PLANE TRIANGULATIONS IN PRODUCT GRAPHS

P. Paulraja and N. Varadarajan
Department of Mathematics, Annamalai University,
Annamalainagar – 608 002, Tamil Nadu, India.

ABSTRACT

A graph G is said to be *locally hamiltonian* if the subgraph induced by the neighbourhood of every vertex is hamiltonian. Alabdullatif conjectured that every connected locally hamiltonian graph contains a spanning plane triangulation. We disprove the conjecture. At the end, we raise a problem about the nonexistence of spanning planar triangulation in a class of graphs.

1. INTRODUCTION

By a graph we mean a finite, simple, undirected and connected graph with at least two vertices. In a graph G, for $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S. A graph G is said to be *traceable* if it contains a hamiltonian path. A graph G is said to be *hamilton cycle decomposable* if the edge set of G can be partitioned into hamilton cycles if G is even regular or, hamilton cycles together with a 1-factor, if G is an odd regular graph. A graph G is said to be *locally hamiltonian* (resp. *traceable*) if the subgraph induced by the neighbourhood of every vertex u, $G[N(u)]$, is hamiltonian (resp. traceable). The *tensor product* of graphs G and H, $G \otimes H$, is the graph with vertex set $V(G \otimes H) = V(G) \times V(H)$ and edge set $E(G \otimes H) = \{ (u,x)(v,y) \mid uv \in E(G)$ and $xy \in E(H) \}$. The tensor product is also called as Kronecker product, direct product, categorical product and graph conjunction. It is well known that tensor product is commutative and associative. The *lexicographic product* of graphs G and H, $G * H$, is the graph with vertex set $V(G * H) = V(G) \times V(H)$ and edge set $E(G * H) = \{(u,x)(v,y) \mid$ either $uv \in E(G)$ or ($u = v$ and $xy \in E(H))\}$. Definitions which are not given here can be found in [2] and [3]. Properties of locally hamiltonian graphs have been studied in [1], [4] and [5]. Properties of locally hamiltonian claw free graphs have been studied in [1] and [5]. Locally hamiltonian planar graphs and locally traceable outer planar graphs have been completely characterized in [1]. A graph G is said to be *locally hamilton cycle decomposable* if $G[N(u)]$ is hamilton cycle decomposable for every $u \in V(G)$. A planar graph G is called a *plane triangulation* if every face of G is of degree 3.

CONJECTURE 1 [1]. Every connected, locally hamiltonian graph contains a spanning plane triangulation as a subgraph.

We disprove this conjecture using tensor product of graphs. Infact, we prove that both $C_3 \otimes K_4$ and $C_3 \otimes K_5$ are counterexamples to Conjecture 1. One may be tempted to check if $C_3 \otimes K_n$, n ≥ 6, is a counterexample to Conjecture 1; but this is not the case. We show that the graph $C_3 \otimes K_n$, n ≥ 6, contains a spanning plane triangulation.

At the end, we raise a problem on the nonexistence of plane triangulation in certain class of graphs. If this conjecture is true, then this would yield a class of locally hamilton cycle decomposable graphs in which no graph has a spanning plane triangulation.

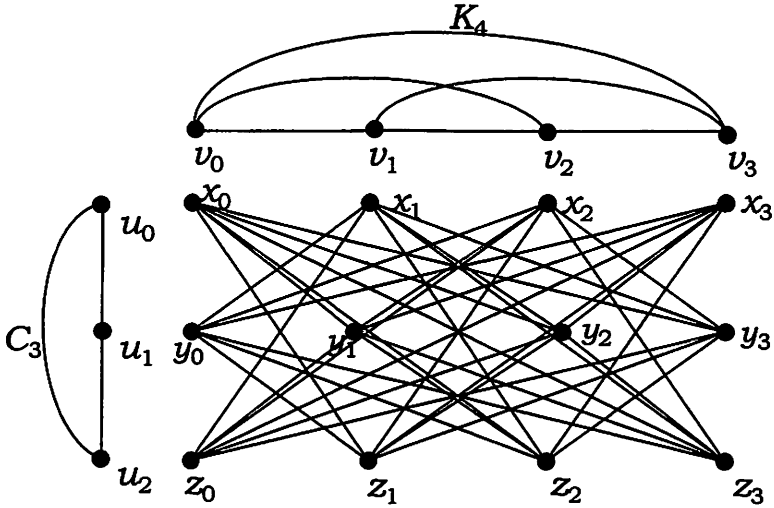
2. COUNTEREXAMPLES

Let $G = C_3 \otimes K_4$ (see Figure 1). We shall show that G is locally hamiltonian and it does not contain a spanning plane triangulation as a subgraph.

