Clique Graphs of Planar Graphs

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Abstract

We are studying clique graphs of planar graphs, $K(Planar)$, this means the graphs which are the intersection of the clique family of some planar graph. In this paper we characterize the $K_3 - free$ and $K_4 - free$ graphs which are in $K(Planar)$.

1 Introduction

Let G be a simple finite undirected graph. The clique graph of G, denoted by $K(G)$, is the intersection graph of the family $C(G)$ of cliques of G.
G is a clique graph if there exists a graph H such that $G = K(H)$. The complexity of the recognition of clique graphs is an open problem [7].
Given a class $Class$ of graphs, denote by $K(Class)$ the class containing exactly the clique graphs of the graphs in $Class$. Over the last decade, many papers appeared characterizing and solving the recognition problem for clique graphs of different classes of graphs [7]. We are studying $K(Planar)$, where $Planar$ is the class of graphs admitting an embedding in the plane. This paper contains a characterization of the $K_3 - free$ and $K_4 - free$ graphs in $K(Planar)$. Although one could expect for a full characterization to be presented, this problem may not be easy. In fact, as it has been remarked by the referee, one of the positive aspects of the present paper is to reveal that characterizing clique graph of planar graph may not be that simple.
Let us introduce the matter in the following way: a first step to know if a given graph belongs to $K(Planar)$ could be checking if the given graph is or not a clique graph. But, so far there is no polynomial time algorithm

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to recognize clique graphs. However, if the given graph is $K_4 - free$ the problem becomes easier because a $K_4 - free$ graph is a clique graph if and only if it is a Helly graph [4] and Helly graphs have polynomial time recognition [6]. Hence given a $K_4 - free$ graph G, to determine if $G \in K(Planar)$, first we can check if G is a Helly graph. If it is not, clearly $G \notin K(Planar)$. But, if it is, there exist infinite different graphs H such that $G = K(H)$, then the problem is to show when at least one of these graphs H is planar, or in other case to show that none of those graphs H is a planar graph.

Notice that if G is a $K_4 - free$ Helly graph and $K(H) = G$ then H is a Helly graph (if H is not a Helly graph then H contains four pairwise intersecting cliques), thus $K(K(H)) = K(G)$ is a subgraph of H [1]. It follows that $G \in K(Planar)$ implies $K(G) \in Planar$. The converse is in general not true. However we have described extra conditions that lead to a complete characterization of $K_4 - free$ graphs in $K(Planar)$. On the other hand we show that for $K_3 - free$ graphs the converse is true without extra conditions. Then a $K_3 - free$ graph comes (through K) from a planar graph if and only if it goes (through K) to a planar graph. We use a result about line graphs to characterize these graphs.

2 Definitions and previous results

We consider finite, undirected and simple graphs. $V(G)$ and $E(G)$ denote respectively the vertex set and the edge set of the graph G. A *complete* of G is a subset of $V(G)$ inducing a complete subgraph. A *clique* is a maximal complete. We also use the term *clique* referring to the corresponding subgraph. If $v \in V(G)$ the *closed neighborhood* of v, $N[v]$, is the set of all vertices adjacent to v, and v. We say that v is *dominated* if there exists another vertex $u \neq v$ such that $N[v] \subseteq N[u]$. An edge with endvertices u and v is denoted uv. We say that G is a supergraph of H if H is a subgraph of G.

The *union* of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ satisfying

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \text{ and } E(G_1 \cup G_2) = E(G_1) \cup E(G_2).$$

