ON SELF-COMPLEMENTARY VERTEX-TRANSITIVE GRAPHS OF ORDER A PRODUCT OF DISTINCT PRIMES

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ABSTRACT. We show that a self-complementary vertex-transitive graph of order pq, p and q distinct primes, is isomorphic to a circulant graph of order pq. We will also show that if Γ is a self-complementary Cayley graph of the nonabelian group G of order pq, then Γ and the complement of Γ are not isomorphic by a group automorphism of G.

In [5], it was shown that there exists a self-complementary vertex-transitive graph of order pq, p and q distinct primes, if and only if $q, p \equiv 1 \pmod 4$. In [12], this result was generalized and it was shown that there exist self-complementary vertex-transitive graphs of order n if and only if whenever $p^a|n$ but $p^{a+1} \not| n$, p a prime, then $p^a \equiv 1 \pmod 4$. We will show that every self-complementary vertex-transitive graph of order pq is isomorphic to a circulant graph.

Liskovets and Pöschel [8], and Jajcay and Li [6], both independently showed that there exist self-complementary circulant graphs of order p^2 which are not isomorphic to their complements by a group automorphism of $\mathbb{Z}_{p^2}$, and gave all such graphs. This work was motivated, in part, by a construction of Suprunenko [14] for constructing self-complementary Cayley graphs using group automorphisms. Their work shows that, at least in some sense, not every self-complementary Cayley graph can be constructed using Suprunenko's construction. It is worth while to point out that the self-complementary circulant graphs of order p^2 which cannot be constructed using Suprunenko's construction given in the above results are also isomorphic to Cayley graphs of $\mathbb{Z}_p^2$, and *can* be constructed using Suprunenko's construction with a group automorphism of $\mathbb{Z}_p^2$ as $\mathbb{Z}_p^2$ is a CI-group with respect to graphs [4]. Thus every self-complementary Cayley graph of order p^2 (and hence every self-complementary vertex-transitive graph of order

p^2, see [9]) can be constructed *upto isomorphism* using Suprunenko's construction. We will show that if Γ is a self-complementary Cayley graph of the nonabelian group G of order pq, then Γ and it's complement are not isomorphic by a group automorphism of G. As every self-complementary vertex-transitive graph of order pq is isomorphic to a circulant graph and, as Alspach and Parsons [1] have shown that $\mathbb{Z}_{pq}$ is a CI-group with respect to graphs, every self-complementary vertex-transitive graph of order pq can be constructed upto isomorphism using Suprunenko's construction. Finally, the interested reader is referred to [7], for a general study of the structure of self-complementary vertex-transitive graphs is undertaken, and an infinite family of non-Cayley self-complementary vertex-transitive graphs is constructed.

Throughout this paper, all graphs and groups are finite. Furthermore, all graphs are loopless without multiple edges. For permutation group theory terminology not defined in this paper, see [15].

Definition 1. A graph Γ is *vertex-transitive* if $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ, is transitive on $V(\Gamma)$. The *complement of* Γ, denoted $\bar{\Gamma}$, is defined by $V(\bar{\Gamma}) = V(\Gamma)$ and $E(\bar{\Gamma}) = \{xy : xy \notin E(\Gamma)\}$. A graph is *self-complementary* if $\bar{\Gamma} \cong \Gamma$.

It is easy to see that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\bar{\Gamma})$.

Let $\mathbb{Z}_n$ be the ring of integers modulo n, and $\mathbb{Z}_n^*$ be the units of $\mathbb{Z}_n$. Let m, n be positive integers and set $\mu = \lfloor m/2 \rfloor$. Let $V = V(\Gamma) = \{v_j^i : i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$, and $\alpha \in \mathbb{Z}_n^*$. Let $S_0, S_1, \dots, S_\mu$ be subsets of $\mathbb{Z}_n$ satisfying the following conditions:

1. $0 \notin S_0 = -S_0$,
2. $\alpha^m S_r = S_r$ for $0 \le r \le \mu$,
3. if m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Let $E = \{(v_j^i, v_h^{i+r}) : 0 \le r \le \mu \text{ and } h - j \in \alpha^i S_r\}$. We define the *metacirculant graph* $\Gamma = \Gamma(m, n, \alpha, S_0, \dots, S_\mu)$ to be the graph with vertex set V and edge set E. We will also refer to Γ as an (m,n)*-metacirculant*. Define two permutations ρ, τ on V by

$\rho(v_j^i) = v_{j+1}^i$

and

$\tau(v_j^i) = v_{\alpha j}^{i+1}$.

Metacirculant graphs were first introduced by Alspach and Parsons [2], where their elementary properties were discussed.

