Generalized orthogonal designs

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Abstract

Orthogonal designs and their special cases such as weighing matrices and Hadamard matrices have many applications in combinatorics, statistics, and coding theory as well as in signal processing. In this paper we generalize the definition of orthogonal designs, we give many constructions for these designs and we prove some multiplication theorems that, most of them, can also be applied in the special case of orthogonal designs. Some necessary conditions for the existence of generalized orthogonal designs are also given.

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1 Introduction

In this section we give some basic known facts and definitions on orthogonal designs that are necessary for our approach in generalized orthogonal designs. Then we present the generalized orthogonal designs and some other definitions we shall need.

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [1], where this was first defined, it was mentioned that

$$A^T A = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A. It was also shown in [1] that $u \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \leq d < 4$ see [3]. An orthogonal design is said to be *full* iff it contains no zeros.

Example 1.1 *The following matrix* $D = OD(4; 1, 1, 1, 1)$ *is an orthogonal design of order* 4 *and type* $(1, 1, 1, 1)$.

$$D = OD(4; 1, 1, 1, 1) = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}$$

A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n; k)$. The number k is called the *weight* of W. If $k = n$, that is, all the entries of W are ± 1 and $WW^T = nI_n$, then W is called an Hadamard matrix of order n. In this case $n = 1, 2$ or $n \equiv 0 (mod\ 4)$.

A set of matrices $\bigcup_{j=1}^{\ell} \{B_j\}$, is said to be *disjoint* if $B_i * B_j = 0$ for all $i \neq j$, $i, j = 1, 2, \ldots, \ell$ where $*$ denotes the Hadamard product.

Let $A = \{A_j : A_j = \{a_{j1}, a_{j2}, ..., a_{jn}\},\ j = 1, \ldots, \ell\}$, be a set of ℓ sequences of length n. These sequences is said to be a set of *disjoint sequences* if the set of the corresponding circulant matrices $B_j = \cup_j \{circ(A_j)\}$, $j = 1, \ldots, \ell$ is disjoint.

The *non-periodic autocorrelation function* $N_A(s)$ (abbreviated as NPAF) of the above sequences is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{1}$$

If $A_j(z) = a_{j1} + a_{j2}z + \ldots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}). \tag{2}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function* $P_A(s)$ (abbreviated as PAF) is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{3}$$

For the results of this paper generally zero PAF is sufficient. However zero NPAF sequences imply zero PAF sequences exist, the zero NPAF sequence being padded at the end with sufficient zeros to make longer lengths. Hence zero NPAF can give more general results.

Four $(0, \pm 1)$ disjoint sequences T_1, T_2, T_3, T_4 of length t are called T-sequences if they have NPAF=0. Four $(0, \pm 1)$ disjoint circulant matrices T_1, T_2, T_3, T_4 of order t are called T-matrices if they satisfy $T_1T_1^T + T_2T_2^T + T_3T_3^T + T_4T_4^T = tI_t$.

Two $(1, -1)$ sequences of length n are called *Golay sequences* iff they have NPAF=0. For the undefined terms we refer to [2].

The following theorem which uses four circulant matrices is very useful in the construction of orthogonal designs.

Theorem 1.2 *[2, Theorem 4.49] Suppose there exist four circulant matrices A, B, C, D of order n satisfying*

$$AA^T + BB^T + CC^T + DD^T = fI_n$$

Let R be the back diagonal matrix. Then

$$G = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix}$$

is a $W(4n, f)$ *when* A, B, C, D *are* $(0, 1, -1)$ *matrices, and an orthogonal design*
$OD(4n; s_1, s_2, \ldots, s_u)$ *on* $x_1, x_2, \ldots, x_u$ *when* A, B, C, D *have entries from* $\{0, \pm x_1, \ldots, \pm x_u\}$ *and* $f = \sum_{j=1}^{u}(s_j x_j^2)$. □

Corollary 1.3 *If there are four sequences* E, F, G, H *of length* n *with entries from* $\{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$ *with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices (we write* $A = circ(E)$, $B = circ(F)$, $C = circ(G)$ *and* $D = circ(H)$*) which can be used in the Goethals-Seidel array to form an* $OD(4n; s_1, s_2, s_3, s_4)$. *We note that if there are sequences of length* n *with zero non-periodic autocorrelation function, then there are sequences of length* $n + m$ *for all* $m \geq 0$. □

Orthogonal designs can exist under some necessary conditions. When these conditions are not satisfied the orthogonal design cannot exist. For example a full orthogonal design of order n cannot exist if $n \not\equiv 0(\ mod\ 4)$ and $n > 2$ or n odd. Thus we were naturally led to the next generalization.