If $T \subseteq E(G)$, $G - T$ denotes the graph such that

$$V(G - T) = V(G) \text{ and } E(G - T) = E(G) - T.$$

Given a *set family* $\mathcal{F} = (F_i)_{i \in I}$, we call *members* the sets F_i and *elements* the elements of $\cup_{i \in I} F_i$. The family $\mathcal{F}$ has the *Helly property* or is a *Helly family*, if any pairwise intersecting subfamily has nonempty total intersection. We say that the family is *separating* when for each pair of elements u and v, there is a member F_u such $u \in F_u$ and $v \notin F_u$. Notice that it is equivalent to say that for every element u the intersection of all members containing u is the set $\{u\}$. The *union* of two set families $\mathcal{F} = (F_i)_{i\in I}$ and $\mathcal{F}' = (F_j)_{j\in J}$ is the set family

$$\mathcal{F} \cup \mathcal{F}' = (F_s)_{s\in I\cup J}.$$

If G is a graph, $\mathcal{C}(G)$ denotes the *clique family* of G. G is said to be a *Helly graph* if $\mathcal{C}(G)$ is a Helly family. Let $Helly$ be the class of all Helly graphs. The *intersection operator*, L, maps a set family $\mathcal{F} = (F_i)_{i\in I}$ into the graph $L(\mathcal{F})$ satisfying

$$V(L(\mathcal{F})) = \{F_i, i \in I\} \text{ and } E(L(\mathcal{F})) = \{F_iF_{i'}/F_i \cap F_{i'} \neq \emptyset\}.$$

Notice that the members of $\mathcal{F}$ and the vertices of $L(\mathcal{F})$ are named in the same manner. *The clique graph* of G, denoted $K(G)$, is the graph $L(\mathcal{C}(G))$. G is *a clique graph* if there exists another graph H such that $G = K(H)$. If $Class$ is a class of graphs, then $K(Class) = \{K(G)/G \in Class\}$.
An **R-S** family of a graph G is a family of completes of G, covering the edges of G and with the Helly property. An **E-R-S** family of G is a separating **R-S** family.
The followings are known results about clique graphs that we use in the next section:

- $K(Helly) = Helly$ [1].
- G is a clique graph if and only if there exists an **R-S** family of G [4] .
- If C is a clique of G with at most three vertices, and $\mathcal{F}$ is an **R-S** family of G, then C is a member of $\mathcal{F}$ [4].
- $G = K(H)$ if and only if there exists an **E-R-S** family of G such that $H = L(\mathcal{F})$ [2].

It is said that a graph is *planar* if it can be drawn in the plane, in such a way that no two edges intersect except at an endvertex in common. Let $Planar$ be the class of all planar graphs. A graph G' is said to be a *subdivision* of a graph G if G' is obtained from G by subdividing some of its edges, that is, by replacing the edges by paths having at most their endvertices in common. Kuratowski's theorem gives a characterization of planar graphs in terms of forbidden subgraphs: A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ [3, Kuratowski's theorem].

3 The main results

The following theorem describes a total characterization of the $K_4 - free$ graphs which are the clique graph of a planar graph. The simple lemma below is used in the proof of the theorem.

Lemma 1 *Let G be a graph and $(H_i)_{i\in I}$ a family of connected subgraphs of G. The intersection graph $L(\mathcal{F})$ of the family $\mathcal{F} = (V(H_i))_{i\in I}$ is connected, if and only if $\cup_{i\in I} H_i$ is connected.*

Theorem 1 *Let G be a $K_4 - free$ clique graph. $G \in K(Planar)$ if and only if the following three conditions are satisfied*

(1) $K(G)$ is planar.

(2) Every edge of G is in at most 3 cliques of G.

(3) If an edge uv of G is in 3 cliques of G: $A = \{u, v, a\}$, $B = \{u, v, b\}$ and $C = \{u, v, c\}$; and T is the set of the edges of the cliques of G containing u or v, then the vertices a, b and c are not in the same connected component of G-T.

