CLOSED PATHS IN THE COSET DIAGRAMS FOR $< y,t : y^6 = t^6 = 1 >$ ACTING ON REAL QUADRATIC FIELDS

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We are interested in linear-fractional transformations y,t satisfying the relations $y^6 = t^6 = 1$, with a view to studying an action of the subgroup $H =< y,t >$ on $Q(\sqrt{n}) \cup \{\infty\}$ by using coset diagrams.

For a fixed non-square positive integer n, if an element $\alpha = \frac{a+\sqrt{n}}{c}$ and its algebraic conjugate have different signs, then α is called an ambiguous number. They play an important role in the study of action of the group H on $Q(\sqrt{n}) \cup \{\infty\}$. In the action of H on $Q(\sqrt{n}) \cup \{\infty\}$, $Stab_\alpha(H)$ are the only non-trivial stabilizers and in the orbit αH; there is only one (up to isomorphism). We classify all the ambiguous numbers in the orbit and use this information to see whether the action is transitive or not.

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1. INTRODUCTION

It is worthwhile to consider linear-fractional transformations x, y satisfying the relations $x^2 = y^m = 1$, with a view to studying an action of the group $< x, y >$ on real quadratic fields. If $y: z \to \frac{az+b}{cz+d}$ is to act on all real quadratic fields then a, b, c, d must be rational numbers and can be taken to be integers, so that $\frac{(a+d)^2}{ad-bc}$ is rational. But if $y: z \to \frac{az+b}{cz+d}$ is of order m one must have $\frac{(a+d)^2}{ad-bc} = \omega + \omega^{-1} + 2$, where ω is a primitive $m-th$ root of unity. Now $\omega + \omega^{-1}$ is rational, for a primitive $m-th$ root ω, only if $m = 1, 2, 3, 4, 6$. So these are the only possible orders of y. The group $< x, y >$ is trivial when $m = 1$. When $m = 2$, it is an infinite dihedral group and does not give inspiring information while studying its action on the real quadratic irrational numbers. For $m = 3$, the group $< x, y >$ is the modular group $PSL(2, Z)$ and its action on real quadratic irrational numbers has been discussed in detail in [3] and [5].

For a fixed non-square positive integer n, an element $\alpha = \frac{a+\sqrt{n}}{c}$ and its algebraic conjugate $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ may have different signs. If such is the case then we shall call such α an ambiguous number. If α and $\bar{\alpha}$ are both positive (negative), then we shall call α a totally positive (negative) number. Ambiguous numbers play an important role in the study of actions of the groups $M = < x, y : x^2 = y^m = 1 >$, for $m = 1, 2, 3, 4$ or 6, on $Q(\sqrt{n})$.

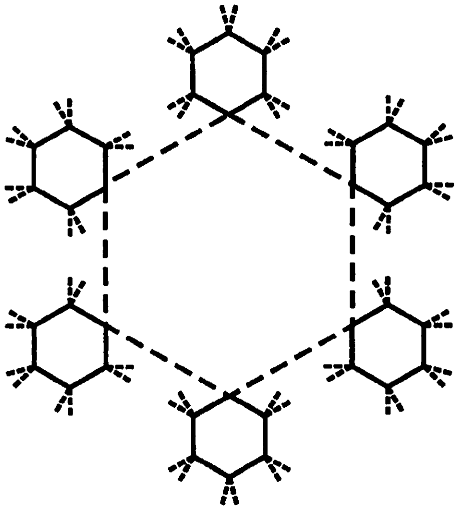
In this note we are interested in the action of a subgroup of $G = < x, y : x^2 = y^6 = 1 >$, where $(z)x = \frac{-1}{3z}$ and $(z)y = \frac{-1}{3(z+1)}$ are linear-fractional transformations, on the real quadratic irrational numbers.

If we let $t = xyx$ then t can be considered as the linear-fractional transformation defined by $(z)t = 1 - \frac{1}{3z}$ and $t^6 = 1$. Some number-theoretic properties of the ambiguous numbers belonging to the orbit of G when acting on $Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a\ , c \in Z,\ b = \frac{a^2-n}{c} \in Z,\ (a,b,c)=1\}$ have been discussed in [4]. In this paper we explore group-theoretic properties of this action vis-à-vis the orbit of α in $H = \langle y,t \rangle$. We shall show that the set of ambiguous numbers is finite and that part of the coset diagram containing these numbers form a single closed path and it is the only closed path in the orbit of α. We shall show here that in the action of H on $Q(\sqrt{n})$, $Stab_\alpha(H)$ are the only non-trivial stabilizers and in the orbit αH, there is only one (up to isomorphism) non-trivial stabilizer.

2. COSET DIAGRAMS

We use coset diagrams for the group H and study its action on the projective line over real quadratic fields. The coset diagrams for the group H are defined as follows. The six cycles of the transformation y are denoted by six unbroken edges of a hexagon (may be irregular) permuted anti-clockwise by y and the six cycles of the transformation t are denoted by six broken edges of a hexagon (may be irregular) permuted anti-clockwise by t. Fixed points of y and t, if they exist, are denoted by heavy dots. This graph can be interpreted as a coset diagram, with the vertices identified with the cosets of $Stab_v(H)$, the stabilizer of some vertex v of the graph, or as 1-skeleton of the cover of the fundamental complex of the presentation which corresponds to the subgroup $Stab_v(H)$.

A general fragment of the coset diagram of the action of H on $Q(\sqrt{n})$ will look as follows.