Let $V(C_3) = \{ u_0, u_1, u_2 \}$, and let $V(K_4) = \{ v_0, v_1, v_2, v_3 \}$. For our convenience, we partition the vertex set of G as follows: $X = \{ x_i = (u_0, v_i) \mid 0 \le i \le 3\}$, $Y = \{ y_i = (u_1, v_i) \mid 0 \le i \le 3 \}$ and $Z = \{ z_i = (u_2, v_i) \mid 0 \le i \le 3\}$. Clearly, by the definition of the tensor product of graphs, X, Y, Z and $\{ x_i, y_i, z_i \}$, $0 \le i \le 3$, are independent sets of vertices of G (see Figure 1).

First we shall show that G is locally hamiltonian. Because of symmetry, it is enough to show that the neighbour set of x_0 induces a hamilton cycle in $G[N(x_0)]$. Clearly, the neighbours of x_0 are y_1, y_2, y_3, z_1, z_2 and z_3. The hamilton cycle in $G[N(x_0)]$ is $y_1z_2y_3z_1y_2z_3y_1$, see Figure 1. This proves that $G = C_3 \otimes K_4$ is locally hamiltonian.

Next we prove that $C_3 \otimes K_4$ does not contain a spanning plane triangulation. If possible assume that M is a spanning plane triangulation of G. As M is a spanning plane triangulation of G, M is 3-connected. Clearly, by the definition of tensor product of graphs, the subgraph induced by $Y \cup Z$ in M, $M[Y \cup Z] = H$ is a bipartite subgraph of M. As no two vertices of X are



Here x_i, y_i and z_i stand for (u_0,v_i), (u_1,v_i) and (u_2,v_i), $0 \leq i \leq 3$, respectively.

$$G = C_3 \otimes K_4$$

Figure 1

adjacent in *M*, the vertices of *X* should be in different faces of *H* in the plane embedding of *M*, where we assume that the plane embedding of *H* is inherited from the plane embedding of *M*. Consequently, number of faces of *H* should be the number of vertices of *X*, that is 4, otherwise *M* would contain a face of degree at least 4, which is not the case. Note that *M* can be reconstructed from *H* by placing a vertex of *X* in *M* "inside" each face of *H* and joining it to all the vertices of its boundary; in the sequel, we call this as "reconstruction of *M* from *H*". Clearly, H has exactly 10 edges, using Euler's formula, that is, the number of edges between *Y* and *Z* in *M* is 10. Similarly, we can show that the number of edges between *X* and *Y* and, *X* and *Z* in *M* is 10 each.

First we claim that *H* has no face of degree 8. If *H* has a face of degree 8, then no vertex of *X* of *M* can be placed inside this face to reconstruct *M*, as each vertex of *G*, and in particular vertices of *X* in *M*, has degree 6. Thus degree of any face of *H* is either 4 or 6 as *H* is a bipartite graph. Next we claim that *H* has no face of degree 4. For, if there exists a face *f* of degree 4 in *H*, then the cycle bounding it must contain vertices having all the four suffixes, namely, { 0, 1, 2, 3 } and hence no vertex of $X = \{ x_0, x_1, x_2, x_3 \}$ can be placed inside f to reconstruct M as x_i is nonadjacent to both y_i and z_i, $0 \le i \le 3$, in M. This proves our claim. Thus H contains only faces of degree 6. Again, $\sum_{f \in F} d(f) = 2|E(H)| = 20$, where $d(f)$ denotes the degree of the face f and F is the set of faces of H, a contradiction to $|F| = 4$ and $d(f) = 6$. This proves that $G = C_3 \otimes K_4$ has no spanning plane triangulation. Therefore, $G = C_3 \otimes K_4$ is a counterexample to the conjecture of Alabdullatif [1].