2 Generalized orthogonal designs

Definition 1 Let D be a matrix on the commuting variables $x_1, x_2, \ldots, x_t$ where each variable can appear (in each column or row) in the form $\pm a_{ij} x_i$, $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, u_i$ and $\sum_{i=0}^{t} u_i = n$, where u_0 is the number of zeros in each row or column. Set $s_i = \sum_{j=1}^{u_i} a_{ij}^2$. Then D is a *generalized orthogonal design (in short GOD)* iff

$$DD^T = \left(\sum_{i=1}^{t} s_i x_i^2\right) I_n.$$

D will be denoted as

$$D = GOD(n; a_{1,1}, a_{1,2}, \ldots, a_{1,u_1}; a_{2,1}, a_{2,2}, \ldots, a_{2,u_2}; \ldots; a_{t,1}, a_{t,2}, \ldots, a_{t,u_t}).$$

Alternate notation of a generalized orthogonal design will be
$D = GOD(n; < k_{1,1}, a_{1,1} >, \ldots, < k_{1,u_1}, a_{1u_1} > ; \ldots; < k_{t,1}, a_{t,1} >, \ldots,$

$< k_{t,u_t}, a_{t,u_t} >$) where $k_{i,j}$ denotes how many times the variable x_i has the coefficient $a_{i,j}$. If $k_{i,j} = 1$ we write $(\ldots, a_{ij}, \ldots)$ otherwise we write $(\ldots, < k_{i,j}, a_{ij} >, \ldots)$.

Remark 2.1 If $a_{ij} = 1$ for all $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, u_i$ then the generalized orthogonal design $GOD(n; a_{11}, a_{12}, \ldots, a_{1u_1}; \ldots; a_{t1}, a_{t2}, \ldots, a_{tu_t})$ is an orthogonal design $OD(n; u_1, u_2, \ldots, u_t)$. Thus orthogonal designs are a special case of the generalized orthogonal designs we have defined above.

Example 2.2 1. Let $D = GOD(4; 1, 1; 1; 1)$. We have that $n = 4$, $u_1 = 2, u_2 = 1, u_3 = 1$ and $a_{11} = a_{12} = a_{21} = a_{31} = 1$. Thus

$$D = GOD(4; 1, 1; 1; 1) = \begin{bmatrix} a & b & a & c \\ -b & a & c & -a \\ -a & -c & a & b \\ -c & a & -b & a \end{bmatrix} = OD(4; 2, 1, 1)$$

is an orthogonal design of order 4 and type $(2, 1, 1)$.

2. As we mention above full orthogonal designs of odd order cannot exist but this is not forbidden for full (with no zeros) generalized orthogonal designs. The next matrix is a generalized orthogonal design of one variable. In this case $n = 3$, $t = 1$, $u_1 = 3$, $a_{11} = 2$, $a_{12} = 2$, $a_{13} = 1$. Thus

$$D = \begin{bmatrix} 2b & 2b & -b \\ -b & 2b & 2b \\ 2b & -b & 2b \end{bmatrix}$$

is a circulant $GOD(4; 2, 2, 1)$ and $DD^T = \left(\underbrace{b^2 \sum_{j=1}^{3} a_{1j}^2}_{s_1} \right) I_n = 9b^2 I_n$.

Some necessary conditions for the existence of a generalized orthogonal designs are given in the next theorem.

Theorem 2.3 *Let D be a generalized orthogonal design of order n and type $(a_{1,1}, a_{1,2}, \ldots, a_{1,u_1}; \ldots; a_{\ell,1}, a_{\ell,2}, \ldots, a_{\ell,u_\ell})$ on the set of commuting variables $\{x_1, x_2, \ldots, x_\ell\}$, where each variable can appear in the form $\pm a_{ij} x_i$, $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, u_i$. Then $\ell \leq \rho(n)$.*

Proof. We write D in the form $D = D_1x_1 + \ldots + D_\ell x_\ell$ where D_i are real or integer matrices of order n. Set $s_i = \sum_{i=1}^{u_i} a_{i,j}^2$. Using the fact that $DD^T = \left(\sum_{i=1}^{\ell} s_i x_i^2\right) I_n$ we have that $D_iD_i^T = s_iI_n$, $i = 1, 2, \ldots, \ell$, and $D_iD_j^T + D_jD_i^T = 0$ for all $i \neq j$. If we replace integer matrices D_i by the matrices $B_i = \left(\frac{1}{\sqrt{s_i}}\right) D_i$, then B_i are real and orthogonal matrices satisfying $B_iB_j^T + B_jB_i^T = 0$ for all $i \neq j$, and Radon [3] had shown that there are no more than $\rho(n)$ such real matrices. This completes the proof. □

Thus, generalized orthogonal designs (because of their orthogonality) cannot have more variables than the upper bound that Radon's function gives.

