0}(x) = \mathcal{H}_{12\dots(k-2)\text{-}k\text{-}(k-1);0}(x) = \mathcal{F}_{(k-2)\dots21k\text{-}(k-1);0}(x).$$

Using the bijection Φ we get easily other results as follows.

Theorem 4.4. *(i) The number of permutations in S_n containing* 12-3 *exactly once is the same number of permutations containing* 21-3 *exactly once;*

(ii) The number of permutations in S_n containing 12-3 *exactly once and containing $(k-1)\dots21\text{-}k$ (resp. $(k-1)\dots21k$) exactly $r = 0, 1$ times, is the same number of permutations in S_n containing* 21-3 *exactly once and containing $12\dots(k-1)\text{-}k$ (resp. $(12\dots(k-1)k$) exactly $r = 0, 1$ times.*

4.2. $\tau = 12\text{-}3$. The following lemma is the base of the main result in this subsection, which holds immediately from definitions.

Lemma 4.5. *Let $n \geq 1$; then*

$$h_{12\text{-}3;r}(n) = h_{12\text{-}3;r}(n-1) + \sum_{j=1}^{n-1} h_{12\text{-}3;r}(n;j),$$
$$h_{12\text{-}3;r}(n;j) = h_{12\text{-}3;r}(n-2) + \sum_{i=j}^{n-1} h_{12\text{-}3;r-(n-i-1)}(n-1;i),$$

where $1 \leq j \leq n-1$.

Theorem 4.6. *For all $n \geq 1$,*

$$h_{12\text{-}3;0}(n) = h_{12\text{-}3;0}(n-1) + (n-1)h_{12\text{-}3;0}(n-2);$$

$$h_{12\text{-}3;1}(n) = h_{12\text{-}3;1}(n-1) + (n-1)h_{12\text{-}3;1}(n-2) + \\ +(n-2)h_{12\text{-}3;0}(n-3);$$

$$h_{12\text{-}3;2}(n) = h_{12\text{-}3;2}(n-1) + (n-1)h_{12\text{-}3;2}(n-2) + \\ +(n-2)h_{12\text{-}3;1}(n-3) + (n-3)h_{12\text{-}3;0}(n-3).$$

Proof. Case $r = 0$: Lemma 4.5 yields $h_{12\text{-}3;0}(n;j) = h_{12\text{-}3;0}(n-2)$ where $1 \leq j \leq n-1$, hence

$$h_{12\text{-}3;0}(n) = h_{12\text{-}3;0}(n-1) + (n-1)h_{12\text{-}3;0}(n-2).$$

Case $r = 1$: Lemma 4.5 yields

$$h_{12\text{-}3;1}(n;j) = h_{12\text{-}3;1}(n-2) + h_{12\text{-}3;0}(n-1;n-2)$$

where $1 \leq j \leq n-2$, and $h_{12\text{-}3;1}(n;n-1) = h_{12\text{-}3;1}(n-2)$, which means that

$$h_{12\text{-}3;1}(n) = h_{12\text{-}3;1}(n-1) + (n-1)h_{12\text{-}3;1}(n-2) + (n-2)h_{12\text{-}3;0}(n-3).$$

Case $r = 2$: similarly as the above cases. □

4.3. $\tau = 13\text{-}2$.

Theorem 4.7. *Let r be a nonnegative integer. Then*

$$H_{13\text{-}2;r}(x) = \frac{1-x}{1-2x}\delta_{r,0} + \frac{x^2}{1-2x}\sum_{d=1}^{r}\frac{H_{13\text{-}2;r-d}(x) - \sum_{j=0}^{d-1} h_{13\text{-}2;r-d}(j)x^j}{(1-x)^d}.$$

Proof. Definitions imply $h_{13\text{-}2;r}(n;n) = h_{13\text{-}2;r}(n-1)$, and for $1 \le j \le n-1$,

$$h_{13\text{-}2;r}(n;j) = \sum_{i=j}^{n-1} h_{13\text{-}2;r+i-j}(n-1;j).$$

By means of induction it is easy to obtain for $1 \le m \le n-1$

$$h_{13\text{-}2;r}(n;n-m) = \sum_{j=0}^{m-1} \binom{m-1}{j} h_{13\text{-}2;r-j}(n-1-m+j).$$

Now, if summing $h_{13\text{-}2;r}(n;n-m)$ over all $0 \le m \le n-1$, then we get

$$h_{13\text{-}2;r}(n) = h_{13\text{-}2;r}(n-1) + \sum_{d=0}^{r} \sum_{j=0}^{n-2-d} \binom{n-2-j}{d} h_{13\text{-}2;r-d}(j+d).$$

To find the desired result, we transfer the last equation to terms of ordinary generating functions by use [CM2, Lemma 7]. □