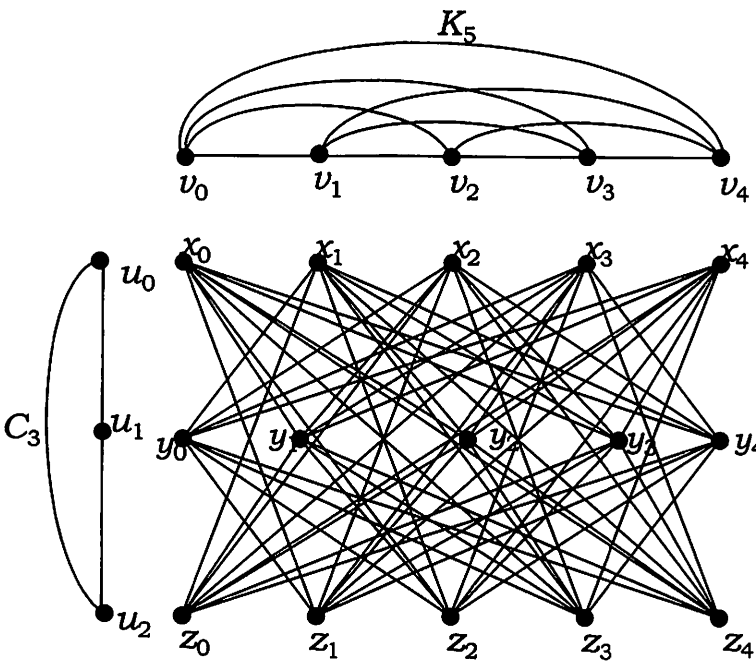
Next we shall prove that $C_3 \otimes K_5$ is also counterexample to Conjecture 1.

Let $G = C_3 \otimes K_5$ (see Figure 2). We shall show that G is locally hamiltonian and it does not contain a spanning plane triangulation.

Let $V(C_3) = \{ u_0, u_1, u_2 \}$ and let $V(K_5) = \{ v_0, v_1, v_2, v_3, v_4 \}$. For our convenience, we partition the vertex set of G as follows: $X = \{ x_i = (u_0, v_i) \mid 0 \le i \le 4 \}$, $Y = \{ y_i = (u_1, v_i) \mid 0 \le i \le 4 \}$ and $Z = \{ z_i = (u_2, v_i) \mid 0 \le i \le 4 \}$. Clearly, by the definition of the tensor product of graphs, X, Y, Z and $\{ x_i, y_i, z_i \}$, $0 \le i \le 4$, are independent sets of vertices of G.

First we observe that G is locally hamiltonian. Because of symmetry, it is enough to show that neighbour set of x_0 in G induces a hamilton cycle in $G[N(x_0)]$. Clearly, the neighbour set of x_0 is $\{y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4\}$. A hamilton cycle in $G[N(x_0)]$ is $y_1z_3y_2z_4y_3z_1y_4z_2y_1$. This proves that $C_3 \otimes K_5$ is locally hamiltonian.

Next we prove that $G = C_3 \otimes K_5$ does not contain a spanning plane triangulation. If possible assume that M is a spanning plane triangulation of G. As M is a plane triangulation, M is 3-connected. Clearly, $M[Y \cup Z] = H$ is an induced bipartite subgraph of M. As no two vertices of X are adjacent in M, the vertices of X should be in different faces of H in the plane embedding of M, where we assume that the embedding of H is inherited from the embedding of M, see the proof of the previous counterexample. Consequently, number of faces of H is same as the number of vertices of X, that is 5. As M is 3-connected and X is an independent set of G, H cannot contain a cut vertex; for, if y_i of H were a cut vertex, then it must be incident with a face, say, f, of degree at least 4 in H. As above, let x_j the unique vertex of M inside f in the plane embedding of M then $\{x_j, y_i\}$ would be a vertex cut of M, a contradiction. This proves that H is 2-connected. Clearly, H has exactly 13 edges, using Euler's formula, that is, the number of edges between Y and Z in M is 13. Similarly we can show that the number of edges between X and Y and, X and Z in M is 13 each.