3 Some Construction Theorems

The theorem of this section gives us many general construction methods for generalized orthogonal designs. Some of them are a straight forward generalization of the theorems given in [1] for orthogonal designs and some other can only be applied in generalized orthogonal designs.

We use one, two or four suitable circulant matrices to construct circulant generalized orthogonal designs and block circulant generalized orthogonal designs.

Theorem 3.1 (The two circulant construction) *Let A_1, A_2 be two circulant matrices of order n with entries of the form $\pm a_{ij}x_i$, $i = 1, 2$ and $j = 1, 2, \ldots, u_i$. where x_1, x_2 are commuting variables, satisfying*

$$A_1A_1^T + A_2A_2^T = fI.$$

If f is the quadratic form $\sum_{i=1}^{2} s_i x_i^2$, there exist $u_1, u_2 \geq 1$ and a_{ij}, $i = 1, 2$, $j = 1, \ldots, u_i$ such that $s_i = \sum_{j=1}^{u_i} a_{ij}^2$ then there is a generalized orthogonal design

$$GOD(2n; a_{11}, a_{12}, \ldots, a_{1u_1}; a_{21}, a_{22}, \ldots, a_{2u_2}).$$

Moreover if $u_1 = u_2 = 1$ *and* $a_{11} = a_{21} = 1$ *then there exist an orthogonal design (special case)* $OD(2n; s_1, s_2)$. *If* f *is an integer then there exists a* $W(2n, f)$.

Proof. We use the matrices as follows

$$D = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} A_1 & A_2R \\ -A_2R & A_1 \end{pmatrix}.$$

□

Corollary 3.2 *If there are two sequences* E, F, *of length* n *with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices (we write* $A_1 = circ(E)$ *and* $A_2 = circ(F)$*) which can be used in the above arrays to form a*

$$GOD(2n; a_{11}, a_{12}, \ldots, a_{1u_1}; a_{21}, a_{22}, \ldots, a_{2u_2}).$$

We note that if there are sequences of length n *with zero non-periodic autocorrelation function, then there are sequences of length* $n+m$ *for all* $m \geq 0$.
□

Example 3.3 Using the sequences A and B as are given below in Corollary 3.2 we give some examples of full

$$GOD(2n; a_{11}, a_{12}, \ldots, a_{1u_1}; a_{21}, a_{22}, \ldots, a_{2u_2})$$

1. $A = \{a, -a, -b\}$ $B = \{2a, 3a, -a\}$ gives a $GOD(6; 1, 1, 1, 2, 3; 1)$.
2. $A = \{a, 3b, b\}$ $B = \{-b, 3b, -2a\}$ gives a $GOD(6; 1, 2; 1, 1, 3, 3)$.
3. $A = \{a, -2a, -2a\}$ $B = \{b, -2b, -2b\}$ gives a $GOD(6; 1, 2, 2; 1, 2, 2)$.
4. $A = \{a, -2a, -3b\}$ $B = \{a, 2a, -b\}$ gives a $GOD(6; 1, 1, 2, 2; 1, 3)$.
5. $A = \{a, 4b, -a\}$ $B = \{3a, -a, 2a\}$ gives a $GOD(6; 1, 1, 1, 2, 3; 4)$.
6. $A = \{a, a, a, a, a\}$ $B = \{-3a, -4a, a, -a, 3a\}$ gives a $GOD(10; 1, 1, 1, 1, 1, 1, 1, 3, 3, 4)$.

Theorem 3.4 *Suppose there exist four circulant matrices* A, B, C, D *of order* n *with entries from the set of commuting variables* $\{x_1, x_2, x_2, x_4\}$,

where each variable can appear to the form $\pm a_{ij}x_i$, $i = 1,2,3,4$ *and* $j = 1,2,\ldots,u_i$, *satisfying*

$$AA^T + BB^T + CC^T + DD^T = fI_n$$

Let R *be the back diagonal matrix and set*

$$G = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix}.$$

Now if f *is the quadratic form* $\sum_{i=1}^{2} s_i x_i^2$, *there exist* $u_1, u_2, u_3, u_4 \geq 1$ *and* a_{ij}, $i = 1,2,3,4$, $j = 1,\ldots,u_i$ *such that* $s_i = \sum_{j=1}^{u_i} a_{ij}^2$ *then there exist a generalized orthogonal design* $GOD(4n; a_{11}, a_{12}, \ldots, a_{1u_1}; a_{21}, a_{22}, \ldots, a_{2u_2}; a_{31}, a_{32}, \ldots, a_{3u_3}; a_{41}, a_{42}, \ldots, a_{4u_4})$. *Moreover if* $u_1 = u_2 = u_4 = u_3 = 1$ *and* $a_{11} = a_{21} = a_{31} = a_{41} = 1$ *then there exist an orthogonal design (special case)* $OD(4n; s_1, s_2, s_3, s_4)$. *If* f *is an integer there exists a* $W(4n, f)$. □