In [4], it has been observed that if $k \neq -1, \frac{-2}{3}, \frac{-1}{2}, \frac{-1}{3}, 0, \infty$ is one of the six vertices of a hexagon in the coset diagram, then

(i) $z < -1$ implies that $(z)y > 0$,

(ii) $z > 0$ implies that $\frac{-1}{3} < (z)y < 0$,

(iii) $\frac{-1}{3} < z < 0$ implies that $\frac{-1}{2} < (z)y < \frac{-1}{3}$,

(iv) $\frac{-1}{2} < z < \frac{-1}{3}$ implies that $\frac{-2}{3} < (z)y < \frac{-1}{2}$,

(v) $\frac{-2}{3} < z < \frac{-1}{2}$ implies that $-1 < (z)y < \frac{-2}{3}$, and

(vi) $-1 < z < \frac{-2}{3}$ implies that $(z)y < -1$.

Also if $k \neq 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, 0, \infty$ is one of the six vertices of a hexagon in the coset diagram, then

(i) $z < 0$ implies that $(z)t > 1$,

(ii) $z > 1$ implies that $\frac{2}{3} < (z)t < 1$,

(iii) $\frac{2}{3} < z < 1$ implies that $\frac{1}{2} < (z)t < \frac{2}{3}$,

(iv) $\frac{1}{2} < z < \frac{2}{3}$ implies that $\frac{1}{3} < (z)t < \frac{1}{2}$,

(v) $\frac{1}{3} < z < \frac{1}{2}$ implies that $0 < (z)t < \frac{1}{3}$, and

(vi) $0 < z < \frac{1}{3}$ implies that $(z)t < 0$.

By $Q^*(\sqrt{n})$ we shall mean a subset $\{\frac{a+\sqrt{n}}{c} : a, c \in Z,\ b = \frac{a^2 - n}{c} \in Z, (a,b,c) = 1\}$ of $Q(\sqrt{n})$.

We state here the following lemmas from [2] for later use.

Lemma 2.1 An $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is a totally positive number if and only if either $a,b,c > 0$ or $a,b,c < 0$.

Lemma 2.2 An $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is a totally negative number if and only if:

(i) either $a < 0$ and $b > 0, c > 0$, or

(ii) $a > 0$ and $b < 0, c < 0$.

Lemma 2.3 An $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is an ambiguous number if and only if $bc < 0$.

3. EXISTENCE OF AMBIGUOUS NUMBERS

Ambiguous numbers play an important role in the study of actions of the group H on $Q(\sqrt{n})$. We shall see here that $Stab_{\alpha}(H)$ are the only non-trivial stabilizers in the action of H on $Q(\sqrt{n})$ and that there is only one (up to isomorphism) non-trivial stabilizer in the orbit αH.

Theorem 3.1 If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^{*}(\sqrt{n})$ is a totally negative real quadratic irrational number then $(\alpha)t^{i}$ is totally positive for $i = 1,2,3,4$ or 5.

Proof. If $\alpha = \frac{a+\sqrt{n}}{c}$, where $b = \frac{a^2-n}{c}$, then $(\alpha)t = 1 - \frac{1}{3\alpha} = 1 - \frac{c}{3(a+\sqrt{n})}$ $= \frac{(3a-c)+3\sqrt{n}}{3(a+\sqrt{n})} \times \frac{a-\sqrt{n}}{a-\sqrt{n}} = \frac{-a+3b+\sqrt{n}}{3b}$. Hence the new values of a and c are $-a+3b$ and $3b$. Using these values, we then obtain the new value for b. That is, $\frac{(3b-a)^2-n}{3b} = \frac{-6a+9b+c}{3}$. Similarly, the new values for a,b and c with respect to $(\alpha)t^{i}$ are:

α	a	b	c
$i=1$	$-a+3b$	$\frac{-6a+9b+c}{3}$	$3b$
$i=2$	$-5a+6b+c$	$-4a+4b+c$	$-6a+9b+c$
$i=3$	$-7a+6b+2c$	$\frac{-12a+9b+4c}{3}$	$3(-4a+4b+c)$
$i=4$	$-5a+3b+2c$	$-2a+b+c$	$-12a+9b+4c$
$i=5$	$-a+c$	$\frac{c}{3}$	$3(-2a+b+c)$

If α is a totally negative number then by lemma 2.2, either a, b, c satisfy (i) or (ii).

If (i) is the case then from the above table the new a,b,c for $(\alpha)t^i$ are all positive. Hence by lemma 2.1 $(\alpha)t^i$ are totally positive.

Similarly, if (ii) is the case then it is easy to see that the new values of a,b,c for $(\alpha)t^i$ are all negative. Then by lemma 2.1 $(\alpha)t^i$ are totally positive.

Example 3.2 Let $\alpha=-2+\sqrt{3}$ then $a=-2, c=1, n=3$ and $b=\dfrac{a^2-n}{c}=1$. Because a is negative and b,c are positive therefore α is a totally negative real quadratic irrational number. We can easily tabulate the following information.

α	-2	1	1
$(\alpha)t$	5	$\frac{22}{3}$	3
$(\alpha)t^2$	17	13	22
$(\alpha)t^3$	22	$\frac{37}{3}$	39
$(\alpha)t^4$	15	6	37
$(\alpha)t^5$	3	$\frac{1}{3}$	18

As we seen from the above information the values of a,b and c for $(\alpha)t^i$, where $i=1,2,3,4$ and 5, are positive, therefore, $(\alpha)t^i$, for $i=1,2,3,4$ and 5 are all totally positive.

