Total domination in K_r-covered graphs

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Abstract

A graph G is K_r-covered if each vertex of G is contained in a clique K_r. Let $\gamma(G)$ and $\gamma_t(G)$ respectively denote the domination and the total domination number of G. We prove the following results for any graph G of order n:

if G is K_6-covered, then $\gamma_t(G) \le \frac{n}{3}$,

if G is K_r-covered with $r = 3$ or 4 and has no component isomorphic to K_r, then $\gamma_t(G) \le \frac{2n}{r+1}$,

if G is K_3-covered and has no component isomorphic to K_3, then $\gamma(G) + \gamma_t(G) \le \frac{7n}{9}$.

Corollaries of the last two results are that every claw-free graph of order n and minimum degree at least 3 satisfies $\gamma_t(G) \le \frac{n}{2}$ and $\gamma(G) + \gamma_t(G) \le \frac{7n}{9}$. For general values of r, we give conjectures which would generalise the previous results. They are inspired by conjectures of Henning and Swart related to less classical parameters γ_{K_r} and $\gamma^t_{K_r}$.

1 Introduction

We follow the terminology and notation of [2], and of [10] where domination is concerned. The graphs G we consider in this paper are simple with vertex set V, edge set E (if necessary we specify $V(G)$ or $E(G)$), and order $|V| = n$. The neighbourhood of a vertex u is denoted by $N(u)$ and its closed neighbourhood $N(u) \cup \{u\}$ by $N[u]$. If S is a set of vertices of G, then $N(S) = \cup_{u \in S} N(u)$ and $N[S] = N(S) \cup S$. For the sake of simplicity we write $N(u,v)$ for $N(\{u,v\})$. The subgraph induced by S in G is denoted by $G[S]$. We may not always know the exact nature of such a set S; if S happens to be empty, then $G[S]$ is not defined.

A set $D \subseteq V$ is a *dominating set* if every vertex in $V \backslash D$ is adjacent to a vertex in D, and a *total dominating set* if every vertex in V is adjacent to a vertex in D. Every graph without isolated vertices has a total dominating set.

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The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set and the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. If the graph G has q components G_i, then $\gamma(G) = \Sigma_{i=1}^{q}\gamma(G_i)$ and $\gamma_t(G) = \Sigma_{i=1}^{q}\gamma_t(G_i)$.

A *K_r-component* of G is a component isomorphic to a clique K_r. Following the notation of Favaron, Li and Plummer in [8], we say that a graph G is *K_r-covered*, $r \geq 2$, if every vertex of G is contained in a clique K_r, and *minimally K_r-covered* if it is K_r-covered but $G - e$ is not K_r-covered for any edge e of G. These properties were already considered by Henning and Swart in [12, 13, 14] under the terms "with no K_r-isolated vertex" or "Property $C(1,r)$", and "Property $C(2,r)$", respectively. They also introduced the concepts of K_r-adjacency, K_r-domination and K_r-connectedness. We repeat their definitions. If r is an integer, $r \geq 2$, and u and v are distinct vertices of G, then u and v are said to be *K_r-adjacent* if there is a subgraph of G, isomorphic to K_r, containing u and v. For $r \geq 2$, a *K_r-dominating set* (*total K_r-dominating set*, respectively) of a graph G is a set D of vertices such that every vertex in $V\backslash D$ (in V, respectively) is K_r-adjacent to a vertex in D. Note that total K_r-dominating sets only exist in K_r-covered graphs. The *K_r-domination number* $\gamma_{K_r}(G)$ (*total K_r-domination number* $\gamma^t_{K_r}(G)$, respectively) of G is the minimum cardinality among the K_r-dominating sets (total K_r-dominating sets, respectively) of G.

Remark 1 A K_2-covered graph is just a graph without isolated vertices (the K_2-adjacency being the usual adjacency) and thus $\gamma_{K_2}(G) = \gamma(G)$ and $\gamma^t_{K_2}(G) = \gamma_t(G)$. Moreover, for $2 \leq r \leq s$, a K_s-dominating set must be a K_r-dominating set and similarly a total K_s-dominating set (if it exists) must be a total K_r-dominating set. From this it follows that

$$\gamma(G) = \gamma_{K_2}(G) \leq \gamma_{K_3}(G) \leq \cdots \leq \gamma_{K_r}(G) \leq \cdots$$

and

$$\gamma_t(G) = \gamma^t_{K_2}(G) \leq \gamma^t_{K_3}(G) \leq \cdots \leq \gamma^t_{K_r}(G) \leq \cdots$$

(The latter chain goes up to the largest r such that $\gamma^t_{K_r}$ exists.)

A *u-v K_r-path* of G is a finite, alternating sequence of vertices and subgraphs isomorphic to K_r, beginning with u and ending with v, such that the vertices of the sequence are distinct, the subgraphs of the sequence are distinct and every subgraph of the sequence is immediately preceded and succeeded by a vertex that is contained in that subgraph. The vertex u is said to be *K_r-connected* to the vertex v if there is a $u - v$ K_r-path in G. A graph G is *K_r-connected* if every pair of its vertices are K_r-connected. If G is K_r-connected, then it is K_r-covered and thus $\gamma^t_{K_r}(G)$ is well defined.