Theorem 1 (Alspach and Parsons, [2]). *The (m,n)-metacirculant graph $\Gamma = \Gamma(m,n,\alpha,S_0,\dots,S_\mu)$ is vertex-transitive with $\langle\rho,\tau\rangle \le \mathrm{Aut}(\Gamma)$. Conversely, any graph Γ' with vertex set V and $\langle\rho,\tau\rangle \le \mathrm{Aut}(\Gamma')$ is an (m,n)-metacirculant.*

We remark that if $q < p$ are distinct primes and $\Gamma(q,p,\alpha,S_0,\dots,S_\mu)$ is a metacirculant graph, then by raising τ to an appropriate power relatively prime to q, we may assume that $|\alpha| = q^k$, $k \ge 0$, in which case $|\langle\rho,\tau\rangle| = pq^k$.

Marušič and Scapellato have shown [11] that $SL(2,2^k)$ has an imprimitive representation with $PG(1,2^k)$ as the complete block system of imprimitivity. Furthermore, in the same paper, they gave the following description of all graphs admitting the previously mentioned imprimitive representation of $SL(2,2^k)$ (see [10] for the following concise definition of these graphs).

Definition 2 ([10], Definition 1.3). Let m be a divisor of $2^k - 1$. For a symmetric subset S of $\mathbb{Z}_m - \{0\}$ and a subset T of $\mathbb{Z}_m$, we let $X(k,m,S,T)$ denote the graph with vertex set $PG(1,2^k)\times\mathbb{Z}_m$ such that, for each $r \in \mathbb{Z}_m$ and each $x \in PG(1,2^k)$, the neighbors of (x,r) are all vertices of the form $(\infty, r+s)$, $s \in S$, and $(y, r+t)$, $y \in GF(2^k)$, $t \in T$ if $x = \infty$ and all the vertices of the form $(x, r+s)$, $s \in S$, $(\infty, r-t)$, $t \in T$, and $(x+w^i, -r+t+2i)$, $i \in \mathbb{Z}_m$, $t \in T$ if $x \ne \infty$.

Metacirculant graphs and the Marušič-Scapelleto graphs defined above are important in this context for the following reason.

Theorem 2 (Theorem, [10]). *Let Γ be a vertex-transitive graph of order pq, where $q < p$ are distinct primes, such that $\mathrm{Aut}(\Gamma)$ contains an imprimitive subgroup H. Then either Γ is a metacirculant graph or H admits a complete block system of p blocks of size q, $p = 2^{2^a} + 1$ is a Fermat prime, q divides $2^{2^a} - 1$, and there is a symmetric subset S of $\mathbb{Z}_q^*$ and a proper, nonempty subset T of $\mathbb{Z}_q$ such that $\Gamma \cong X(2^a, q, S, T)$.*

Lemma 3. *No graph $X(k,m,S,T)$ is self-complementary.*

Proof. It was shown in [11] that the imprimitive representation of $G = SL(2, 2^k)$ with $PG(1, 2^k)$ as the complete block system of imprimitivity of degree $m(2^k+1)$ is unique. Furthermore, by [11, Lemma 2.3], the stabilizer of a point in this representation has m orbits of length 1 and m orbits of length 2^k. As m is a divisor of $2^k - 1$, $m < 2^k$. Let $x \in V(X(k, m, S, T))$. As $\mathrm{Stab}_G(x)$ has m orbits of length 1 and m orbits of length 2^k, if x is adjacent in $X(k, m, S, T)$ to y and y is contained in an orbit of length 2^k, then x is adjacent to every vertex in the orbit of $\mathrm{Stab}_G(x)$ that contains y. Let r be the number of orbits $\mathcal{O}_y$ of $\mathrm{Stab}_G(x)$ whose length is 2^k such that $xy \in E(X(k, m, S, T))$ for some $y \in \mathcal{O}_y$. Thus x is adjacent to $r2^k + s$ vertices of $X(k, m, S, T)$, where $s < m$. Then x is adjacent in $\overline{X(k, m, S, T)}$ to $(m - r)2^k + (m - 1 - s)$ vertices. As m divides $2^k - 1$, it is odd, and r and $m - r$ are not equal, with, say, $m - r > r$. Then $r2^k + s < r2^k + 2^k = (r + 1)2^k \leq (m - r)2^k + (m - 1 - s)$, so that $r2^k + s < (m - r)2^k + (m - s - 1)$. Thus $\deg_{X(k,m,S,T)}(x) < \deg_{\overline{X(k,m,S,T)}}(x)$. Of course, if $X(k, m, S, T)$ is self-complementary, then every vertex of $X(k, m, S, T)$ and $\overline{X(k, m, S, T)}$ must have degree $(m(2^k + 1) - 1)/2$. Thus $X(k, m, S, T)$ is not self-complementary. □