Lemma 3.3 If $\alpha=\dfrac{a+\sqrt{n}}{c}\in Q^*(\sqrt{n})$ is an ambiguous number then one of $(\alpha)t^i$, where $i=1,2,3,4$ or 5 is ambiguous and the other four are totally positive.

Proof First we suppose that α is a negative number. Then the possibilities for $\overline{\alpha}$ to be positive or negative are as follows:

α	$(\alpha)t$	$(\alpha)t^2$	$(\alpha)t^3$	$(\alpha)t^4$	$(\alpha)t^5$	$\overline{\alpha}$	$\overline{(\alpha)t}$	$\overline{(\alpha)t^2}$	$\overline{(\alpha)t^3}$	$\overline{(\alpha)t^4}$	$\overline{(\alpha)t^5}$
-	+	+	+	+	+	+	-	+	+	+	+
						+	+	-	+	+	+
						+	+	+	-	+	+
						+	+	+	+	-	+
						+	+	+	+	+	-

Similarly if α is a positive number then:

α	$(\alpha)t$	$(\alpha)t^2$	$(\alpha)t^3$	$(\alpha)t^4$	$(\alpha)t^5$	$\overline{\alpha}$	$\overline{(\alpha)t}$	$\overline{(\alpha)t^2}$	$\overline{(\alpha)t^3}$	$\overline{(\alpha)t^4}$	$\overline{(\alpha)t^5}$
+	-	+	+	+	+	-	+	+	+	+	+
+	+	-	+	+	+						
+	+	+	-	+	+						
+	+	+	+	-	+						
+	+	+	+	+	-						

Therefore from the above tables we can easily deduce that one of $(\alpha)t^i$, for $i = 1,2,3,4,$ or 5 is ambiguous and the other four are totally positive.

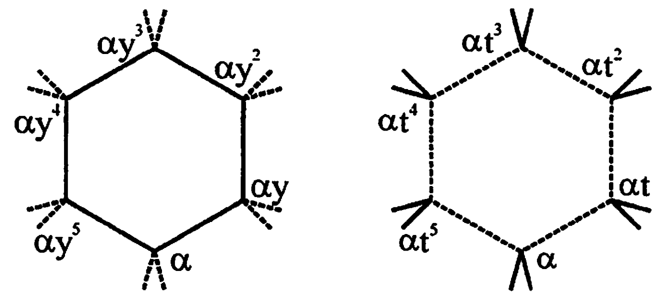
Example 3.4 Let $\alpha = 1+\sqrt{2}$ then $a = 1, c = 1, n = 2$ and $b = \dfrac{a^2 - n}{c} = -1$.

Since $bc < 0$, therefore α is an ambiguous real quadratic irrational number. We can easily tabulate the following information.

α	1	-1	1
$(\alpha)t$	-4	$\dfrac{-14}{3}$	-3
$(\alpha)t^2$	-10	-7	-14
$(\alpha)t^3$	-11	$\dfrac{-17}{3}$	-21
$(\alpha)t^4$	-6	-2	-17
$(\alpha)t^5$	0	$\dfrac{1}{3}$	-6

As we seen from the above information the values of a, b and c for $(\alpha)t^i$, where $i = 1,2,3$ and 4, are negative, therefore, $(\alpha)t^i$, for $i = 1,2,3$ and 4 are all totally positive. Since for $(\alpha)t^5$, $bc < 0$, therefore, $(\alpha)t^5$ is an ambiguous real quadratic irrational number.

Diagrammatically, the six vertices representing the six cycles of y and t will be as follows.



Lemma 3.5 If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is an ambiguous number then one of $(\alpha)y^j$, for $j = 1,2,3,4,$ or 5 is ambiguous and the other four are totally positive numbers.

Proof The proof of this lemma is given in [1].

If the norm of $\alpha = \frac{a+\sqrt{n}}{c}$ is defined as $\|\alpha\| = |a|$, then:

Theorem 3.6 If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is totally positive then:

(i) $\|(\alpha)y^j\| > \|\alpha\|$, for $j = 1,2,3,4,$ and 5.

(ii) $\|(\alpha)t^i\| < \|\alpha\|$ if $(\alpha)t^i$ is totally negative for $i = 1,2,3,4,$ or 5

Proof

(i) If $\alpha = \dfrac{a+\sqrt{n}}{c}$, where $b = \dfrac{a^2-n}{c}$, then we can easily calculate new values of a, b, c for $(\alpha)y^j$, where $j = 1,2,3,4,$ and 5, as follows:

α	a	b	c
$j=1$	$-a-c$	$\dfrac{c}{3}$	$3(2a+b+c)$
$j=2$	$-5a-3b-2c$	$2a+b+c$	$12a+9b+4c$
$j=3$	$-7a-6b-2c$	$\dfrac{12a+9b+4c}{3}$	$3(4a+4b+c)$
$j=4$	$-5a-6b-c$	$4a+4b+c$	$6a+9b+c$
$j=5$	$-a-3b$	$\dfrac{6a+9b+c}{3}$	$3b$

Since α is a totally positive number, therefore, either $a,b,c>0$ or $a,b,c<0$.

If $a,b,c>0$ (or $a,b,c<0$), $\|(\alpha)y\| = |a+c|$, $\|(\alpha)y^2\| = |5a+3b+2c|$, $\|(\alpha)y^3\| = |7a+6b+2c|$, $\|(\alpha)y^4\| = |5a+6b+c|$ and $\|(\alpha)y^5\| = |a+3b|$. Thus, $\|(\alpha)y^j\| > \|\alpha\|$, for $j = 1,2,3,4,$ and 5.