Proof. Observe that $GG^T = fI_{4n}$. □

Corollary 3.5 *If there are four sequences* E, F, G, H *of length* n *with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices (we write* $A = circ(E)$, $B = circ(F)$, $C = circ(G)$ *and* $D = circ(H)$*) which can be used in the Goethals-Seidel array to form an* $GOD(4n; a_{11}, a_{12}, \ldots, a_{1u_1}; a_{21}, a_{22}, \ldots, a_{2u_2}; a_{31}, a_{32}, \ldots, a_{3u_3}; a_{41}, a_{42}, \ldots, a_{4u_4})$. *We note that if there are sequences of length* n *with zero non-periodic autocorrelation function, then there are sequences of length* $n + m$ *for all* $m \geq 0$. □

Example 3.6 Using Corollary 3.5 we give some examples of full

$$GOD(4n; a_{11}, a_{12}, \ldots, a_{1u_1}; \ldots; a_{t1}, a_{22}, \ldots, a_{tu_t})$$

1. Sequences $E = \{a, b\}$, $F = \{a, b\}$, $G = \{a, b\}$, and $H = \{a, -3b\}$ have zero NPAF and thus gives a $GOD(4(2+s); < 4,1 >; < 3,1 >, 3)$, $s = 0,1,\ldots$.

2. Sequences $A = \{a_{1,1}a, a_{2,1}b\}$, $F = \{a_{3,1}c, a_{4,1}d\}$, $G = \{a_{2,1}a, -a_{1,1}b\}$, and $H = \{a_{4,1}c, -a_{3,1}d\}$ gives a $GOD(4(2+s); a_{11}, a_{21}; a_{11}, a_{21}; a_{31}, a_{41}; a_{31}, a_{41})$, $s = 0, 1, \ldots$.

Theorem 3.7 *Let X_1, X_2, X_3, X_4 be four disjoint sequences of length t with PAF=0 (or NPAF=0) and a, b, c, d are commuting variables. Then*

$$\begin{array}{llll} X = & aX_1 + bX_2 + cX_3 + dX_4 & Y = & -bX_1 + aX_2 + dX_3 - cX_4 \\ Z = & -cX_1 - dX_2 + aX_3 + bX_4 & W = & -dX_1 + cX_2 - bX_3 + aX_4 \end{array}$$

have elements from the set $\{0\} \bigcup_{i=1}^{4} \bigcup_{j=1}^{u_i} \{\pm x_{i,j}a, \pm x_{i,j}b, \pm x_{i,j}c, \pm x_{i,j}d\}$ *and can be used for the construction of a generalized orthogonal design*

$$GOD(4t; g; g; g; g)$$

and $g = x_{1,1}, \ldots, x_{1,u_1}, x_{2,1}, \ldots, x_{2,u_2}, x_{3,1}, \ldots, x_{3,u_3}, x_{4,1}, \ldots, x_{4,u_4}$ *where $x_{i,j}$, $i = 1, 2, 3, 4$, $j = 1, 2, \ldots, u_i$ are the non zero elements of sequences X_i, $i = 1, 2, 3, 4$.*

Proof. Set $A = circ(X)$, $B = circ(Y)$, $C = circ(Z)$, $D = circ(W)$ and $s_i = \sum_{j=1}^{u_i} x_{i,j}^2$ for all $i = 1, 2, 3, 4$. Then we have

$$AA^T + BB^T + CC^T + DD^T = (\sum_{i=1}^{4} s_i^2) I_n$$

and thus we can use theorem 3.4 to construct the desirable generalized orthogonal design.