Definition 3. Let G be a group. For $g \in G$, define $g_L : G \to G$ by $g_L(h) = gh$. Then $G_L = \{g_L : g \in G\}$ is itself a group, the *left regular representation of* G and is isomorphic to G. Let $S \subset G - \{1\}$ such that $S^{-1} = S$. Define a graph $\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(j, h) : j^{-1}h \in S\}$. We say that $\Gamma(G, S)$ is a *Cayley graph of* G. Clearly $j^{-1}h \in S$ and $(gj)^{-1}gh \in S$ are equivalent statements, so that $g_L \in \mathrm{Aut}(\Gamma)$. Hence $G_L \leq \mathrm{Aut}(\Gamma)$. A *circulant graph of order* n is a Cayley graph of $\mathbb{Z}_n$.

We remark that if $\Gamma(q, p, \alpha, S_0, S_1, \ldots, S_\mu)$ is a metacirculant graph with $|\alpha| = 1$, then Γ is canonically isomorphic to a circulant graph.

Definition 4. Let Γ be a vertex-transitive graph of order k such that $\Gamma \not\cong Y \wr E^n$, where E^n is the graph on n vertices with no edges and $n \neq 1$. We say that Γ is *irreducible.* Let $G \leq \mathrm{Aut}(\Gamma)$ with $|\mathrm{Stab}_G(x)| = n$, x is any vertex of Γ. In [13] Sabidussi showed that for every transitive group $G \leq \mathrm{Aut}(\Gamma)$, the graph $\Gamma \wr E^n$ is a Cayley graph of G. We will refer to Γ

as an n-Cayley graph of G (so $n = 1$ means that Γ is a Cayley graph of G). Furthermore, Sabidussi also showed that $\text{Aut}(\Gamma \wr E^n) = \text{Aut}(\Gamma) \wr S_n$. Hence $\text{Aut}(\Gamma \wr E^n)$ admits a complete block system $\mathcal{B}$ of k blocks of size n, formed by the orbits of $1_{S_k} \wr S_n$. There thus exists a correspondence between the vertex set of Γ and the complete block system $\mathcal{B}$. For w a vertex of $\Gamma \wr E^n$, we denote by w_* the vertex of Γ which corresponds to the block of $\mathcal{B}$ that contains w. Note that if Γ' is an another irreducible vertex-transitive graph, and $\delta : \Gamma \wr E^n \to \Gamma' \wr E^n$ is an isomorphism, then $\delta_* : \Gamma \to \Gamma'$ is an isomorphism where $\delta_*(x_*) = y_*$ if and only if $\delta(x) \in y_*$. Further, if $\delta : V(\Gamma) \to V(\Gamma')$ is an isomorphism, then $n\delta : V(\Gamma \wr E^n) \to V(\Gamma' \wr E^n)$ where $n\delta((x, a)) = (\delta(x), a)$ is also an isomorphism. Finally, $(n\delta)_* = \delta$.

Definition 5. Let G be a group and Γ a Cayley graph of G. We say that G is a *CI-group with respect to graphs* if whenever Γ' is a Cayley graph of G, then $\Gamma \cong \Gamma'$ if and only if there exists $\alpha \in \text{Aut}(G)$ such that $\alpha(\Gamma) = \Gamma'$.

Definition 6. Assume that if Γ and Γ' are isomorphic n-Cayley graphs of G then Γ and Γ' are isomorphic by α_*, for some $\alpha \in \text{Aut}(G)$. We then say that Γ is a *(n, G)-CI-graph.* If every n-Cayley graph of G is an (n, G)-CI-graph, we say that G is a *(n, G)-CI-group.* If every n-Cayley graph of G that is not a (n, G)-CI-graph is a Cayley graph of G' and G' is a CI-group with respect to graphs, we say that G is a *weak (n, G)-CI group via G'.*

Definition 7. Let G be a transitive permutation group admitting a complete block system $\mathcal{B}$ of m blocks of size k. For $g \in G$, define $g/\mathcal{B}$ in the symmetric group S^m by $(g/\mathcal{B})(i) = j$ if and only if $g(B_i) = B_j$, $B_i, B_j \in \mathcal{B}$. There is thus a canonical homomorphism $\pi : G \to S^m$ given by $\pi(g) = g/\mathcal{B}$. We denote the kernel of this homomorphism by $\text{fix}_G(\mathcal{B})$. Hence $\text{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$.