(ii) The new values of a,b,c for $(\alpha)t^i$, where $i = 1,2,3,4,$ or 5, are tabulated as follows:

α	a	b	c
$i=1$	$-a+3b$	$\dfrac{-6a+9b+c}{3}$	$3b$
$i=2$	$-5a+6b+c$	$-4a+4b+c$	$-6a+9b+c$
$i=3$	$-7a+6b+2c$	$\dfrac{-12a+9b+4c}{3}$	$3(-4a+4b+c)$
$i=4$	$-5a+3b+2c$	$-2a+b+c$	$-12a+9b+4c$

$i=5$	$-a+c$	$\frac{c}{3}$	$3(-2a+b+c)$

By theorem 3.1, one of $(\alpha)t^i$, for $i=1,2,3,4$, or 5 is totally negative. Note also that since α is totally positive, there are only two possibilities, namely, either $a,b,c>0$ or $a,b,c<0$. We deal with these possibilities separately.

First, we suppose that $(\alpha)t$ is totally negative. If $a,b,c>0$ then from the above information we can see that $-a+3b<0$. Hence $-a<-a+3b<a$ or $|-a+3b|<a$ or $\|(\alpha)t\|<\|\alpha\|$. Similarly for $a,b,c<0$ we note that $\|(\alpha)t\|<\|\alpha\|$.

Secondly, suppose that $(\alpha)t^2$ is totally negative, therefore, $(\alpha)t$ must be totally positive. If $a,b,c>0$ then $-a+3b>0, -6a+9b+c>0$, $-5a+6b+c<0$ and $-4a+4b+c>0$. Since $-5a+6b+c=(-4a+4b+c)$ $-a+2b$, therefore, $-a+2b < -5a+6b+c$ or $-a < -5a+6b+c < a$ or $|-5a+6b+c|<a$ or $\|(\alpha)t^2\|<\|\alpha\|$. Similarly, for $a,b,c<0$, $\|(\alpha)t^2\|<\|\alpha\|$.

Now, suppose that $(\alpha)t^3$ is totally negative. Then $(\alpha)t$ and $(\alpha)t^2$ must be totally positive. If $a,b,c>0$ then $-a+3b>0$, $-6a+9b+c>0$, $-5a+6b+c>0$, $-4a+4b+c>0$, $-7a+6b+2c<0$ and $-12a+9b+4c>0$. Since $-14a+12b+4c = (-12a+9b+4c)-2a+3b$, therefore, $-2a<-14a+12b+4c$ or $-a<-7a+6b+2c$ or $-a<-7a+6b+2c<a$ or $|-7a+6b+2c|<a$ or $\|(\alpha)t^3\|<\|\alpha\|$. Similarly, for $a,b,c<0$ we obtain $\|(\alpha)t^3\|<\|\alpha\|$.

Next, let $(\alpha)t^4$ be totally negative. Therefore, $(\alpha)t$, $(\alpha)t^2$ and $(\alpha)t^3$ are totally positive. If $a,b,c>0$ then $-5a+3b+2c<0$, $-2a+b+c>0$. Since $-5a+3b+2c=$ $(-4a+2b+2c)-a+b$, therefore $-a<-5a+3b+2c<a$ or

$|-5a+3b+2c|<a$ or $\|(\alpha)t^4\|<\|\alpha\|$. Similarly, for $a,b,c<0$ we get $\|(\alpha)t^4\|<\|\alpha\|$.

Finally, suppose that $(\alpha)t^5$ is totally negative. Therefore $(\alpha)t,(\alpha)t^2,(\alpha)t^3$ and $(\alpha)t^4$ are totally positive. If $a,b,c>0$ then $-a+c<0$, and $-a<-a+c<a$ or $|-a+c|<a$ or $\|(\alpha)t^5\|<\|\alpha\|$. Similarly, for $a,b,c<0$ we have $\|(\alpha)t^5\|<\|\alpha\|$.

Example 3.7 Let $\alpha=3+\sqrt{3}$ then $a=3, c=1, n=3$ and $b=\dfrac{a^2-n}{c}=6$. As a,b and c are positive therefore α is a totally positive real quadratic irrational number. We can easily tabulate the following information.

α	3	6	1
$(\alpha)y$	-4	$\frac{1}{3}$	39
$(\alpha)y^2$	-35	13	94
$(\alpha)y^3$	-59	$\frac{94}{3}$	111
$(\alpha)y^4$	-52	37	73
$(\alpha)y^5$	-24	$\frac{73}{3}$	18
$(\alpha)t$	15	$\frac{37}{3}$	18
$(\alpha)t^2$	22	13	37
$(\alpha)t^3$	17	$\frac{22}{3}$	39
$(\alpha)t^4$	5	1	22
$(\alpha)t^5$	-2	$\frac{1}{3}$	3

We can see from the above information that $(\alpha)y^j$, where $j = 1,2,3,4$ and 5, are totally negative numbers and $\|(\alpha)y^j\| > \|\alpha\|$ for $j = 1,2,3,4$ and 5. Also $(\alpha)t^5$ is a totally negative number such that $\|(\alpha)t^5\| < \|\alpha\|$.

Theorem 3.8 If $\alpha = \dfrac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is a totally negative number. Then

(i) $\|(\alpha)t^i\| > \|\alpha\|$, for $i = 1,2,3,4,$ or 5, and

(ii) $\|(\alpha)y^j\| < \|\alpha\|$ if $(\alpha)y^j$ is totally positive for $j = 1,2,3,4,$ and 5.