Here x_i, y_i and z_i stand for (u_0,v_i), (u_1,v_i) and (u_2,v_i), $0 \leq i \leq 4$, respectively.

$G = C_3 \otimes K_5$

Figure 2

First we claim that H has no face of degree 10. If there exists a face of degree 10, no vertex of X in M can be placed "inside" this face to reconstruct M, as each vertex of G has degree exactly 8. Next we claim that H has no face of degree 8. Suppose there exists a face f_0 of degree 8, then the vertices of the cycle bounding the face should have the suffixes in the set $\{0, 1, 2, 3, 4\} \backslash \{i\}$, $0 \leq i \leq 4$, for otherwise, no vertex of X can be placed "inside" the face f_0 to reconstruct M. Without loss of generality we assume that $y_0z_2y_1z_3y_2z_0y_3z_1y_0$ is the cycle bounding the face f_0 of H. As $\sum_{f \in F} d(f) = 2|E(H)|$ where $d(f)$ denotes the degree of the face f in H and F is the set of faces in the bipartite graph H, we have

$$4\varphi_4 + 6\varphi_6 + 8\varphi_8 = 26 \text{ and } \sum_{i=4,6,8} \varphi_i = 5 \qquad (2.1)$$

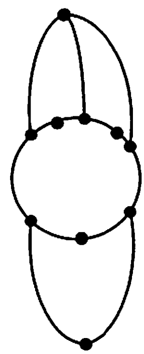


Figure 3(a)

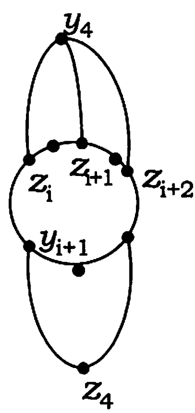


Figure 3(b)

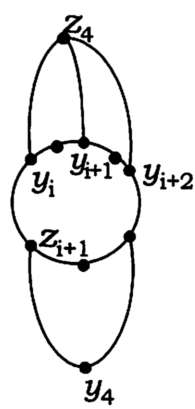


Figure 3(c)

where φ_i denotes the number of faces of degree i in H. Since H is a bipartite graph and it has exactly 5 faces, the equation (2.1) has a unique solution, namely, $\varphi_8 = 1$, $\varphi_6 = 1$ and $\varphi_4 = 3$. Then H must be isomorphic to the graph of Figure 3 (a) and the labels of the vertices of H must be as in Figure 3(b) or 3(c).

Then either y_4 is common to both the boundaries of the four degree faces of H (see Figure 3(b)) or z_4 is common to both the boundaries of four degree faces (see Figure 3(c)). In the first (second) case y_4 (z_4) is adjacent to three consecutive z_i's (y_i's), namely, z_i, z_{i+1} and z_{i+2} (y_i, y_{i+1} and y_{i+2}). Consequently, boundary of every face of H contains either z_{i+1} or y_{i+1}. Thus x_{i+1} cannot be placed inside any of the faces of H to reconstruct M, a contradiction. Hence $\varphi_8 = 0$, and therefore equation (2.1) can be rewritten as follows:

$$\sum_{f \in F} d(f) = 2|E(H)|$$

$$4\varphi_4 + 6\varphi_6 = 26, \text{ and } \varphi_4 + \varphi_6 = 5 \qquad (2.2)$$

The equation (2.2) has a unique solution, namely, $\varphi_4 = 2$ and $\varphi_6 = 3$. The possible 2-connected bipartite graphs with bipartition (A, B) with $|A| = |B| = 5$ and $\varphi_4 = 2$ and $\varphi_6 = 3$ are shown in the list of graphs of Figures 4, 5, 6 and 7. Hence H must be isomorphic to one of these graphs of Figures 4, 5, 6 and 7. We shall show that M cannot be reconstructed from none of the graphs in the Figures 4, 5, 6 and 7. (The graphs of these figures have been obtained by considering the number of vertices or edges in the intersection of the boundaries of the two four degree faces of H).