Proof: Since $G \in K(Planar)$, there is a planar graph H such that $G = K(H)$. Let $\mathcal{F}$ be an **E-R-S** family of G satisfying $L(\mathcal{F}) = H$.
Since G is $K_4 - free$, $\mathcal{C}(G)$ is a subfamily of $\mathcal{F}$, then $L(\mathcal{C}(G)) = K(G)$ is an induced subgraph of $L(\mathcal{F}) = H$. This prove that $K(G)$ is planar.
Suppose there is an edge uv of G in four different cliques. Since $\mathcal{F}$ is a separating family, it must contain a member F_u such that $u \in F_u$ and $v \notin F_u$. Since the four cliques and F_u are different members of $\mathcal{F}$, and all of them contain the vertex u, their intersection generates a K_5 in $L(\mathcal{F}) = H$. This contradicts the planarity of H. We have proved that every edge of G is in at most three cliques.
We will prove (3) by contradiction. Suppose (3) is not true, then there exists an edge uv in three cliques of G: $A = \{u, v, a\}$, $B = \{u, v, b\}$ and $C = \{u, v, c\}$; and the vertices a, b and c are in a same connected component X of $G - T$, where T is the set of the edges of the cliques of G containing u or v. Notice that if $e \in E(G - T)$ and e is an edge of a clique C then

$$u \notin C \text{ and } v \notin C \qquad \text{(I)}$$

Since $\mathcal{F}$ is a separating family, it must contain a set F_u such that $u \in F_u$ and $v \notin F_u$ and another set F_v such that $v \in F_v$ and $u \notin F_v$. The intersection of the three cliques A, B, C and these two sets generate in $H = L(\mathcal{F})$ the subgraph S of the Figure 1 which is planar, but we will prove that there is a supergraph of S in $L(\mathcal{F})$ (it means in H) which is not planar because it has a subdivision of $K_{3,3}$. More precisely, we will show that there is a subgraph S' of $L(\mathcal{F})$, connected and disjoint from S, which contains vertices adjacent to A, B and C. Thus clearly there is a subdivision of $K_{3,3}$.
Let $\mathcal{C}_X = (C_i)_{i\in I}$ be the family of cliques of G, containing any edge of the connected component X. $\mathcal{C}_X$ is a subfamily of $\mathcal{F}$, then $S' = L(\mathcal{C}_X)$ is an induced subgraph of $L(\mathcal{F})$.

If C_i is a member of $\mathcal{C}_X$, by definition of $\mathcal{C}_X$ and (I), C_i does not contain u or v, then C_i is not A, B, nor C, and it is not F_u neither F_v, in case these sets are cliques of G. It follows that S' and S are disjoint subgraphs.
On the other hand every C_i and $\cup_{i\in I}C_i$ are connected then, by the previous lemma, $S' = L(\mathcal{C}_X)$ is connected. We have proved that S' is a connected subgraph of H disjoint from S. Now, since a, b and c are vertices of the connected component X, there exist edges of X incident to those vertices, then there are members of $\mathcal{C}_X$ containing a, b and c. Clearly these members of $\mathcal{C}_X$ are vertices of S' adjacent in $L(\mathcal{F})$ to A, B, and C respectively, then the proof follows.

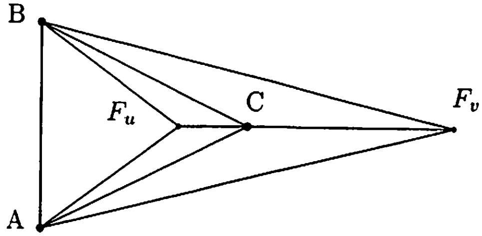

Fig.1: Subgraph S of $H = L(\mathcal{F})$

To prove the converse let G be a $K_4 - free$ clique graph, then G is a Helly graph, so $\mathcal{C}(G)$ is an **R-S** family of G. Let $Dom(G) = \{u_1, u_2, ..., u_s\}$ be the set of dominated vertices of G, clearly the family $\mathcal{F} = \mathcal{C}(G) \cup (\{u_i\})_{1\leq i\leq s}$ is an **E-R-S** family of G [2], then $H = L(\mathcal{F})$ is a graph such that $G = K(H)$. We will prove that if (1), (2) and (3) are satisfied, H is a planar graph. Exactly we will prove by induction that for every j, $0 \leq j \leq s$, the graph $H_j = L(\mathcal{F}_j)$ is planar, where

$$\mathcal{F}_0 = \mathcal{C}(G) \text{ and, if } j > 0,\ \mathcal{F}_j = \mathcal{C}(G) \cup (\{u_i\})_{1\leq i\leq j})$$