Proof

(i) If $\alpha = \dfrac{a+\sqrt{n}}{c}$, where $b = \dfrac{a^2-n}{c}$, then we can easily calculate new a,b,c for $(\alpha)t^i$ as follows:

α	a	b	c
$i=1$	$-a+3b$	$\dfrac{-6a+9b+c}{3}$	$3b$
$i=2$	$-5a+6b+c$	$-a+4b+c$	$-6a+b+c$
$i=3$	$-7a+6b+2c$	$\dfrac{-12a+9b+4c}{3}$	$3(-4a+4b+c)$
$i=4$	$-5a+3b+2c$	$-2a+b+c$	$-12a+9b+4c$
$i=5$	$-a+c$	$\dfrac{c}{3}$	$3(-2a+b+c)$

Since α is a totally negative number, therefore, $a>0$, and $b<0, c<0$ or $a<0$, and $b>0, c>0$. If $a>0$, and $b<0, c<0$ (or $a<0$, and $b>0, c>0$) then $\|(\alpha)t\| = |-a+3b|$, $\|(\alpha)t^2\| = |-5a+6b+c|$, $\|(\alpha)t^3\| = |-7a+6b+2c|$, $\|(\alpha)t^4\| = |-5a+3b+2c|$ and $\|(\alpha)t^5\| = |-a+c|$. Hence $\|(\alpha)t^i\| > \|\alpha\|$, for $i = 1,2,3,4,$ and 5.

(i) Again, we can write information about $(\alpha)y^j$, as follows.

α	a	b	c
$j=1$	$-a-c$	$\frac{c}{3}$	$3(2a+b+c)$
$j=2$	$-5a-3b-2c$	$2a+b+c$	$12a+9b+4c$
$j=3$	$-7a-6b-2c$	$\frac{12a+9b+4c}{3}$	$3(4a+4b+c)$
$j=4$	$-5a-6b-c$	$4a+4b+c$	$6a+9b+c$
$j=5$	$-a-3b$	$\frac{6a+9b+c}{3}$	$3b$

Analogous to Theorem 3.1, if α is totally positive then $(\alpha)y^j$, $j=1,2,3,4,$ and 5 are totally negative.

First, let us suppose that $(\alpha)y$ is totally positive. As α is totally negative, there are two possibilities, either $a<0$ and $b>0, c>0$ or $a>0$ and $b<0, c<0$. If $a<0$ and $b>0, c>0$ then $-a-c>0$. Hence $-a>-a-c>a$ or $|-a-c|<|a|$ or $\|(\alpha)y\|<\|\alpha\|$. Similarly, $\|(\alpha)y\|<\|\alpha\|$ for $a>0$ and $b<0, c<0$.

Now, suppose that $(\alpha)y^2$ is totally positive. Then $(\alpha)y$ must be totally negative.

If $a<0$ and $b>0, c>0$ then $2a+b+c>0, -a-c<0, -5a-3b-2c>0$ and $12a+9b+4c>0$. Since $-15a-9b-6c=(-12a-9b-4c)-3a-2c$, therefore $-15a-9b-6c<-3a$. Hence $a<-5a-3b-2c<-a$ or $|-5a-3b-2c|<|a|$ or $\|(\alpha)y^2\|<\|\alpha\|$. Similarly, $\|(\alpha)y^2\|<\|\alpha\|$ for $a>0$ and $b<0, c<0$.

Let $(\alpha)y^3$ be totally positive. Then $(\alpha)y$ and $(\alpha)y^2$ are totally negative. If $a<0$ and $b>0, c>0$ then $-a-c<0$, $2a+b+c>0$, $-5a-3b-2c<0$, $12a+9b+4c>0$, $-7a-6b-2c>0$ and $4a+4b+c>0$.

Since $-14a-12b-4c=(-12a-9b-4c)-2a-3b<-2a$. Then $-7a-6b-2c<-a$ or $a<-7a-6b-2c<-a$ or $|-7a-6b-2c|<|a|$ or $\|(\alpha)y^3\|<\|\alpha\|$. Similarly, $\|(\alpha)y^3\|<\|\alpha\|$ for $a>0$ and $b<0, c<0$.

Next, suppose that $(\alpha)y^4$ is totally positive. Then $(\alpha)y, (\alpha)y^2$ and $(\alpha)y^3$ are totally negative. If $a<0$ and $b>0, c>0$ then $-a-c<0$, $2a+b+c>0$, $-5a-3b-2c<0$, $12a+9b+4c>0$, $-7a-6b-2c<0$, $4a+4b+c>0$, $-5a-6b-c>0$ and $6a+9b+c>0$. Since $-10a-12b-2c=(-8a-8b-2c)-2a-4b$. Then $-10a-12b-2c<-2a$ or $a<-5a-6b-c<-a$ or $|-5a-6b-c|<|a|$ or $\|(\alpha)y^4\|<\|\alpha\|$. Similarly, $\|(\alpha)y^4\|<\|\alpha\|$ for $a>0$ and $b<0, c<0$.

Finally, we suppose that $(\alpha)y^5$ is totally positive. If $a<0$ and $b>0, c>0$ then $-a-3b>0$. This implies that $a<-a-3b<-a$ or $|-a-3b|<|a|$ or $\|(\alpha)y^5\|<\|\alpha\|$. Similarly, $\|(\alpha)y^5\|<\|\alpha\|$ for $a>0$ and $b<0, c<0$.