In view of Theorem 2.3 and Theorem 4.7 we have that the number of permutations in $S_n(12\text{-}3)$ containing 13-2 exactly r times is the same number of permutations in $S_n(21\text{-}3)$ containing 13-2 exactly r times. To verify that by combinatorial bijective proof let π be any 12-3-avoiding permutation; it is easy to see $\pi = (\pi_1, \dots, \pi_{j-1}, n, \pi')$ where $\pi_1 > \cdots > \pi_{j-1}$, so the number of occurrences of 13-2 in π is given by $N := n-1-\pi_{j-1}-(j-2)+N'$ where N' the number occurrences of 13-2 in π'. On the other hand, let $\beta = \Phi(\pi)$, so by definitions of Φ with induction hypothesis (induction on length of π) we get that β contains the same number N of occurrences of 13-2. Hence, by means of induction we shall showed that Φ is a bijection, and $H_{13\text{-}2;r}(x) = F_{13\text{-}2;r}(x)$ for all $r \ge 0$.

4.4. $\tau = 23\text{-}1$. The following lemma is the base of the main result in this subsection, which holds immediately from definitions.

Lemma 4.8. *Let $n \ge 1$; then*

$$h_{23\text{-}1;r}(n) = h_{23\text{-}1;r}(n-1) + \sum_{j=1}^{n-1} h_{23\text{-}1;r}(n;j),$$
$$h_{23\text{-}1;r}(n;j) = \sum_{i=j}^{n-1} h_{23\text{-}1;r-(j-1)}(n-1;i),$$

where $1 \le j \le n-1$.

Theorem 4.9. *Let r be any nonnegative integer, then there exists a polynomial $p_r(n)$ of degree at most r with coefficients in Q such that for all $n \ge r$*

$$h_{23\text{-}1;r}(n) = p_r(n)2^n.$$

Proof. Lemma 4.8 yields, $h_{23\text{-}1;r}(n;n) = h_{23\text{-}1;r}(n;1) = h_{23\text{-}1;r}(n-1)$, $h_{23\text{-}1;r}(n;j) = 0$ for all $r+2 \leq j \leq n-1$, and

$$h_{23\text{-}1;r}(n;j) = h_{23\text{-}1;r}(n;j-1) - h_{23\text{-}1;r+1-j}(n-1;j-1)$$

for all $2 \leq j \leq r+1$.

Assume that $h_{23\text{-}1;d}(n)$ can be expressed as $p_d(n)2^n$ and $h_{23\text{-}1;d}(n;j)$ can be expressed as $p_{d-1}(n)2^n$ where $2 \leq j \leq r+1$ for all $0 \leq d \leq r-1$. The statement is trivial for $r=0$, and by using the principle of induction with the above explanation we get, immediately, the desired result. □

Theorem 4.9 with the initial values of the sequences $h_{23\text{-}1;r}(n)$ for $r = 0,1,2$ yield

Corollary 4.10.
(i) *For all* $n \geq 1$, $h_{23\text{-}1;0}(n) = 2^{n-1}$;

(ii) *For all* $n \geq 2$, $h_{23\text{-}1;1}(n) = (n-2)2^{n-3}$;

(iii) *For all* $n \geq 3$, $h_{23\text{-}1;2}(n) = (n-3)(n+8)2^{n-6}$.

4.5. $\tau = 31\text{-}2$.

Theorem 4.11. *Let r be a nonnegative integer. Then*

$$H_{31\text{-}2;r}(x) = \frac{1-x}{1-2x}\delta_{r,0} + \frac{x^2}{1-2x}\sum_{d=1}^{r}\frac{H_{31\text{-}2;r-d}(x) - \sum_{j=0}^{d-1} h_{31\text{-}2;r-d}(j)x^j}{(1-x)^d}.$$

Proof. Definitions imply $h_{31\text{-}2;r}(n;1) = h_{13\text{-}2;r}(n-1)$, and for $2 \leq j \leq n-1$,

$$h_{31\text{-}2;r}(n;j) = \sum_{i=j}^{n-1} h_{31\text{-}2;r}(n-1;i) = h_{31\text{-}2;r}(n-1) - \sum_{i=1}^{j-1} h_{31\text{-}2;r}(n-1;i).$$

By means of induction it is easy to obtain for $1 \leq m \leq n-1$

$$h_{31\text{-}2;r}(n;m) = \sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}h_{31\text{-}2;r}(n-1-j).$$

Similarly as Theorem 2.3 (or Theorem 4.7), by using the above equation with (it is easy to check by definitions)

$$h_{31\text{-}2;r}(n;n) = \sum_{j=0}^{r} h_{31\text{-}2;r-j}(n-1;n-1-j),$$

we get the desired result. □

Again, we have $H_{31\text{-}2;r}(x) = F_{13\text{-}2;r}(x)$ for all $r \geq 0$. But here we failed to find a combinatorial explanation that the number of permutations in $S_n(12\text{-}3)$ containing 13-2 exactly r times is the same number of permutations in $S_n(21\text{-}3)$ containing 31-2 exactly r times.