The proposition is true when $j = 0$, by (1) and $H_0 = K(G)$. Suppose that for a given $l - 1 \geq 0$, H_{l-1} is planar, then we will prove that H_l is also planar. Clearly H_l is the planar graph H_{l-1} plus a new vertex $\{u_l\}$ adjacent only to all those cliques of G containing the dominated vertex u_l. Let us show that these cliques are at most three. Since u_l is a dominated vertex, there exists a vertex $v \neq u_l$ such that $N[u_l] \subseteq N[v]$. It follows that the edge vu_l is in every clique containing u_l, then by (2), u_l is in at most three cliques. Now, let us consider separately the case when u_l is in one or two cliques and the case when u_l is exactly in three cliques of G.
If u_l is in one or two cliques of G, to obtain H_l we have to add to H_{l-1} the vertex $\{u_l\}$ adjacent only to one vertex of H_{l-1} or adjacent to both extremes of an edge of H_{l-1}, then is clear that H_l is planar if H_{l-1} is so.
If u_l is in three cliques of G, these cliques must be triangles because G is

$K_4 - free$, so we can denote these cliques A= $\{u_l, v, a\}$, B= $\{u_l, v, b\}$ and C= $\{u_l, v, c\}$. If H_{l-1} admits a planar representation with the triangle A, B, C forming a face, we can add to H_{l-1} the vertex $\{u_l\}$ in the center of that face and do $\{u_l\}$ adjacent to A, B and C maintaining planarity. If there is not such representation we will prove in the following that (3) is contradicted, i.e. we will prove that a, b and c are in a connected subgraph of $G - T$, where T is the set of the edges of the cliques of G containing u_l or v.
If H_{l-1} does not admit a planar representation with the triangle A, B, C forming a face, it must be because in H_{l-1} there are disjoint induced subgraphs S and S', not containing the vertices A, B or C, and containing vertices adjacent to A, B and C, as it is shown in Figure 2.

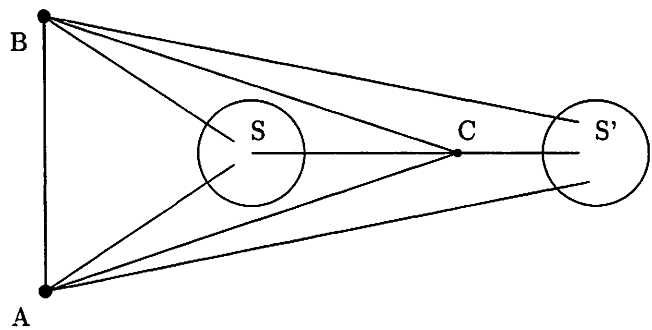

Fig.2: The graph $H_{l-1} = L(\mathcal{F}_{l-1})$

It is clear that the vertices of S and S' are members of $\mathcal{F}_{l-1}$. Since u_l is not in a member of $\mathcal{F}_{l-1}$ except A, B and C, and since v is in A, B,C and in at most one other member of $\mathcal{F}_{l-1}$, we can assume that:

$$\text{the vertices of } S' \text{ does not contain } u_l \text{ or } v \qquad \text{(II)}$$

Call $\mathcal{F}'$ the subfamily of $\mathcal{F}_{l-1}$ such that $L(\mathcal{F}') = S'$. Since each member of the subfamily $\mathcal{F}'$ is the vertex set of a connected subgraph of G, and since $S' = L(\mathcal{F}')$ is connected, by the previous lemma, the union of those subgraphs is a connected subgraph of G, call X that union. Then we have proved that X is connected. We have to show that a, b and c are vertices of X. In S' there is a vertex adjacent to $A = \{u_l, v, a\}$, by (II) that vertex contains a, thus by definition of X, a is a vertex of X. In a similar way we prove that b and c are vertices of X.
Finally we will prove that X is a subgraph of $G-T$. Let e be an edge of X, this means e is an edge of a clique of G such that its vertex set is a vertex of S'. Clearly the endvertices of e are not u_l or v. It follows that e is not in a clique containing u_l since the only cliques containing that vertex are