Example 3.9 Let $\alpha=-3+\sqrt{2}$ then $a=-3, c=1, n=2$ and $b=\dfrac{a^2-n}{c}=7$. As a is negative and b, c are positive therefore α is a totally negative real quadratic irrational number. We can easily tabulate the following information.

α	-3	7	1
$(\alpha)y$	2	$\frac{1}{3}$	6
$(\alpha)y^2$	-8	2	31
$(\alpha)y^3$	-23	$\frac{31}{3}$	51
$(\alpha)y^4$	-28	17	46
$(\alpha)y^5$	-18	$\frac{46}{3}$	21

$(\alpha)t$	24	$\frac{82}{3}$	21
$(\alpha)t^2$	58	41	82
$(\alpha)t^3$	65	$\frac{103}{3}$	123
$(\alpha)t^4$	38	14	103
$(\alpha)t^5$	4	$\frac{1}{3}$	42

We can see from the above information that $(\alpha)t^i$, where $i = 1,2,3,4$ and 5, are totally positive numbers and $\|(\alpha)t^i\| > \|\alpha\|$ for $i = 1,2,3,4$ and 5. Also $(\alpha)y$ is a totally positive number such that $\|(\alpha)y\| < \|\alpha\|$.

Theorem 3.10 If $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$ is a totally positive number. Then there exists a sequence $\alpha(=\alpha_1), \alpha_2, \alpha_3, ..., \alpha_m$ such that α_i is alternately totally positive and totally negative number, for $i = 1,2,3,...,m-1$ and α_m, is an ambiguous number.

Proof Since $\alpha = \alpha_1 = \frac{a+\sqrt{n}}{c}$ is a totally positive number, therefore, by theorem 3.1, one of $(\alpha)t^i$, for $i = 1,2,3,4$, or 5 is totally negative. If $(\alpha)t^i$ is totally negative then by theorem 3.5, $\|(\alpha)t^i\| < \|\alpha\|$. Also $(\alpha)t^i$ is totally negative, then, one of $(\alpha)t^i y^j$, for $j = 1,2,3,4$, and 5 is totally positive. If $(\alpha)t^i y^j$ is totally positive then by theorem 3.4, $\|(\alpha)t^i y^j\| < \|(\alpha)t^i\| < \|\alpha\|$. If we let $\alpha = \alpha_1$, $(\alpha)t^i = \alpha_2$, and $(\alpha)t^i y^j = \alpha_3$ and continue in this way we obtain an alternate sequence $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m$ of totally positive and totally negative numbers such that $\|\alpha_1\| > \|\alpha_2\| > \|\alpha_3\| > ... > \|\alpha_m\|$. Since

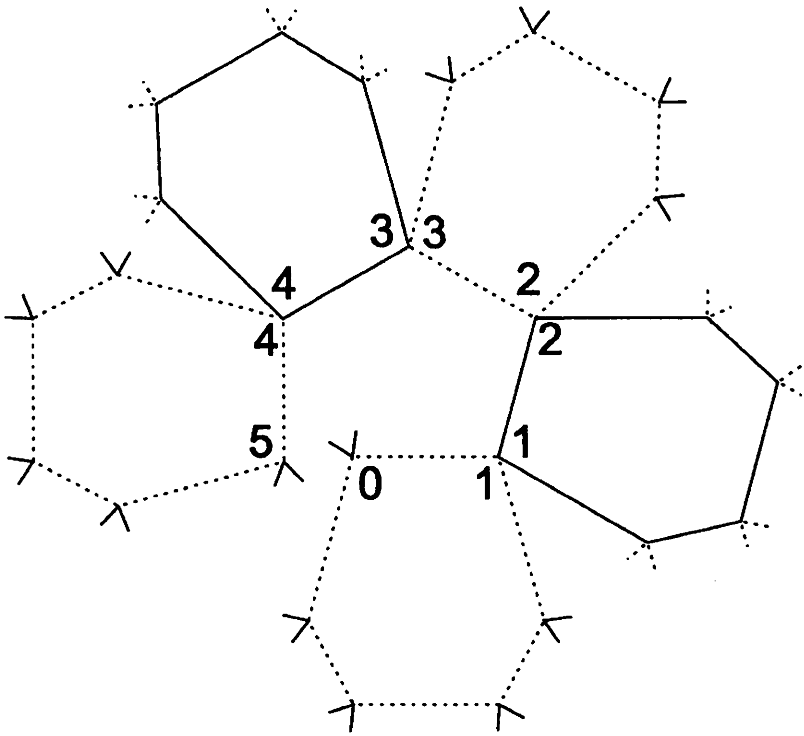
$\|\alpha_1\|, \|\alpha_2\|, \|\alpha_3\|, \ldots, \|\alpha_m\|$ is a decreasing sequence of non-negative integers therefore it must terminate. That is, after a finite number of steps we reach to α_m such that $\|\alpha_m\| < \sqrt{n}$. This means that, if $\alpha_m = \frac{a_1 + \sqrt{n}}{c_1}$ then $\|\alpha_m\| = |a_1| < \sqrt{n}$. Thus $a_1^2 < n$ or $a_1^2 - n < 0$ or $\frac{a_1^2 - n}{c_1^2} < 0$ or $\alpha_m \overline{\alpha}_m < 0$. Hence α_m is an ambiguous number.

Example 3.11 Let $\alpha = 6 + \sqrt{3}$ then $a = 6, c = 1, n = 3$ and $b = \frac{a^2 - n}{c} = 33$. As a, b, c are positive therefore α is a totally positive real quadratic irrational number. We can easily tabulate the following information.