Domination in K_r-covered graphs was studied in [8] and [12]. Henning and Swart studied total domination in [13] and [14]. They gave interesting conjectures related to $\gamma^t_{K_r}(G)$ and $\gamma_{K_r}(G) + \gamma^t_{K_r}(G)$ in a K_r-covered graph G, and proved them for small values of r. Our purpose in this paper is to give similar conjectures and results, still holding for K_r-covered graphs but related to the usual parameters γ and γ_t and the usual connectedness rather than to the specific parameters γ_{K_r} and γ_{K_r} and the specific concept of K_r-connectedness.

A graph is claw-free if it does not contain any induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. Claw-free graphs of minimum degree at least 3 form a special class of K_3-covered graphs with no K_3-component. Hence results on K_3-covered graphs with no K_3-component give as corollaries results on claw-free graphs of minimum degree at least 3

The conjectures and results of Henning and Swart are related to the following properties.

Definitions

- Property $\mathcal{A}(r)$: every K_r-covered graph G of order n satisfies $\gamma^t_{K_r}(G) \leq \frac{2n}{r}$.
- Property $\mathcal{B}(r)$: every K_r-connected graph G of order $n \geq r+1$ satisfies $\gamma^t_{K_r}(G) \leq \frac{2n}{r+1}$.
- Property $\mathcal{C}(r)$: every K_r-connected minimally K_r-covered graph G of order $n \geq r+1$ satisfies $\gamma_{K_r}(G) + \gamma^t_{K_r}(G) \leq \frac{3n}{r+1}$.
- Property $\mathcal{D}(r)$: every K_r-connected graph G of order $n \geq r+1$ satisfies $\gamma_{K_r}(G) + \gamma^t_{K_r}(G) \leq \frac{3r-2}{r^2}n$.

Conjecture A [13] Property $\mathcal{A}(r)$ is true for all $r \geq 3$.

Conjecture B [13] Property $\mathcal{B}(r)$ is true for all $r \geq 3$.

Conjecture C [14] Property $\mathcal{C}(r)$ is true for all $r \geq 3$.

Conjecture D [14] Property $\mathcal{D}(r)$ is true for all $r \geq 3$.

Theorem A [13] *Property $\mathcal{A}(r)$ is true for $r = 3, 4$ and 5.*

Theorem B [13] *Property $\mathcal{B}(r)$ is true for $r = 3$ and 4.*

Theorem C [14] *Property $\mathcal{C}(r)$ is true for $r = 3$.*

Theorem D [14] *Property $\mathcal{D}(r)$ is true for $r = 3$.*

Note that for $r = 2$, Property $\mathcal{A}(2)$ is obvious, and Properties $\mathcal{B}(2)$ and $\mathcal{D}(2)$ (and thus $\mathcal{C}(2)$) are respectively proved in [3] and in [1].

In a K_r-covered graph, a *good vertex* is a vertex of degree $r-1$ and a *good clique* is a clique K_r containing a good vertex. The following result, independently proved in [6] and in [8], will be of constant use throughout the paper.

Theorem E [6, 8] *Every edge of a minimally K_r-covered graph is contained in a good clique.*

Remark 2 In minimally K_r-covered graphs, adjacent vertices are K_r-adjacent since every edge is contained in a K_r. Hence a minimally K_r-covered graph G satisfies $\gamma(G) = \gamma_{K_r}(G)$ and $\gamma_t(G) = \gamma^t_{K_r}(G)$, and is K_r-connected whenever it is connected.

Upper bounds for the total domination number γ_t are given in [10, pp. 160-161], [11] (for graphs with minimum degree 2) and [7] (for graphs with minimum degree 3). The following conjecture also appears in [7].

Conjecture F [7] If G has order n and $\delta(G) \geq 3$, then $\gamma_t(G) \leq n/2$.

Some progress towards proving Conjecture F was made in [4]:

Theorem F [4] *If G is a claw-free cubic graph of order n, then $\gamma_t(G) \leq n/2$.*

We obtain an improvement of Theorem F as a corollary to our results on K_r-covered graphs in Section 4. But recently we became aware of the existence of three preprints proving Conjecture F for cubic graphs [9] and more generally for any graph [15, 16].

2 Conjectures on γ and γ_t in K_r-covered graphs

In this section we present four conjectures related to γ_t and $\gamma + \gamma_t$ and begin to discuss their relationships with Conjectures A to D. First we define four properties similar to Properties $\mathcal{A}$ to $\mathcal{D}$.