Theorem 4. *Let Γ be a self-complimentary q^k-Cayley graph of the non-abelian group $\langle \rho, \tau \rangle$ of order pq^{k+1}, where ρ, τ are defined as above with $m = q$, $n = p$, and $|\alpha| = q^{k+1}$. Then Γ is not a $(q^k, \langle \rho, \tau \rangle)$-CI-graph. In particular, there does not exist $\alpha \in \text{Aut}(\langle \rho, \tau \rangle)$ such that $\alpha_*(\Gamma) = \bar{\Gamma}$.*

Proof. By Theorem 1, $\Gamma = \Gamma(q,p,\alpha,S_0,\dots,S_\mu)$ is a (q,p)-metacirculant graph so that $\bar{\Gamma} = \bar{\Gamma}(q,p,\alpha,\mathbb{Z}_p^* - S_0,\mathbb{Z}_p - S_1,\dots,\mathbb{Z}_p - S_\mu)$ is also a (q,p)-metacirculant graph. Assume there exists $\omega_* : \Gamma \to \bar{\Gamma}$ such that $\omega \in \mathrm{Aut}(\langle\rho,\tau\rangle$ and ω_* is an isomorphism (so that $\omega_* \in N_{S^V}(\langle\rho,\tau\rangle))$. Let $\mathcal{B}$ be the unique complete block system of q blocks of size p of $\langle\rho,\tau\rangle$ formed by the orbits of $\langle\rho\rangle$. As $\langle\rho\rangle$ is the unique Sylow p-subgroup of $\langle\rho,\tau\rangle$, $\omega_*^{-1}\langle\rho\rangle\omega_* = \langle\rho\rangle$. Furthermore, as ρ,τ generate $\langle\rho,\tau\rangle$, we must also have that $\omega_*^{-1}\langle\tau\rangle\omega_* = \langle\tau\rho^\ell\rangle$, for some $\ell \in \mathbb{Z}_p$. Thus $\omega_*^{-1}\langle\tau\rangle\omega_*/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$. As ω_* is a bijection from V to V, we have that $\omega_* \in S^V$. As $\tau(V^i) = V^{i+1}$ (where $V^i = \{v_j^i : j \in \mathbb{Z}_p\}$), we have that $\langle\tau\rangle/\mathcal{B}$ is canonically isomorphic to $(\mathbb{Z}_q)_L$ and $\tau/\mathcal{B} = \sigma \in N_{S^q}((\mathbb{Z}_q)_L)$. It is well known that $N_{S^q}((\mathbb{Z}_q)_L) = \{x \to ax+b : a \in \mathbb{Z}_q^*, b \in \mathbb{Z}_q\}$, so that $\sigma(i) = ri+b$, $r \in \mathbb{Z}_q^*$, $b \in \mathbb{Z}_q$. Let $\iota = \omega_*\tau^{-b}$. Note $\iota \in N_{S^V}(\langle\rho\rangle)$. As $N_{S^V}(\langle\rho\rangle) = \{v_j^i \to v_{\beta j+b_i}^{\sigma(i)} : \sigma \in S^q, \beta \in \mathbb{Z}_p^*, b_i \in \mathbb{Z}_p\}$, we have that $\iota(v_j^i) = v_{\beta j+b_i}^{ri}$ for some $\beta \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. Then $\iota(v_j^i) = v_{\beta j+b_i}^{ri}$ so that $\iota^{-1}(v_j^i) = v_{\beta^{-1}j-\beta^{-1}b_{r^{-1}i}}^{r^{-1}i}$. Then

$$\iota^{-1}\tau\iota(v_j^i) = v_{\alpha j+\beta^{-1}\alpha b_i-\beta^{-1}b_{i+r^{-1}}}^{i+r^{-1}}$$

and $\iota^{-1}\tau\iota \in \langle\rho,\tau\rangle$. Thus there exists $c \in \mathbb{Z}_p$ such that $\iota^{-1}\tau\iota\rho^c(v_j^i) = v_{\alpha j}^{i+r^{-1}}$. Hence $\tau^{-r^{-1}}\iota^{-1}\tau\iota\rho^c(v_j^i) = v^i_{\alpha^{-r^{-1}+1}j}$, and $\tau^{-r^{-1}}\iota^{-1}\tau\iota\rho^c \in \mathrm{fix}_{\langle\rho,\tau\rangle}(\mathcal{B})$. If $r \neq 1$, then $\langle\alpha\rangle = \langle\alpha^{-r^{-1}+1}\rangle$ and $|\tau^{-r^{-1}}\iota^{-1}\tau\iota\rho^c| \geq q^{k+1}$. Thus $|\mathrm{fix}_{\langle\rho,\tau\rangle}(\mathcal{B})| \geq p\cdot q^{k+1}$ and $|\langle\rho,\tau\rangle| \geq p\cdot q^{k+2}$. As $\langle\rho,\tau\rangle$ is a q^k-Cayley graph of $\langle\rho,\tau\rangle$, we have that $|\langle\rho,\tau\rangle| = p\cdot q^{k+1}$, a contradiction. Thus $r = 1$.