$\alpha = \alpha_0$	6	33	1	(totally positive)
$\alpha_1 = (\alpha_0)t^5$	-5	$\frac{1}{3}$	66	(totally negative)
$\alpha_2 = (\alpha_1)y^5$	4	13	1	(totally positive)
$\alpha_3 = (\alpha_2)t^5$	-3	$\frac{1}{3}$	18	(totally negative)
$\alpha_4 = (\alpha_3)y^5$	2	1	1	(totally positive)
$\alpha_5 = (\alpha_4)t^5$	-1	$\frac{1}{3}$	-6	(ambiguous)

We can see from the above information that $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ is an alternating sequence of totally positive and totally negative numbers and α_5 is an ambiguous number.

The above information are shown by the following coset diagram in which 0,1,2,3,4 and 5 represent $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 respectively.



4. CLOSED PATHS

The ambiguous numbers play an important role in studying the action of H on $Q^*(\sqrt{n})\cup\{\infty\}$. Let $\alpha\in Q^*(\sqrt{n})$ and αH denote an orbit of $Q^*(\sqrt{n})$. The existence of an ambiguous number in αH is related to the stabilizers of H. We describe the action of H on $Q^*(\sqrt{n})\cup\{\infty\}$ in the following theorems.

Theorem 4.1 The ambiguous numbers in the coset diagram for the orbit αH, where $\alpha=\dfrac{a+\sqrt{n}}{3c}\in Q^*(\sqrt{n})$, form a closed path and it is the only closed path contained in it.

Proof Let k_0 be an arbitrary ambiguous number in αH. We pass on to another ambiguous number by successive applications of either y^j or t^i, for

$i, j = 1,2,3,4$ or 5. Without loss of generality, we assume that $(k_0)y^j$ is another ambiguous number.

Since each hexagon, representing six edges of y or t contains two ambiguous numbers (by virtue of lemmas 3.2 and 3.3), therefore at the second ambiguous number within the $k-th$ hexagon, we successively apply the second generator, namely t (or y) to reach the next ambiguous number in the $(k+1)-th$ hexagon.

Suppose $k-th$ hexagon (depicting either the six cycles of the generator t or y) contains two ambiguous numbers, namely α_1 and α_2. We assume that the $k-th$ hexagon is the one which depicts the six cycles of the generator t. Then $\alpha_2^{(k-1)} = \alpha_1^{(k-1)} y^{\varepsilon_1}, \alpha_2^{(k)} = \alpha_1^{(k)} t^{\varepsilon_2}$ and $\alpha_2^{(k+1)} = \alpha_1^{(k+1)} y^{\varepsilon_3}$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 = 1,2,3,4,$ or 5. Also, since $\alpha_2^{(k-1)} = \alpha_1^{(k)}$ and $\alpha_2^{(k)} = \alpha_1^{(k+1)}$, therefore, $\alpha_1^{(k-1)} y^{j_1} t^{j_2} y^{j_3} = \alpha_2^{(k+1)}$. We can continue in this way and since by lemma 3 in [2] there are only finite number of ambiguous numbers of the form $\alpha = \frac{a+\sqrt{n}}{3c}$ in $Q^*(\sqrt{n})$, after a finite number of steps we reach to the vertex (ambiguous number) $\alpha_2^{(k+m)} = \alpha_1^{(k-1)}$.

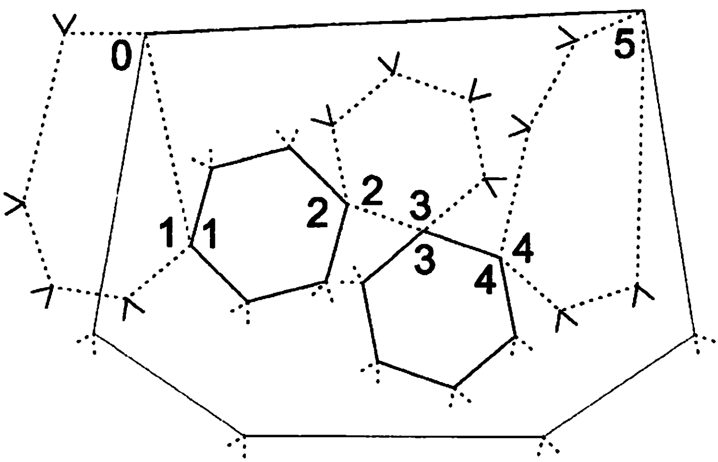
Hence the ambiguous numbers form a path in the coset diagram. The path is closed because there are only finite number of ambiguous numbers in a coset diagram. Since only ambiguous numbers form a closed path and these are the only ambiguous numbers therefore all the ambiguous numbers form a single closed path in the coset diagram of the orbit αH.

Example 4.2 Let $\alpha = \frac{2+\sqrt{7}}{3}$ then $a = 2, c = 3, n = 7$ and

$b = \frac{a^2 - n}{c} = -1$. As $bc < 0$ therefore α is an ambiguous number. We can easily tabulate the following information.

$\alpha = \alpha_0$	2	−1	3	(ambiguous)
$\alpha_1 = (\alpha_0)t^5$	1	1	−6	(ambiguous)
$\alpha_2 = (\alpha_1)y^3$	−1	−1	6	(ambiguous)
$\alpha_3 = (\alpha_2)t$	−2	1	−3	(ambiguous)
$\alpha_4 = (\alpha_3)y^5$	−1	−2	3	(ambiguous)
$\alpha_5 = (\alpha_4)t^3$	1	2	−3	(ambiguous)
$\alpha_0 = (\alpha_5)y$	2	−1	3	(ambiguous)

We can see from the above information that $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 form a closed path. The above information are shown by the following coset diagram in which 0,1,2,3,4 and 5 represent $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 respectively.