Definitions

- Property $\mathcal{P}_1(r)$: every K_r-covered graph G of order n satisfies $\gamma_t(G) \leq \frac{2n}{r}$.
- Property $\mathcal{P}_2(r)$: every K_r-covered graph G of order n with no K_r-component satisfies $\gamma_t(G) \leq \frac{2n}{r+1}$.
- Property $\mathcal{P}_3(r)$: every minimally K_r-covered graph G of order n with no K_r-component satisfies $\gamma(G) + \gamma_t(G) \leq \frac{3n}{r+1}$.
- Property $\mathcal{P}_4(r)$: every K_r-covered graph G of order n with no K_r-component satisfies $\gamma(G) + \gamma_t(G) \leq \frac{3r-2}{r^2}n$.

Conjecture 1 Property $\mathcal{P}_1(r)$ is true for all $r \geq 3$.

Conjecture 2 Property $\mathcal{P}_2(r)$ is true for all $r \geq 3$.

Conjecture 3 Property $\mathcal{P}_3(r)$ is true for all $r \geq 3$.

Conjecture 4 Property $\mathcal{P}_4(r)$ is true for all $r \geq 3$.

For $r = 2$, Properties $\mathcal{P}_1(2)$ to $\mathcal{P}_4(2)$ are respectively the same as Properties $\mathcal{A}(2)$ to $\mathcal{D}(2)$, and are true as already noticed.

Conjecture 2 is stronger than Conjecture 1. The difference comes from the K_r-components of G since $\gamma_t(K_r) = 2 = \frac{2n}{r}$, and disjoint unions of cliques K_r show that Conjecture 1 is sharp. Similarly, disjoint unions of $K_{r+1} - e$ (a clique K_{r+1} minus one edge) show that Conjectures 2 and 3 are sharp. The sharpness of Conjecture 4 is shown for instance by the following graphs. Let $G(r)$ consist of r cliques K_r, say K^0, K^1, ..., K^{r-1}, plus $r-1$ edges u_iv_i of a matching with $u_i \in K^0$ and $v_i \in K^i$ for $1 \leq i \leq r-1$ (the graph $G(3)$ is represented in Section 5, Figure 2). Then $n = r^2$, $\gamma(G_r) = r$, $\gamma_t(G_r) = 2(r-1)$ and $\gamma(G_r) + \gamma_t(G_r) = \frac{3r-2}{r^2}n$. Obviously, this equality is also satisfied by every disjoint union of graphs $G(r)$.

The first theorem shows that Conjectures 1 and 3 are respectively equivalent to Conjectures A and C.

Theorem 1 *Properties $\mathcal{P}_1(r)$ and $\mathcal{P}_3(r)$ are respectively equivalent to Properties $\mathcal{A}(r)$ and $\mathcal{C}(r)$.*

Proof. If $\mathcal{A}(r)$ holds, then $\mathcal{P}_1(r)$ holds since $\gamma(G) \leq \gamma_{K_r}(G)$ for every K_r-covered graph G by Remark 1. Similarly, suppose Property $\mathcal{C}(r)$ to be true and let G be a minimally K_r-covered graph of order n and with no K_r-component. Each component G^i of G is minimally K_r-covered of order at least $r+1$ and is K_r-connected by Remark 2. Adding the inequalities of Property $\mathcal{C}(r)$ for all the components G^i shows that Property $\mathcal{P}_3(r)$ is true.

Conversely, $\mathcal{P}_3(r)$ implies $\mathcal{C}(r)$ by Remark 2. Suppose $\mathcal{P}_1(r)$ to be true and let G be a K_r-covered graph of order n and F a minimally K_r-covered spanning subgraph of G. Then, by Remark 2 and $\mathcal{P}_1(r)$, $\gamma^t_{K_r}(G) \leq \gamma^t_{K_r}(F) = \gamma_t(F) \leq \frac{2n}{r}$. Hence $\mathcal{A}(r)$ is true. ∎

An immediate consequence of Theorems 1, A and C is

Proposition 2

1. *For $r \in \{3,4,5\}$, every K_r-covered graph G of order n satisfies $\gamma_t(G) \leq \frac{2n}{r}$.*
2. *Every connected minimally K_3-covered graph G of order $n \geq 4$ satisfies $\gamma(G) + \gamma_t(G) \leq \frac{3n}{4}$.*

In Section 3 we improve Theorem A by proving that $\mathcal{P}_1(6)$, and thus by Theorem 1 also $\mathcal{A}(6)$, is true. In Section 4 we prove that Conjectures 2 and B are equivalent and deduce $\mathcal{P}_2(3)$ and $\mathcal{P}_2(4)$ from Theorem B. In Section 5 we show that Property $\mathcal{P}_4(r)$ implies Property $\mathcal{D}(r)$ for all r and that Property $\mathcal{D}(3)$ implies Property $\mathcal{P}_4(3)$. This allows us to deduce $\mathcal{P}_4(3)$ from Theorem D. Corollaries on claw-free graphs of minimum degree at least 3 are deduced from $\mathcal{P}_2(3)$ and $\mathcal{P}_4(3)$.

3 Some results on Conjecture 1

The proof of Property $\mathcal{P}_1(6)$ uses a particular family $\mathcal{F}_r$ of minimally K_r-covered graphs which was already considered in [8]. Recall that the corona of a graph H is obtained from H by adding a pendant edge at each vertex of H and that the middle graph of H is the line graph of the corona of H (see for instance [5]).