First we consider the graphs 1 to 15 of Figure 4. If H is isomorphic to any one of the fifteen graphs, then in each of them there are faces of degree 4 and 6, say, F_1 and F_2, respectively, such that the vertex set of the cycle bounding F_1 is contained in the vertex set of the cycle bounding F_2. Observe that the suffixes of the vertices of the cycle of length 4 bounding F_1 should be $\{0, 1, 2, 3, 4\} \setminus \{i\}$ $0 \le i \le 4$. As $\{0, 1, 2, 3, 4\} \setminus \{i\}$ is common suffixes of the vertices to both the faces of F_1 and F_2, x_i is the only vertex that can be placed inside both F_1 and F_2 to reconstruct M from H, which is impossible.

Next we shall prove that M cannot be reconstructed from H, if H is isomorphic to the graphs 16 to 23 of Figure 5.

First we observe that $d_H(y_i,z_i) \ge 3$, $0 \le i \le 4$. Further, any $y_i - z_i$, $0 \le i \le 4$, path in H is of odd length and diameter of H is 4, implies $d_H(y_i,z_i) = 3$. Without loss

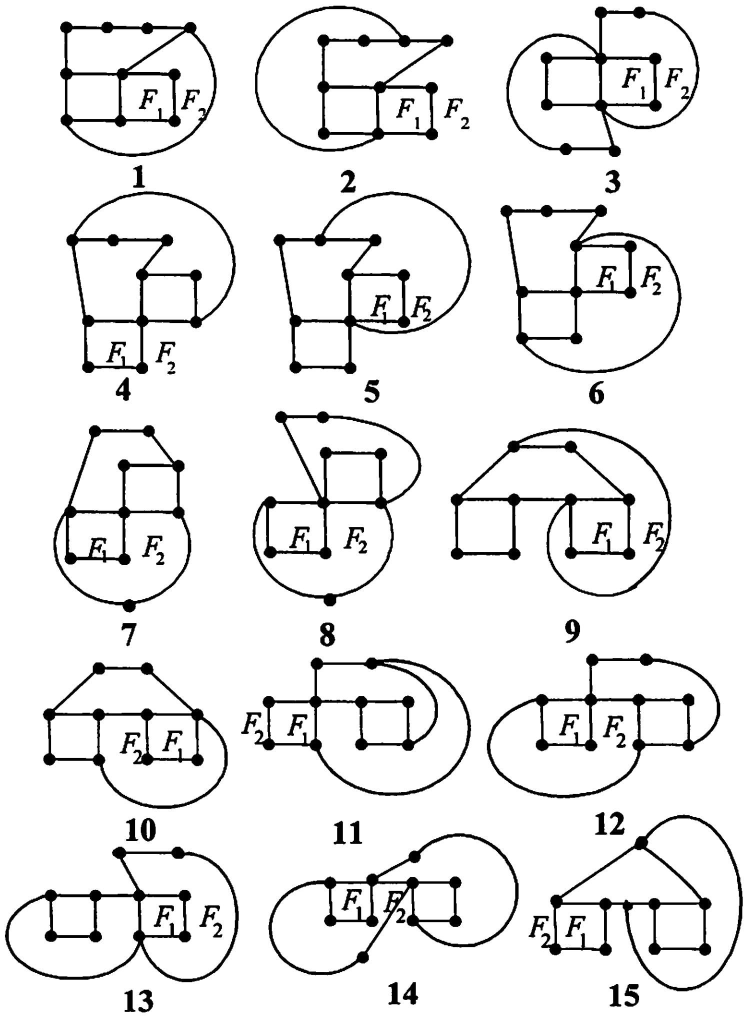


Figure 4

of generality assume that y_i is fixed as shown in the graphs 16 to 23. Then z_i must be as marked in the graphs of Figure 5 as $d_H(y_i,z_i) = 3$ (if there are two labels z_i, then they are possible representatives for z_i). But all the faces of H are incident with at least one of these two vertices y_i and z_i. Hence x_i cannot be placed inside any of the faces of H to reconstruct M.