Example 3.8 Let $T_1 = \{x_{1,1}, 0, 0\}$, $T_2 = \{0, x_{2,2}, 0\}$, $T_3 = \{0, 0, x_{3,3}\}$, $T_4 = \{0, 0, 0\}$. This are four disjoint sequences of length $t = 3$ and can be used in theorem 3.7 to obtain a

$$GOD(12; x_{1,1}, x_{2,2}, x_{3,3}; x_{1,1}, x_{2,2}, x_{3,3}; x_{1,1}, x_{2,2}, x_{3,3}).$$

These sequences have NPAF=0. Thus (by adding p zeros at the end of each of them) we obtain a

$$GOD(4(3+p); x_{1,1}, x_{2,2}, x_{3,3}; x_{1,1}, x_{2,2}, x_{3,3}; x_{1,1}, x_{2,2}, x_{3,3}).$$

Corollary 3.9 *Let X_1, X_2, X_3, X_4 be T-sequences of length t (or the first rows of T-matrices of order t) and a, b, c, d are commuting variables. Then if b_1, b_2, b_3, b_4 are real numbers then*

$$\begin{aligned} X = &\ b_1 a X_1 + b_2 b X_2 + b_3 c X_3 + b_4 d X_4 \\ Y = &\ -b_2 b X_1 + b_1 a X_2 + b_3 d X_3 - b_3 c X_4 \\ Z = &\ -b_3 c X_1 - b_3 d X_2 + b_1 a X_3 + b_2 b X_4 \\ W = &\ -b_3 d X_1 + b_3 c X_2 - b_2 b X_3 + b_1 a X_4 \end{aligned}$$

have elements from the set $\{\pm b_1 a, \pm b_2 b, \pm b_3 c, \pm b_4 d\}$ and can be used for the construction of a generalized orthogonal design

$$GOD(4t; < t, b_1 >; < t, b_2 >; < t, b_3 >; < t, b_4 >)$$

Proof. Set $A = circ(X)$, $B = circ(X)$, $C = circ(X)$, $D = circ(X)$. Then we have

$$AA^T + BB^T + CC^T + DD^T = (nb_1^2 a^2 + nb_2^2 b^2 + nb_3^2 c^2 + nb_4^2 d^2) I_n$$

and thus we can use theorem 3.4 to construct the desirable generalized orthogonal design.

Example 3.10 Using T-sequences of length 3 :

$$X_1 = \{1,\ 0,\ 0\},\ \ X_2 = \{0,\ 1,\ 0\},\ \ X_3 = \{0,\ 0,\ 1\}\ \ X_4 = \{0,\ 0,\ 0\}$$

we obtain the sequences

$$X = \{a, 2b, 3c\},\ Y = \{-2b, a, 5d\},\ Z = \{-3c, -5d, a\},\ W = \{-5d, 3c, -2b\}$$

which, for all $s = 0, 1, \ldots$ (we just add s zeros at the end of each sequence), can be used to construct the corresponding circulant matrices and therefore the desirable generalized orthogonal design

$$GOD(4(3+s); < 3, 1 >; < 3, 2 >; < 3, 3 >; < 3, 5 >).$$

We give the full matrix of the $GOD(12; < 3, 1 >; < 3, 2 >; < 3, 3 >; < 3, 5 >)$ we can construct from these sequences.

$$\begin{bmatrix}
a & 2b & 3c & -2b & a & 5d & -3c & -5d & a & -5d & 3c & -2b \\
3c & a & 2b & a & 5d & -2b & -5d & a & -3c & 3c & -2b & -5d \\
2b & 3c & a & 5d & -2b & a & a & -3c & -5d & -2b & -5d & 3c \\
2b & -a & -5d & a & 2b & 3c & 5d & 2b & -3c & -3c & a & -5d \\
-a & -5d & 2b & 3c & a & 2b & 2b & -3c & 5d & a & -5d & -3c \\
-5d & 2b & -a & 2b & 3c & a & -3c & 5d & 2b & -5d & -3c & a \\
3c & 5d & -a & -5d & -2b & 3c & a & 2b & 3c & 2b & -5d & a \\
5d & -a & 3c & -2b & 3c & -5d & 3c & a & 2b & -5d & -a & -2b \\
-a & 3c & 5d & 3c & -5d & -2b & 2b & 3c & a & -a & 2b & -5d \\
5d & -3c & 2b & 3c & -a & 5d & -2b & 5d & a & a & 2b & 3c \\
-3c & 2b & 5d & -a & 5d & 3c & 5d & a & -2b & 3c & a & 2b \\
2b & 5d & -3c & 5d & 3c & -a & a & -2b & 5d & 2b & 3c & a
\end{bmatrix}$$

Example 3.11 Using the T-sequences of length 11

$$\begin{aligned}
X_1 &= \{\ 1,\ 1,\ -1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0\ \} \\
X_2 &= \{\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1,\ -1,\ -1,\ 0\ \} \\
X_3 &= \{\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\ \} \\
X_4 &= \{\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ \}
\end{aligned}$$

we get

$$\begin{aligned}
X &= \{2a, 2a, -2a, 2a, 3b, 2a, 3b, 3b, -3b, -3b, 5c\} \\
Y &= \{3b, 3b, -3b, 3b, -2a, 3b, -2a, -2a, 2a, 2a, -d\} \\
Z &= \{5c, 5c, -5c, 5c, d, 5c, d, d, -d, -d, -2a\} \\
W &= \{d, d, -d, d, -5c, d, -5c, -5c, 5c, 5c, 3b\}
\end{aligned}$$

from which, for all $s = 0, 1, \ldots$ (just add s zeros at the end of each sequence), we obtain $GOD(4(11+s); <11,1>; <11,2>; <11,3>; <11,5>)$.