Definition: $\mathcal{F}_r$ is the family of the middle graphs of $(r-1)$-regular graphs.

From this definition, $\mathcal{F}_r$ is the collection of graphs G consisting of edge-disjoint cliques of order r, where each such clique contains exactly one vertex of degree $r-1$ and the remaining $r-1$ vertices have degree $2(r-1)$. Let S be the set of these edge-disjoint cliques. Then each vertex of G of degree $r-1$ belongs to exactly one K_r in S and each vertex of degree $2(r-1)$ belongs to exactly two K_r's in S.

We first show that for $r \geq 3$, Property $\mathcal{P}_2(r)$, and thus also $\mathcal{P}_1(r)$, holds for every graph of $\mathcal{F}_r$. We need a lemma.

Lemma 3 *Every d-regular graph H, $d \geq 2$, has a spanning subgraph consisting of p_1 isolated vertices, p_2 paths of length one and p_3 paths of length two such that $p_1 \leq \frac{d-2}{d} p_3$.*

Proof. Let L be a spanning subgraph of H of the required form such that $p_2 + p_3 - p_1$ is maximum. Let X be the set of isolated vertices of L, Y (Z, respectively) the set of the vertices of degree two (one, respectively) of the paths of length two, and T the set of vertices of the paths of length one. Let u be any vertex in X. The vertex u has no neighbour in T, for otherwise we could find a spanning subgraph L' of the required form with $p'_3 - p'_1 = p_3 - p_1 + 2$ and $p'_2 = p_2 - 1$. Similarly, u has no neighbour in Z. Hence X sends dp_1 edges to Y and Y sends at most $(d-2)p_3$ edges to X. Therefore $p_1 \leq \frac{d-2}{d} p_3$. ■

Theorem 4 *For $r \geq 3$, every graph G of order n of $\mathcal{F}_r$ satisfies $\gamma_t(G) < \frac{2n}{r+1}$.*

Proof. Let V be the disjoint union $V_1 \cup V_2$, where V_1 and V_2 are respectively the sets of vertices of degree $r-1$ and $2(r-1)$ of G. Each vertex in V_1 has $r-1$ neighbours in V_2 and each vertex in V_2 has two neighbours in V_1. Hence $2|V_2| = (r-1)|V_1|$. Since $|V_1| + |V_2| = n$, we get $|V_1| = \frac{2n}{r+1}$. Now consider the graph H whose vertices are the elements of S and where two vertices A and B of H are adjacent if the cliques A and B of G share (exactly) one vertex x_{AB}. The graph H has order $N(H) = |V_1| = \frac{2n}{r+1}$ and is $(r-1)$-regular. Consider a spanning subgraph L of H as in Lemma 3. We construct a set D of vertices of G as follows: for each path ABC of L, let D contain the vertices x_{AB} and x_{BC}; for each path AB of L, let D contain the vertex x_{AB} and one of its neighbours; for each isolated vertex A of L, let D contain two vertices of A. Hence D is a total dominating set of G of order $|D| = 2p_3 + 2p_2 + 2p_1 = N(H) + p_1 - p_3 < N(H)$ by Lemma 3. Therefore $\gamma_t(G) < \frac{2n}{r+1}$. ■

We can now prove $\mathcal{P}_1(6)$.

Theorem 5 *Every K_6-covered graph G of order n satisfies $\gamma_t(G) \leq \frac{n}{3}$.*

Proof. The proof is by induction on $n \geq 6$. If $n = 6$, then $G \cong K_6$ and the property is true. Suppose the property to be true for graphs of order less than n and let G be a K_r-covered graph of order $n > 6$. Let H be a minimally K_6-covered spanning subgraph of G. Since $\gamma_t(G) \leq \gamma_t(H)$, it is sufficient to prove $\gamma_t(H) \leq \frac{n}{3}$, and we henceforth consider H instead of G.

For every pair of adjacent vertices u and v, let

$$P(u,v) = \{x \in N(u,v)\backslash\{u,v\} : N(x) \subseteq N(u,v)\}.$$

By Theorem E, the edge uv is contained in a good clique C. Let u' be a good neighbour of u in C. The $r-1$ neighbours of u' are vertices of C and thus are adjacent to u. Hence every good neighbour of u, and similarly every good neighbour of v, belongs to $P(u,v) \cup \{u,v\}$. Note also that if $t \in N(u,v)\backslash(P(u,v) \cup \{u,v\})$, then t has a neighbour $t_1 \notin N(u,v)$, and thus a good neighbour $t' \notin N(u,v)$ belonging to a good clique containing the edge tt_1. The graph $H[V\backslash(P(u,v) \cup \{u,v\})]$, if it exists, is still K_6-covered and satisfies the induction hypothesis. Note that $\{u,v\}$ is a total dominating set of $H[P(u,v) \cup \{u,v\}]$. Hence if $|P(u,v)| \geq 4$, we are done. We suppose henceforth $|P(u,v)| \leq 3$ for every pair of adjacent vertices of H.