Theorem 4.3 The graph of the action of H on the rational projective line in connected.

Proof To prove this we need only to show that for any rational number k_0 there is a path joining k_0 to ∞.

Let k_0 be a positive rational number, say, $k_0 = \frac{a}{b}$. Then $(k_0)y^j = \frac{-b}{3(a+b)}, \frac{-(a+b)}{3a+2b}, \frac{-(3a+2b)}{3(2a+b)}, \frac{-(2a+b)}{3a+b}$ and $\frac{-(3a+b)}{3a}$, for $j = 1,2,3,4,$ or 5. Let $\|k_0\| = \max(|a|,|b|)$. Then $\|(k)y\| = 3(a+b), \|(k)y^2\| = 3a+2b$, $\|(k)y^3\| = 3(2a+b)$, $\|(k)y^4\| = 3a+b$ and $\|(k)y^5\| = 3a+b$. Therefore, $\|(k)y^j\| > \|k\|$ for $j = 1,2,3,4,$ or 5. Similarly, if k_0 is a negative rational number, say, $k_0 = \frac{a}{b}$ with $b < 0$, then $(k_0)t^i = \frac{3a-b}{3a}, \frac{2a-b}{3a-b}, \frac{3a-2b}{3(2a-b)}, \frac{a-b}{3a-2b}$ and $\frac{-b}{3(a-b)}$. That is $\|(k_0)t\| = 3a-b, \|(k_0)t^2\| = 3a-b, \|(k_0)t^3\| = 3(2a-b), \|(k_0)t^4\| = 3a-2b$ and

$\|(k_0)t^5\| = 3(a-b)$. Hence $\|(k_0)t^i\| > \|k_0\|$ for $i = 1,2,3,4,$ or 5. If k_0 is positive then one of $(k_0)t^i$ is negative. If we let this negative number to be k_1 then $\|k_0\| > \|k_1\|$. As k_1 is negative one of $(k_1)y^j$ is positive. Let it be k_2, that is, $k_2 = (k_1)y^j$ where $j = 1,2,3,4$ or 5. This implies that $\|k_2\| < \|k_1\|$. If we continue in this way, we get a unique alternating sequence of positive and negative rational numbers $k_0, k_1, k_2 \ldots$ such that $\|k_0\| > \|k_1\| > \|k_2\| > \ldots$. The decreasing sequence of positive integers must terminate after a finite number of steps. It will terminate only when ultimately we arrive at a hexagon with vertices $-1, \frac{-2}{3}, \frac{-1}{2}, \frac{-1}{3}, 0, \infty$ or $1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, 0$ and ∞. An alternating sequence of positive and negative rational numbers $k_0, k_1, k_2 \ldots$ such that $\|k_0\| > \|k_1\| > \|k_2\| > \ldots$ shows that there is a path joining k_0 to ∞. Hence every rational number occur in the coset diagram and that the diagram for the action of H on the rational projective line is connected.

Theorem 4.4 The action of H on the rational projective line is transitive.

Proof We shall prove transitivity of the action by showing that there is a path from a rational number p to a rational number q, that is, there exists some h in H such that $ph = q$.

As we have shown in theorem 4.2 that there exists a path joining p to ∞, that is, there exists an element $g_1 = t^{\varepsilon_1} y^{\eta_1} t^{\varepsilon_2} y^{\eta_2} \ldots t^{\varepsilon_n} y^{\eta_n}$ of H such that $\infty = pg_1 = p(t^{\varepsilon_1} y^{\eta_1} t^{\varepsilon_2} y^{\eta_2} \ldots t^{\varepsilon_n} y^{\eta_n})$ where $\varepsilon_1 = 0,1,2,3,4$ or 5, $\varepsilon_i = 1,2,3,4$ or 5, for $i = 2,3,\ldots,n$ and $\eta_n = 0,1,2,3,4$ or 5, $\eta_j = 1,2,3,4$ or 5, where $j = 1,2,\ldots,n-1$. Similarly we can find another element g_2 in H such that $\infty = qg_2$. Hence $pg_1 = qg_2$ or $pg_1g_2^{-1} = q$. That is, the action of H on the rational projective line is transitive.

We conclude with the following observations. If we are given a real quadratic irrational number α, we can find the closed path in the orbit αH. If α is totally negative then one of $(\alpha)y^j$, for $j = 1,2,3,4$ or 5 is totally positive, and we can use theorem 3.10 to find an ambiguous number in the same orbit. When we have an ambiguous number, the proof of theorem 4.1 shows how to construct the closed path. This means that if α and β are two real quadratic irrational numbers then we can test whether or not they belong to the same orbit. We can find closed paths in the orbits αH and βH and see if they are same or not. Note that for a fixed value of n, a non-square positive integer, all possible ambiguous numbers do not lie in the same orbit.

For instance, if we take $n=7$, then $(1+\sqrt{7})t^5y^5t^5y^3t^5y^5t^5y^5t^3y^5 = (1+\sqrt{7})$ and $(1-\sqrt{7})t^5y^5t^5y^3t^5y^5t^5y^5t^3y^5 = (1-\sqrt{7})$. If we let $\alpha = 1+\sqrt{7}$ and $\beta = 1-\sqrt{7}$ then $\alpha H \cap \beta H$ is empty. That is, α and β do not lie in the same orbit.

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