Theorem 3.12 *Set* $a_{1,1} = \frac{n-2}{2}a_{1,2}$. *Then there exist a circulant generalized orthogonal design* D *of one variable and* $D = GOD(n; a_{1,1}, < n-1, a_{1,2} >) = GOD(n; a_{1,1}, a_{1,2}, \ldots, a_{1,2})$.

Proof. Set $D = circ(-a_{1,1}x_1, \underbrace{a_{1,2}x_1, \ldots, a_{1,2}x_1}_{n-1 \text{ times}})$. Then

$$DD^T = [(a_{1,1}^2 + (n-1)a_{1,2}^2)x_1^2]I_n = (\frac{n^2}{2}x_1^2)I_n$$

and thus D is the required $GOD(n; a_{1,1}, < n-1, a_{1,2} >)$.

Example 3.13 (i) Let $n = 5$. From the above theorem 3.12 we have that $D = circ(-3a_{1,1}x_1,\ 2a_{1,1}x_1, 2a_{1,1}x_1, 2a_{1,1}x_1, 2a_{1,1}x_1)$ is a circulant one variable generalized orthogonal design $GOD(5; 3a_{1,1}, < 4, 2a_{1,2} >)$.

(ii) Let $n = 6$. From the above theorem 3.12 we have that $D = circ($ $-2a_{1,1}x_1, a_{1,1}x_1, a_{1,1}x_1,\ a_{1,1}x_1, a_{1,1}x_1, a_{1,1}x_1)$ is a circulant one variable generalized orthogonal design $GOD(6; 2a_{1,1}, < 5, a_{1,1} >)$.

Theorem 3.14 *Let $T_1 = \{c_1, c_2, \ldots, c_n\}$ and $T_2 = \{d_1, d_2, \ldots, d_n\}$ be two disjoint sequences of length n with zero PAF (NPAF). Then the following generalized orthogonal design exist:*

(i) $GOD(2n; a_1, a_2, \ldots, a_{s_1}, b_1, b_2, \ldots, b_{s_2}; a_1, a_2, \ldots, a_{s_1}, b_1, b_2, \ldots, b_{s_2})$ *where $a_i, i = 1, 2, \ldots, s_1$ and $b_i, i = 1, 2, \ldots, s_2$ are the non zero elements of T_1 and T_2 respectively.*

(ii) $GOD(4n; a_1, \ldots, a_{s_1}, b_1, \ldots, b_{s_2}; a_1, \ldots, a_{s_1}, b_1, \ldots, b_{s_2}; a_1, \ldots, a_{s_1}, b_1, \ldots, b_{s_2}; a_1, \ldots, a_{s_1}, b_1, \ldots, b_{s_2})$ *where $a_i,\ i = 1, 2, \ldots, s_1$ and $b_i, i = 1, 2, \ldots, s_2$ are the no n zero elements of T_1 and T_2 respectively.*

Proof.

(i) Use sequences $E = x_1T_1 + x_2T_2$ and $F = x_2T_1 - x_1T_2$ in corollary 3.2 to obtain the result.

(i) Use sequences $E = x_1T_1 + x_2T_2$, $F = x_3T_1 + x_4T_2$, $G = x_2T_1 - x_1T_2$ and $H = x_4T_1 - x_3T_2$ in corollary 3.5 to obtain the result.

□

Example 3.15 Let $T_1 = \{-2, 0, 6, 0, 0\}$ and $T_2 = \{0, 4, 0, 3, 0\}$. These are two disjoint sequences of length $n = 5$ with zero NPAF. Then using theorem 3.14 we obtain a generalized orthogonal design D of two variables of length $2n = 10$, $D = GOD(10; 2, 3, 4, 6; 2, 3, 4, 6)$.

Corollary 3.16 *Let A, B be Golay sequences of order n Then there exist a* $GOD(2n; < n, a_{1,1} >; < n, a_{1,1} >$ *and a* $GOD(4n; < n, a_{1,1} >; < n, a_{1,1} >$ $; < n, a_{1,1} >; < n, a_{1,1} >)$.