Lemma 5.1 *Two good cliques K_6 cannot share more than one vertex.*

Proof of Lemma 5.1. Suppose to the contrary that C and C' are two good K_6's both containing the (non-good) vertices u and v, and let x (respectively x') be a good vertex of C (respectively C'). Since $|(C \cup C')\backslash\{u,v,x,x'\}| \geq 3$ and $|P(u,v)\backslash\{x,x'\}| \leq 1$, at least one vertex t of $(C \cup C')\backslash\{u,v,x,x'\}$ is not in $P(u,v)$. Let t' be a good neighbour of t not in $N(u,v)$. Since $\{x,x',t'\} \subseteq P(v,t)$ and $|P(v,t)| \leq 3$, u is not in $P(v,t)$ and has a good neighbour u' not in $N(v,t)$. Similarly, v has a good neighbour v' not in $N(u,t)$. The four vertices x, x', u', v' are distinct and belong to $P(u,v)$, a contradiction. □

Lemma 5.2 *No vertex can belong to three good cliques K_6.*

Proof of Lemma 5.2. Suppose to the contrary that u belongs to three good cliques C, C', C'', and let x, x', x'' be good vertices of C, C', C'' respectively. Let w and t be two vertices of $C\backslash\{u,x\}$. Since $\{x,x',x''\} \subseteq P(u,t)$, $w \notin P(u,t)$ and w has a good neighbour w' not in $N(u,t)$ and thus distinct from x, x', x''. But then $\{x,x',x'',w'\} \subseteq P(u,w)$, a contradiction. □

We now complete the proof of the theorem. Let C be any good K_6 of H. Then C contains at least two non-good vertices u and v, for otherwise we would have $|P(x,y)| \geq 4$ for a pair of adjacent vertices x and y. Each of u and v belongs to another (exactly one, by Lemma 5.2) good K_6. By Lemma 5.1, these two good K_6's are distinct, say C_u and C_v. Let x, u' and v' be good vertices of C, C_u and C_v respectively. If C contains a second good vertex x', then $\{x,x',u',v'\} \subseteq P(u,v)$, a contradiction. Therefore each good clique contains exactly one good vertex and five non-good ones, and each non-good vertex is contained in exactly two good K_6's. The graph H belongs to the family $\mathcal{F}_6$ described above and thus $\gamma_t(H) < \frac{2n}{7} < \frac{2n}{6}$. This completes the proof by induction. ■

4 Some results on Conjecture 2

In this section we prove the equivalence of Conjectures B and 2.

Theorem 6 *For any $r \geq 3$, Properties $\mathcal{B}(r)$ and $\mathcal{P}_2(r)$ are equivalent.*

Proof. 1. $\mathcal{P}_2(r)$ implies $\mathcal{B}(r)$.
Suppose $\mathcal{P}_2(r)$ to be true and let G be a K_r-connected graph of order $n \geq r+1$. Let $F = (V, E')$ be the spanning subgraph of G obtained by deleting the edges of G which are not contained in a K_r. The graph F is still K_r-connected, and thus connected, since the K_r-paths have not been destroyed. By $\mathcal{P}_2(r)$, $\gamma_t(F) \leq \frac{2n}{r+1}$. Since every edge of F is contained in a K_r, $\gamma^t_{K_r}(F) = \gamma_t(F)$. Therefore $\gamma^t_{K_r}(G) \leq \gamma^t_{K_r}(F) \leq \frac{2n}{r+1}$ and $\mathcal{B}(r)$ is true.

2. $\mathcal{B}(r)$ implies $\mathcal{P}_2(r)$.
We suppose $\mathcal{B}(r)$ to be true and prove $\mathcal{P}_2(r)$ by induction on $n \geq r+1$. If $n = r+1$, then the K_r-covered graph G is isomorphic to K_{r+1} or to $K_{r+1} - e$ and thus $\gamma_t(G) = 2 = \frac{2n}{r+1}$. Suppose the property to be true for graphs of order at most $n-1$ and let G be a K_r-covered graph of order $n \geq r+2$ and with no K_r-component. Let $F = (V, E_1)$ be a minimally K_r-covered spanning subgraph of G. All the components of F are K_r-connected but some of them may be of order r, that is, isomorphic to K_r. If no component of F is isomorphic to K_r, then, by Property $\mathcal{B}(r)$ applied to each component of F, we get $\gamma^t_{K_r}(F) \leq \frac{2n}{r+1}$ and thus $\gamma_t(G) \leq \gamma^t_{K_r}(G) \leq \gamma^t_{K_r}(F) \leq \frac{2n}{r+1}$.