As $r = 1$, ι maps the $|S_1|$ edges between v_0^0 and elements of V^1 in Γ to the $|\mathbb{Z}_p - S_1|$ edges between v_0^0 and elements of V^1 in $\bar{\Gamma}$. However, as $p \neq 2$ is prime, $|S_1| \neq |\mathbb{Z}_p - S_1|$, a contradiction. □

Theorem 5 ([3], Theorem 21). *Let $q|(p-1)$, where q and p are prime, $\alpha \in \mathbb{Z}_p^*$ such that $|\alpha| = q^{k+1}$, and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Then $G = \langle\rho,\tau\rangle$ is a weak (q^k,G)-CI group via $\mathbb{Z}_{qp}$ with respect to graphs, and is a (q^k,G)-CI-group with respect to graphs if and only if $q \leq 3$.*

Corollary 6. *Every self-complementary vertex-transitive graph of order a product of two distinct primes is isomorphic to a circulant graph.*

Proof. Let Γ be a self-complementary vertex-transitive graph of order pq, $q < p$ distinct primes, so that $q \equiv 1 \pmod 4$. Clearly $\mathrm{Aut}(\Gamma)$ is either primitive or imprimitive. If $\mathrm{Aut}(\Gamma)$ is primitive, then Γ is not self-complementary by [5, Proposition 2.2]. Thus $\mathrm{Aut}(\Gamma)$ is imprimitive. By Theorem 2, Γ is either a Marušič-Scapelleto graph, in which case Γ is not self-complementary by Lemma 3, or a metacirculant graph. Thus Γ is a metacirculant graph, and, either the result follows, or Γ is an $(q, p, \alpha, S_0, \ldots, S_\mu)$-metacirculant graph with $|\alpha| = q^{k+1}$, $k \geq 0$. By Theorem 4, Γ is not a $(q^k, \langle \rho, \tau \rangle)$-CI-graph. It then follows from Theorem 5 that Γ is isomorphic to a circulant graph and the result follows. □

Finally, it is now not very difficult to give examples of self-complementary Cayley graphs $\Gamma(G, S)$, where G is a nonabelian group of order pq, p and q odd primes such that $q|(p-1)$, having the property that $\Gamma(G, S)$ and it's complement are not isomorphic by a group automorphism of G. Indeed, if Γ is any self-complementary vertex-transitive graph of order pq, we know by Corollary 6 that Γ is isomorphic to a circulant graph. We thus need only find circulant self-complementary graphs of order pq that are also isomorphic to Cayley graphs of G. As a circulant graph of order pq may be regarded as a $(q, p, 1, S_0, \ldots, S_\mu)$-metacirculant graph, we will henceforth consider only such (q, p)-metacirculant graphs. Let $\rho = z_0 \ldots z_{q-1}$, where each z_i is a p-cycle that permutes v_0^i, and $\alpha \in \mathbb{Z}_p^*$ such that $|\alpha| = q$. Let $\delta = \Pi_{i=0}^{q-1} z_i^{\alpha^i}$. A straightforward computation will then show that $\tau^{-1}\delta\tau = \delta^\alpha$. It is then easy to see that $\langle \tau, \delta \rangle$ is isomorphic to G. Let Γ_q and Γ_p be self-complementary circulant graphs of order q and p, respectively. Then $\Gamma_q \wr \Gamma_p$ is a circulant self-complementary graph of order qp, and $\delta \in \mathrm{Aut}(\Gamma_q \wr \Gamma_p) = \mathrm{Aut}(\Gamma_q) \wr \mathrm{Aut}(\Gamma_p)$. Thus $\Gamma_q \wr \Gamma_p$ is a self-complementary vertex-transitive graph isomorphic to Cayley graphs of both groups of order pq, and so if we label $V(\Gamma_q \wr \Gamma_p)$ so that $\Gamma_q \wr \Gamma_p$ is a Cayley graph of G, then $\Gamma_q \wr \Gamma_p$ is not isomorphic to it's complement by a group automorphism of G.

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