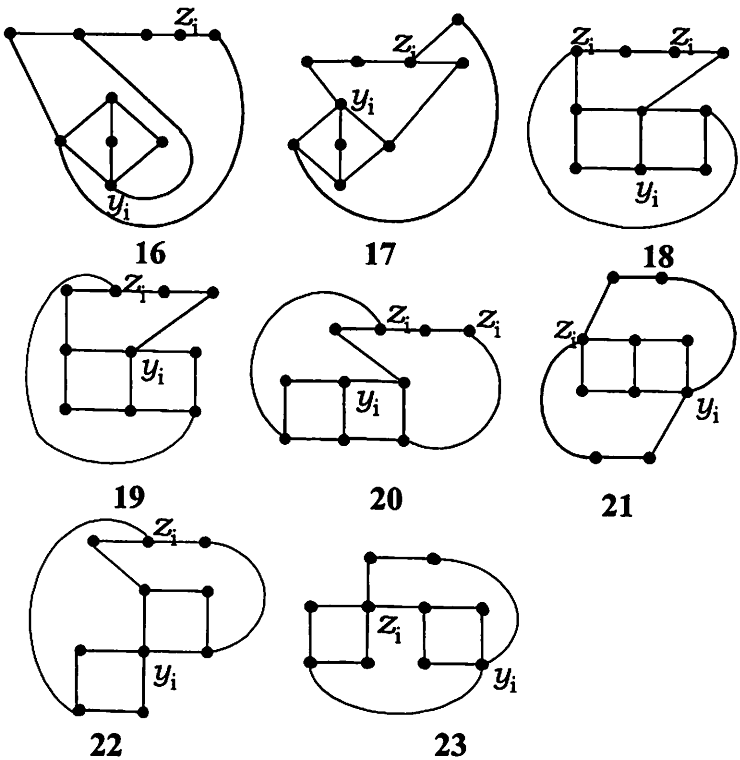


Figure 5

Next we shall show that M cannot be reconstructed from H, if H is isomorphic to graphs 24 and 25 of Figure 6.

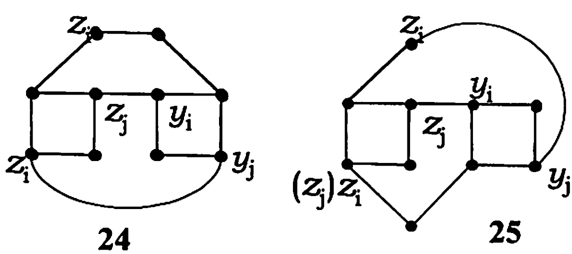


Figure 6

If we fix y_i as shown in the graphs of Figure 6, then there are two choices for z_i (as $d_H(y_i,z_i) = 3$). Having these choices for z_i, fix another vertex y_j as shown in the graphs of Figure 6. This uniquely fixes z_j in graph 24 but two vertices in graph 25 have the choice for z_j also. In either case, all the faces of H are incident with the vertices in $\{y_i,z_i\}$ or $\{y_j,z_j\}$. Consequently, x_i or x_j cannot be placed in any of the faces of H to reconstruct M from i.

Finally we show that H cannot be isomorphic to the graph 26 of Figure 7.