Suppose now that F has some K_r-components and let $H = (V, E_2)$ be a spanning subgraph of G obtained by joining each K_r-component of F to another component by one edge of G (this can be done since G has no K_r-component). Each edge $e \in E_2 \backslash E_1$ is a cut-edge of H such that at least one component of $H - e$ is a K_r. Indeed, since $r \geq 3$, the edges in $E_2 \backslash E_1$ are the only cut-edges of H. Since $\gamma_t(G) \leq \gamma_t(H)$ and since we can apply the induction hypothesis to each component of H, it is sufficient to suppose H to be connected and to prove $\gamma_t(H) \leq \frac{2n}{r+1}$.

Let y be a vertex of H incident to the cut-edges $yx_1, yx_2, \cdots, yx_k$, $k \geq 1$, of H such that each x_i belongs to a K_r-component A_i of F and the other edges incident to y are not cut-edges of H. Let

$$X = \{x_1, x_2, \cdots, x_k\} \text{ and } X' = \bigcup_{i=1}^{k} (V(A_i) \backslash \{x_i\}).$$

We first prove a lemma.

Lemma 6.1 *If y is incident to at least two cut-edges, then $\gamma_t(H) \le \frac{2n}{r+1}$.*

Proof of Lemma 6.1 Let y be incident to the cut-edges yx_1 and yx_2 such that each x_i belongs to a K_r-component A_i of F, and let $H' = H - (V(A_1) \cup V(A_2))$. Clearly, H' is a K_r-covered graph of order $n-2r$ with no K_r-component, and so by the induction hypothesis, H' has a total dominating set S' with $|S'| \le \frac{2n-4r}{r+1}$. Then $S = S' \cup \{x_1, x_2, y\}$ is a total dominating set of G with $|S| \le \frac{2n-4r+3r+3}{r+1} \le \frac{2n}{r+1}$, as required. □

We thus assume that xy is the unique cut-edge of H incident to y, where x belongs to the K_r-component A of F (see Figure 1). Then

$$|\{x,y\}| = 2 = \tfrac{2}{r+1}|\{x,y\} \cup X'|. \tag{1}$$

Let $L = N(y)\backslash\{x\}$ and $P = \{z \in L : N(z) \subseteq L \cup \{y\}\}$. Let Q be the set of the vertices $t \in L\backslash P$ such that all the edges of H incident to t and not in $H[L]$ are cut-edges of H. Let

$$T = N(Q)\backslash(L \cup \{y\}) \text{ and } T' = N(T)\backslash Q.$$

By Lemma 6.1 applied to each vertex in Q, $|T| = |Q|$, $H[T \cup T']$ consists of $|Q|$ isolated cliques K_r and $|T'| = (r-1)|Q|$. Since the sets Q, T and T' are mutually disjoint, we have

$$|Q \cup T \cup T'| = (r+1)|Q|$$

and so

$$|Q \cup T| = 2|Q| = \tfrac{2}{r+1}|Q \cup T \cup T'|. \tag{2}$$

Let $M = T \cup T' \cup \{x,y\} \cup X'$. The vertices of $P \cup Q \cup M$ have no neighbours in $V\backslash(L \cup M)$. Recall that each edge of F is contained in a good K_r of F. Every vertex u in $L\backslash(P \cup Q)$ has at least one neighbour v in $V\backslash(L \cup M)$ such that the edge uv is not a cut-edge of H and thus belongs to a good clique C of F, which cannot contain y since $v \notin L$. Let w be a good vertex of C in F. Note that $w \notin L$, for otherwise yw (which is not a cut-edge) is contained in a good clique $C' \ne C$. But then w belongs to two distinct cliques and is not a good vertex of F, a contradiction. The clique C is entirely contained in $V\backslash(P \cup Q \cup M)$ since w is not in L and thus has no neighbour in M, and w has no neighbour in $P \cup Q$ by the definition of P and Q. Therefore $H[V\backslash(P \cup Q \cup M)]$ (if it exists) is K_r-covered, but may have K_r-components. In this case, all these K_r-components have at least one vertex in $L\backslash(P \cup Q)$. For each K_r-component B_i, $1 \le i \le s$, of $H[V\backslash(P \cup Q \cup M)]$, choose a vertex $b_i \in V(B_i) \cap (L\backslash(P \cup Q))$ and let

$$R = \{b_1, \cdots, b_s\}, \;\; R' = \bigcup_{i=1}^{s}(V(B_i)\backslash\{b_i\}).$$