Proof. Set $X_1 = a_{1,1}\left(\frac{A+B}{2}\right)$ and $X_2 = a_{1,1}\left(\frac{A-B}{2}\right)$ and apply theorem 3.14 to get the designs. □

Remark 3.17 We know Golay sequences for all $n = 2^a \cdot 10^b \cdot 26^c$, where a, b and c are non negative integer numbers. Then from corollary 3.16 we obtain a $GOD(2n; < n, a_{1,1} >; < n, a_{1,1} >)$ and a $GOD(4n; < n, a_{1,1} >; < n, a_{1,1} >; < n, a_{1,1} >; < n, a_{1,1} >)$ where $a_{1,1}$ is any real number.

4 Multiplication Methods

In this section we discuss some results on multiplication of generalized orthogonal designs. Some theorems for multiplying the length of the design or/and increasing the number of variables in the design or/and multiplying some of the variable's coefficients by a number.

Lemma 4.1 *Let D be a*

$$GOD(n; a_{11}, a_{12}, \ldots, a_{1u_1}; \ldots; a_{t1}, a_{t2}, \ldots, a_{tu_t})$$

generalized orthogonal design with variables from the set $S = \{0\} \bigcup_{i=1}^{t} \bigcup_{j=1}^{u_i} \{\pm a_{i,j} x_i\}$, *where $x_1, x_2, \ldots, x_t$ are commuting variables. Then we can construct the following generalized orthogonal design:*

i) $GOD(n; a_{11}, a_{12}, \ldots, a_{1u_1}; \ldots; a_{i1} + a_{j1}, a_{i2} + a_{j2}, \ldots, a_{iu_i} + a_{ju_i}, a_{ju_i+1}, \ldots, a_{ju_j}; \ldots; a_{t1}, a_{t2}, \ldots, a_{tu_t})$ *on $t-1$ variables.*

ii) $GOD(n; a_{11}, a_{12}, \ldots, a_{1u_1}; \ldots; a_{i-1,1}, a_{i-1,2}, \ldots, a_{i-1,u_{i-1}}; a_{i+1,1}, a_{i+1,2}, \ldots, a_{i+1,u_{i+1}}; \ldots; a_{t1}, a_{t2}, \ldots, a_{tu_t})$ *on $t-1$ variables.*

iii) $GOD(2n; a_{11}, a_{12}, \ldots, a_{1u_1}; \ldots; a_{t1}, a_{t2}, \ldots, a_{tu_t})$ *on t variables.*

iv) $GOD(2n; a_{11}, a_{11}, a_{12}, a_{12}, \ldots, a_{1u_1}, a_{1u_1}; \ldots; a_{t1}, a_{t1}, a_{t2}, a_{t2}, \ldots, a_{tu_t}, a_{tu_t})$ *on t variables.*

v) $GOD(2n; a_{10}; a_{11}, a_{12}, \ldots, a_{1u_1}; \ldots; a_{t1}, a_{t2}, \ldots, a_{tu_t})$ *on $t+1$ variables.*

vi) $GOD(2n; a_{10}; a_{11}, a_{12}, \ldots, a_{1u_1}; a_{21}, a_{21}, a_{22}, a_{22}, \ldots, a_{2u_2}, a_{2u_2}; \ldots; a_{t1}, a_{t1}, a_{t2}, a_{t2}, \ldots, a_{tu_t}, a_{tu_t})$ *on $t+1$ variables.*

vii) $GOD(n; c_{11}, c_{12}, \ldots, c_{1u_1}; \ldots; c_{t1}, c_{t1}, c_{t2}, \ldots, c_{tu_t})$ *where $c_{ij} = b_i a_{ij}$ and b_i any real numbers on t variables.*

Proof. In D we do the following replacement of the variables:

i) Set $x_i = x_j$.

ii) Set $x_i = 0$.

iii) Replace x_i by $\begin{bmatrix} x_i & 0 \\ 0 & x_i \end{bmatrix}$, for all $i = 1, 2, \ldots, t$.

iv) Replace x_i by $\begin{bmatrix} x_i & x_i \\ x_i & -x_i \end{bmatrix}$, for all $i = 1, 2, \ldots, t$.

v) Replace x_1 by $\begin{bmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{bmatrix}$, and x_i by $\begin{bmatrix} 0 & x_i \\ x_i & 0 \end{bmatrix}$ for all $i = 2, 3, \ldots, t$.

vi) Replace x_1 by $\begin{bmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{bmatrix}$, and x_i by $\begin{bmatrix} x_i & x_i \\ x_i & -x_i \end{bmatrix}$ for all $i = 2, 3, \ldots, t$.

vii) Replace x_i by $b_i x_i$, where b_i are any real numbers, for all $i = 1, 2, \ldots, t$. □