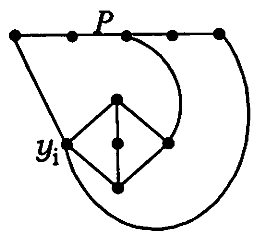


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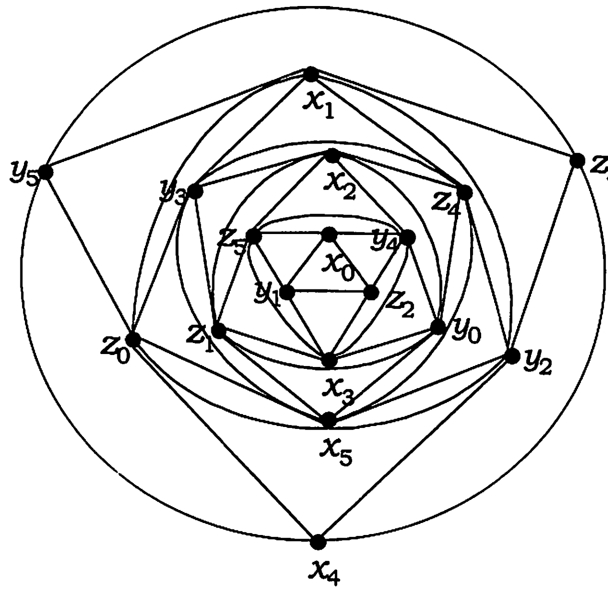
Figure 7

Clearly, the vertices of the subgraph $K_{2,3}$ in the graph of Figure 7 must have all suffixes in {0, 1, 2, 3, 4}. As a consequence, the suffixes of the vertices of the path P in the graph (see Figure 7) must be {0, 1, 2, 3, 4}. Thus the vertices of the exterior face have the suffixes 0, 1, 2, 3, 4. Hence no vertex of X can be placed in the exterior face of H to reconstruct M from H.

Thus we conclude that M cannot be a spanning plane triangulation of $G = C_3 \otimes K_5$. This completes the proof. ■

One may be tempted to see if there is a spanning plane triangulation in $C_3 \otimes K_6$. Infact, we can find a spanning plane triangulation in $C_3 \otimes K_6$. As in the previous cases we write the vertex set of $C_3 \otimes K_6$ as $X = \{ x_0,x_1,x_2,x_3,x_4,x_5 \}$, $Y = \{ y_0,y_1,y_2,y_3,y_4,y_5 \}$ and $Z = \{ z_0,z_1,z_2,z_3,z_4,z_5 \}$. With this vertex set we give below a spanning plane triangulation of $C_3 \otimes K_6$.



A spanning plane triangulation in $C_3 \otimes K_6$.

Figure 8

Next we present a general construction for the existence of a spanning plane triangulation in $C_3 \otimes K_n$, $n \geq 7$.

In the case when $n \equiv 0 \pmod 3$, we consider the array of numbers together with x, y and z as shown in (a) below:

x	y	z	x	y	z	x	y	z
0	1	2	0	1	2	0	1	2
3	4	5	3	4	5	3	4	5
6	7	8	6	7	8	6	7	8
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
n - 3	n - 2	n - 1	n - 1	0	1	n - 2	n - 1	0
2	0	1	2	3	4	1	2	3
5	3	4	5	6	7	4	5	6
8	6	7	8	9	10	7	8	9
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
n - 1	n - 3	n - 2	n - 2	n - 1	0	n - 1	0	1
1	2	0	1	2	3	2	3	4
4	5	3	4	5	6	5	6	7
7	8	6	7	8	9	8	9	10
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
n - 2	n - 1	n - 3	n - 3	n - 2	n - 1	n - 3	n - 2	n - 1
	(a)			(b)			(c)	

We take the rows as suffixes of x, y and z in order, for example, the first row corresponds to x_0, y_1 and z_2 the vertices of the innermost cycle C_3 (see for example, Figure 8). The next innermost C_3 is based on the second row, namely x_3, y_4 and z_5. Proceed further to construct a spanning plane triangulation. Similarly we can find spanning plane triangulation in the cases $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$ using the arrays (b) and (c).

Finally, we raise the following problem.

Problem. If G has no spanning plane triangulation as a subgraph, then $G * \overline{K}_n$, where $\overline{K}_n$ denotes the complement of K_n and $n \geq 3$, does not contain a spanning plane triangulation as a subgraph.

If this problem is true, then this will give families of infinitely many locally hamiltonian graphs not admitting a spanning plane triangulation by taking $G = C_3 \otimes K_4$ or $G = C_3 \otimes K_5$.

If the above problem is true, then $H_1 = (C_3 \otimes K_4) * \overline{K}_n$, $n \geq 3$, is a locally hamilton cycle decomposable graph since the subgraph induced by a vertex of H_1 is isomorphic to $C_6 * \overline{K}_n$ which is hamilton cycle decomposable, see [6].

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