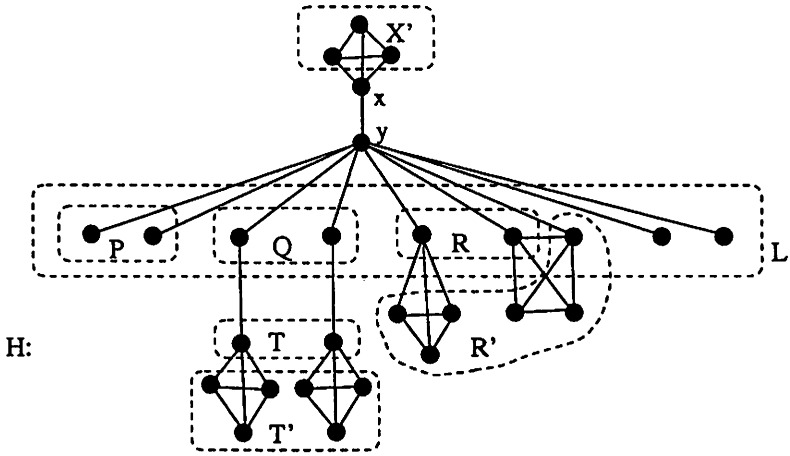

Figure 1

Since the cliques B_i are disjoint,

$$|R| = \frac{|R\cup R'|}{r} \le \frac{2}{r+1}|R \cup R'|. \tag{3}$$

The graph $H[V \backslash (P \cup Q \cup M \cup R \cup R')]$ (assuming that it exists) is K_r-covered and has no K_r-components. By the induction hypothesis it has a total dominating set D_1 such that

$$|D_1| \le \frac{2}{r+1}|V \backslash (P \cup Q \cup M \cup R \cup R')|.$$

On the other hand, since $T = \emptyset$ whenever $Q = \emptyset$, $D_2 = \{x, y\} \cup Q \cup T \cup R$ is a total dominating set of $H[P \cup Q \cup M \cup R \cup R']$. Since the sets $P, Q, T, T', \{x, y\}, X', R, R'$ are mutually disjoint, it follows from (1), (2) and (3) that

$$\begin{aligned} |D_2| &= |\{x,y\}| + |Q \cup T| + |R| \\ &\le \frac{2}{r+1}|\{x,y\} \cup X' \cup Q \cup T \cup T' \cup R \cup R'| \\ &= \frac{2}{r+1}|M \cup Q \cup R \cup R'| \\ &\le \frac{2}{r+1}|P \cup Q \cup M \cup R \cup R'|. \end{aligned}$$

Therefore $D_1 \cup D_2$ is a total dominating set of H of order at most $\frac{2n}{r+1}$. This completes the proof by induction. ∎

A consequence of Theorems 6 and B is that Property $\mathcal{P}_2(r)$ is true for $r = 3$ and 4.

Theorem 7 *For $r = 3$ and 4, every K_r-covered graph G of order n and with no K_r-component satisfies $\gamma_t(G) \le \frac{2n}{r+1}$.*

We have already seen that disjoint unions of $K_{r+1} - e$ satisfy $\gamma_t(G) = \frac{2n}{r+1}$. The following class gives an example of arbitrarily large connected graphs satisfying the equality in Theorem 7. For any integer $p \ge r$, let $G_{p,r}$ be the graph obtained from one clique K isomorphic to K_p and p cliques K^1, K^2,.., K^p isomorphic to K_r by adding p edges u_iv_i of a matching with $u_i \in K$ and $v_i \in K^i$ for $1 \le i \le p$. Then $G_{p,r}$ is K_r-covered, $n(G_{p,r}) = p(r+1)$ and $\gamma_t(G_{p,r}) = 2p = \frac{2n}{r+1}$.

For claw-free graphs, we obtain the following corollary to the case $r = 3$ of Theorem 7.

Corollary 8 *Every claw-free graph of order n and minimum degree at least 3 satisfies* $\gamma_t(G) \le \frac{n}{2}$.

5 Some results on Conjecture 4

Theorem 9 below states that Conjecture 4 is at least as strong as Conjecture D. As for the converse, we only prove in Theorem 10 that Property $\mathcal{D}(3)$, which is true by Theorem D, implies Property $\mathcal{P}_4(3)$.

Theorem 9 *For any* $r \ge 3$*, Property* $\mathcal{P}_4(r)$ *implies Property* $\mathcal{D}(r)$.

The proof of Theorem 9 is exactly the same as the proof of the first part of Theorem 6 and is not repeated.

Theorem 10 *Every* K_3*-covered graph* G *of order* n *and with no* K_3*-component satisfies* $\gamma(G) + \gamma_t(G) \le \frac{7n}{9}$.

Proof. We proceed by induction on n. The beginning of the proof is similar to the proof of the second part of Theorem 6. We recall it briefly for the sake of completeness. If $n = 4$, then G is isomorphic to K_4 or $K_4 - e$ and satisfies $\gamma(G) + \gamma_t(G) = 3 < \frac{7n}{9}$. Suppose the property true for graphs of order less than n and let G be a K_3-covered graph with no K_3-component and of order $n \ge 5$. Let $F = (V, E_1)$ be a minimally K_3-covered spanning subgraph of G. All the components of F are K_3-connected. If no component of F is isomorphic to K_3, then we can apply Theorem D to each of them and add the resulting inequalities. Hence $\gamma(G)+\gamma_t(G) \le \gamma_{K_3}(G)+\gamma^t_{K_3}(G) \le \gamma_{K_3}(F)+\gamma^t_{K_3}(F) \le \frac{7n}{9}$.