A, B or C. For the same reason, if e is in a clique containing v, this clique must be a vertex of S, and then there is a vertex of S adjacent to a vertex of S'. This is not possible since H_{l-1} is planar. □

Corollary 1 *Let G be a $K_4 - free$ clique graph. Denote by $\mathcal{F}$ the family $C(G) \cup (\{u\})_{u \in Dom}$, where Dom is the set of dominated vertices of G. $G \in K(Planar)$ if and only if $L(\mathcal{F})$ is a planar graph.*

3.1 $K_3 - free$ Graphs

Any $K_3 - free$ graph is a Helly graph, so any $K_3 - free$ graph is a clique graph. On the other hand if G is a $K_3 - free$ graph then G satisfies immediately the second and third hypothesis of the preceding theorem, so if G is $K_3 - free$, $G \in K(Planar)$ if and only if $K(G)$ is planar. Notice that for a $K_3 - free$ graph G, $K(G)$ is the line graph of G, this is the graph $L(\mathcal{A}(G))$ where $\mathcal{A}(G)$ is the family of the endvertices of the edges of G. Thus, we use the result in [5] to prove the following theorem:

Theorem 2 *Let G be a $K_3 - free$ graph. The following three statements are equivalent*

1. $G \in K(Planar)$.
2. $K(G) \in Planar$.
3. (a) $G \in Planar$,
   (b) *Every vertex of G has at most degree four, and*
   (c) *If a vertex of G has degree four then it is a cut vertex.*

4 Comments

4.1 Some graphs not in $K(Planar)$

Using the results of the previous section, we show that none of the graphs of the Figure 3 are in $K(Planar)$.
G_1 is $K_3 - free$ and it has a vertex in four cliques which is not a cut vertex, so $G_1 \notin K(Planar)$.
G_2 is $K_4 - free$ and $K(G_2)$ is non planar so $G_2 \notin K(Planar)$.
$K(G_3)$ is planar but G_3 has an edge in four cliques so $G_3 \notin K(Planar)$.
$K(G_4)$ is planar too, and every edge is in at most three cliques, but there is an edge in exactly three cliques and the vertices a, b and c do not satisfy the third hypothesis of Theorem 1 so $G_4 \notin K(Planar)$.

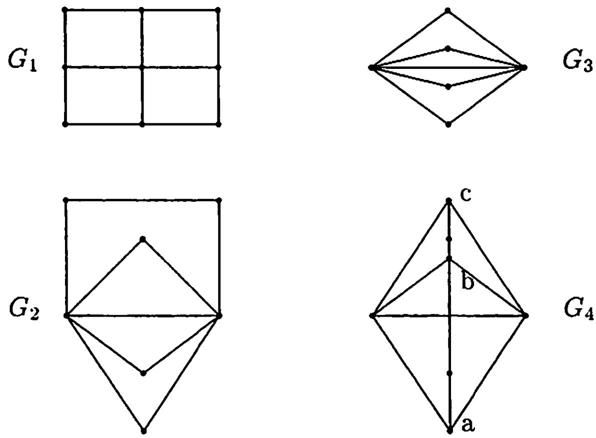

Fig.3: Some graphs not in $K(Planar)$.

The following interesting result is easily deduced from Theorem 2:

$$\textit{If } G \textit{ is a } K_3 - \textit{free non planar graph, then}$$
$$K(G) \notin Planar \textit{ and } G \notin K(Planar),$$

$K_{3,3}$ is an example of such a graph .

4.2 Complexity

The conditions stated in Theorem 1 can be verified in polynomial time. The same applies for the construction of $L(\mathcal{F})$ as well as for checking planarity.

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