Example 4.2 Let $D = GOD(6; 1, 2; 1, 1, 3, 3)$ be the generalized orthogonal design as it is given in example 3.3. Then using theorem 4.1 we have

i) If we set $b = a$ we obtain the generalized orthogonal design $D = GOD(6; 1, 1, 1, 2, 3, 3)$.

ii) If we set $a = 0$ we obtain the generalized orthogonal design $D = GOD(6; 1, 1, 3, 3)$ and if we set $b = 0$ we obtain the generalized orthogonal design $D = GOD(6; 1, 2)$

iii) If we replace a and b by $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and $\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$, respectively we obtain the generalized orthogonal design$GOD(12; 1, 2; 1, 1, 3, 3)$.

iv) If we replace a and b by $\begin{bmatrix} a & a \\ a & -a \end{bmatrix}$ and $\begin{bmatrix} b & b \\ b & -b \end{bmatrix}$, respectively we obtain the generalized orthogonal design $GOD(12; < 2, 1 >, < 2, 2 > ; < 4, 1 >, < 4, 3 >)$.

v) If we replace a by $\begin{bmatrix} c & a \\ -a & c \end{bmatrix}$, and b by $\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$ we obtain the generalized orthogonal design $GOD(12; 1, 2; 1, 2; 1, 1, 3, 3)$.

vi) If we replace a by $\begin{bmatrix} c & a \\ -a & c \end{bmatrix}$, and b by $\begin{bmatrix} b & b \\ b & -b \end{bmatrix}$ we obtain the generalized orthogonal design $GOD(12; 1, 2; 1, 2; < 4, 1 >, < 4, 3 >)$.

vii) If we replace a by ca and b by db, where c, d are any real numbers, we obtain the generalized orthogonal design $GOD(6; c, 2c; d, d, 3d, 3d)$. □

Lemma 4.3 *If there exist two circulant matrices which give a* $GOD(2n; a_{1,1}, \ldots, a_{1,u_1}; a_{2,1}, \ldots, a_{2,u_2})$ *then there exist two circulant matrices which give a* $GOD(2pn; a_{1,1}, \ldots, a_{1,u_1}; a_{2,1}, \ldots, a_{2,u_2})$ *and a* $GOD(2pn; a_{1,1}, a_{1,1}, \ldots, a_{1,u_1}, a_{1,u_1}; a_{2,1}, a_{2,1}, \ldots, a_{2,u_2}, a_{2,u_2})$ *for all integers* $p > 1$.

Proof. Write 0_{p-1} for the sequence of $p-1$ zeros. Suppose $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are the two sequences of length n with zero PAF that can be used as the first rows of the corresponding circulant matrices to construct the $GOD(n; a_{1,1}, \ldots, a_{1,u_1}; a_{2,1}, \ldots, a_{2,u_2})$. Now by considering the sequences $X' = \{x_1, 0_{p-1}, x_2, 0_{p-1}, \ldots, x_n, 0_{p-1}\}$ and $Y' = \{y_1, 0_{p-1}, y_2, 0_{p-1}, \ldots, y_n, 0_{p-1}\}$ of length pn with zero PAF that can be used to construct the $GOD(2n; a_{1,1}, \ldots, a_{1,u_1}; a_{2,1}, \ldots, a_{2,u_2})$. If we now form another sequence Y'' by permuting the first row of Y' by one position (i.e. $y'_i := y''_{i+1}$). Then X' and Y'' are disjoint. Hence $X' + Y''$ and $X' - Y''$ are two sequences of length pn with zero PAF that can be used to construct the desirable $GOD(2pn; a_{1,1}, a_{1,1}, \ldots, a_{1,u_1}, a_{1,u_1}; a_{2,1}, a_{2,1}, \ldots, a_{2,u_2}, a_{2,u_2})$. □

Example 4.4 From example 3.3 we have the sequences $A = \{a, 3b, b\}$, $B = \{-b, 3b, -2a\}$ which give a $GOD(6; 1, 2, 3; 1, 2, 3)$. Now using lemma 4.3 we obtain a $GOD(6p; 1, 2, 3; 1, 2, 3)$ and a $GOD(6p; 1, 1, 2, 2, 3, 3; 1, 1, 2, 2, 3, 3)$ for all $p = 2, 3, \ldots$.

References

[1] A.V.Geramita, J.M.Geramita, and J.Seberry Wallis, Orthogonal designs, *Linear and Multilinear Algebra*, 3, (1976), 281–306.

[2] A.V.Geramita, and J.Seberry, *Orthogonal designs: Quadratic forms and Hadamard matrices*, Marcel Dekker, New York-Basel, 1979.

[3] J. Radon, Lineare Scharen Orhogonaler Matrizen, *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universitat*, Vol. 1 (1922), 1–14.