4.6. $\tau = 32\text{-}1$. The following lemma is the base of the main result in this subsection, which holds immediately from definitions.

Lemma 4.12. *Let $n \geq 1$; then*

$$h_{32\text{-}1;r}(n) = h_{32\text{-}1;r}(n-1) + \sum_{j=2}^{n} h_{32\text{-}1;r}(n;j),$$
$$h_{32\text{-}1;r}(n;j) = h_{32\text{-}1;r}(n-1) - \sum_{i=1}^{j-1} h_{32\text{-}1;r}(n-1;i),$$

where $1 \leq j \leq n-1$, and

$$h_{32\text{-}1;r}(n;n) = h_{32\text{-}1;r}(n-1;1) + h_{32\text{-}1;r-1}(n-1;2) + \cdots + h_{32\text{-}1;0}(n-1;r+1).$$

Theorem 4.13. *For all $n \geq 1$,*

$$h_{32\text{-}1;0}(n) = \sum_{j=0}^{n-2}(-1)^j\binom{n-1}{j+1}h_{32\text{-}1;r}(n-1-j)+$$
$$+\sum_{j=1}^{r+1}\sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}h_{32\text{-}1;r+1-j}(n-2-i).$$

Proof. By means of induction with use Lemma 4.12 we imply that for all $1 \leq m \leq n-1$

$$h_{32\text{-}1;r}(n;m) = \sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}h_{32\text{-}1;r}(n-1-i).$$

On the other hand, using Lemma 4.12 the third equality and then using Lemma 4.12 the first equality we get the desired result. □

For example, Theorem 4.13 with [CM2, Lemma 7] (as Example 2.12) yield the exact formula for $H_{32\text{-}1;r}(x)$ where $r = 0, 1$ (see Claesson and Mansour [CM1] for the case $r = 0$).

Corollary 4.14.

$$H_{32\text{-}1;0}(x) = \sum_{k\geq 0}\frac{x^{2k}}{p_{k-1}(x)p_{k+1}(x)};$$
$$H_{32\text{-}1;1}(x) = \sum_{n\geq 0}\left[\frac{x^2(1-(n+2)x)}{1-(n+1)x}\sum_{k\geq 0}\frac{u^{2(k+n)}}{p_{n+k}(x)p_{n+k+2}(x)}\right].$$

where $p_d(x) = \prod_{j=0}^{d}(1-dx)$.

5. Further results

The first possibility to extend the above result is to count occurrences of two generalized patterns. For example, the number of permutations in S_n containing 12-3 exactly once and containing 13-2 exactly once is given by

$$(n^2 - 7n + 14)2^{n-3} - 2$$

for all $n \geq 1$. As another example, the number of permutations in S_n containing 12-3 exactly twice and containing 13-2 twice is given by

$$(n^4 - 18n^3 + 163n^2 - 826n + 1832)2^{n-7} - 4n - 14$$

for all $n \geq 1$. These results can be extended as follows.

Theorem 5.1. *Let us denote the number of permutations in S_n containing* 12-3 *exactly r times and containing* 13-2 *exactly s times by $a_n^{r,s}$; then there exists polynomials $p(n)$ and $q(n)$ of degree at most $r+s+1-\delta_{r,0}-\delta_{s,0}$ and $r+s-\delta_{r,0}-\delta_{s,0}$, respectively, such that for all $n \geq 1$*

$$a_n^{r,s} = p(n)2^n + q(n).$$

Another direction to extend the results in above sections is to restricted more than two patterns. For example, the number of permutations in $S_n(12\text{-}3, 13\text{-}2, 21\text{-}3)$ is given by the $(n+1)$th Fibonacci number (see [CM1]). Again, this result can be extended as follows.

Theorem 5.2. *(i) The ordinary generating function for the number of permutations in S_n(12-3, 21-3) such containing* 13-2 *exactly $r \geq 1$ times is given by*

$$\frac{x^2(1-x)^{r-1}}{(1-x-x^2)^{r+1}}$$

and for $r = 0$ is given by $\frac{1}{1-x-x^2}$.

(ii) The ordinary generating function for the number of permutations in S_n(12-3, 21-3) such containing 23-1 *exactly $r \geq 1$ times is given by*

$$\frac{x^2(1-x)^{r-1}}{(1-x-x^2)^{r+1}}$$

and for $r = 0$ is given by $\frac{1}{1-x-x^2}$.

In view of Theorem 5.2 suggests that there should exists a bijection between the sets $\{12\text{-}3, 21\text{-}3\}$-avoiding permutations such containing 13-2 exactly r times and $\{12\text{-}3, 21\text{-}3\}$-avoiding permutations such containing 23-1 exactly r times for any $r \geq 0$. However, we failed to produce such a bijection, and to find it remains a challenging open question.

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