Suppose now F has K_3-components and let $H = (V, E_2)$ be a spanning subgraph of G obtained by joining each K_3-component of F to another component by one edge of G. Since $\gamma(G) + \gamma_t(G) \le \gamma(H) + \gamma_t(H)$, we can suppose without loss of generality that H is connected, and prove $\gamma(H) + \gamma_t(H) \le 7n/9$. Let y be any vertex incident to cut-edges of H. We define X, X', L, P, Q, T, T', M, R, R' (some possibly empty), as in Theorem 6. (Note that now $|X| > 1$ is possible, although it can be shown that $|X| \le 2$, and in this case $\{x\}$ is replaced by X in the definition of M.) Then $|T| \ge |Q|$, $T = \emptyset$ if $Q = \emptyset$ and

$$|X \cup X'| = 3|X|, \quad |T \cup T'| = 3|T|, \quad |R \cup R'| = 3|R|. \tag{4}$$

Moreover, $L \ne \emptyset$ since y belongs to a triangle. We state the following result as a lemma for reference.

Lemma 10.1 *Let $t \in L$ be a good vertex of F. If t is good in H, then $t \in P$; otherwise $t \in Q$.*

Proof of Lemma 10.1. If t is good in H, then t is contained in exactly one clique and this clique contains y, hence $t \in P$. If t is not good in H, then all the edges of H incident to t and not in $H[L]$ are in $E_2 \backslash E_1$ and thus cut-edges of H, therefore $t \in Q$. □

Let $S = P \cup Q \cup M \cup R \cup R'$. If it exists, the graph $H[V \backslash S]$ is K_3-covered without K_3-components and, by the induction hypothesis, has a total dominating set D_1 and a dominating set D_1' such that $|D_1| + |D_1'| \leq 7(n - |S|)/9$. Note that

$$D_2 = \{y\} \cup X \cup Q \cup T \cup R$$

is a total dominating set of $H[S]$. We first consider the case $P = \emptyset$.

Lemma 10.2 *If $P = \emptyset$, then $\gamma(H) + \gamma_t(H) \leq 7n/9$.*

Proof of Lemma 10.2. If $P = \emptyset$, then

$$D_2' = X \cup T \cup R$$

is a dominating set of $H[S]$. By (4),

$$|S| = 1 + |Q| + 3(|X| + |T| + |R|),$$

therefore

$$\begin{aligned} 7|S| - 9(|D_2| + |D_2'|) &= -2 - 2|Q| + 3(|X| + |T| + |R|) \\ &> 2(|T| - |Q|) + 2(|X| - 1) \\ &\geq 0 \ \ (\text{since } |T| \geq |Q| \text{ and } |X| > 0). \end{aligned}$$

It follows that $D = D_1 \cup D_2$ (respectively $D' = D_1' \cup D_2'$) is a total dominating set (respectively a dominating set) of H such that $|D| + |D'| \leq 7n/9$. □

We now suppose that $P \neq \emptyset$. Then

$$D_2'' = \{y\} \cup X \cup T \cup R$$

is a dominating set of $H[S]$. By (4),

$$|S| = 1 + |P| + |Q| + 3(|X| + |T| + |R|).$$

Therefore

$$\begin{aligned} 7|S| - 9(|D_2| + |D_2''|) &= -11 + 7|P| - 2|Q| + 3(|X| + |T| + |R|) \\ &= -1 + 7(|P| - 1) + 3(|X| - 1) + 3|T| - 2|Q| + 3|R|, \end{aligned}$$

with $|P| \geq 1$, $|X| \geq 1$, $|T| \geq |Q|$ and $T = \emptyset$ if $Q = \emptyset$. If $|P| > 1$, $|X| > 1$, $|Q| > 0$ or $|R| > 0$, then $|D_2| + |D_2'| \leq 7|S|/9$, in which case $D = D_1 \cup D_2$ (respectively $D' = D_1' \cup D_2''$) is a total dominating set (respectively a dominating set) of H such that $|D| + |D'| \leq 7n/9$.

It remains to consider the case where, for each vertex y incident to a cut-edge of H,

$$|P| = |X| = 1, \quad Q = \emptyset \text{ and } R = \emptyset. \tag{5}$$

Lemma 10.3 $\mathrm{Deg}_H(y) = 3$.

Proof of Lemma 10.3. Note that $\deg_H(y) \geq 3$ since $|X| = 1$ and y is contained in a good triangle C, where C contains the unique good vertex y' in H adjacent to y. If $\deg_H(y) > 3$, let $v \in L \backslash V(C)$. Then yv is contained in a good triangle $C' \neq C$ of F and hence y is adjacent to a good vertex y'' of F, where $y'' \neq y'$. Since y'' is not good in H, it follows from Lemma 10.1 that $y'' \in Q$, a contradiction since $Q = \emptyset$. □

Let $L = \{u, z\}$ with u good and z not. If z is incident to a cut-edge of H, then exchanging the roles of y and z shows that H consists of three triangles joined by two edges (see the graph G_3 in Figure 2) and verifies $\gamma(H) + \gamma_t(H) = 7 = 7n/9$.

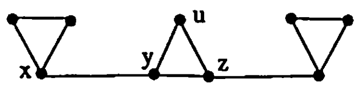

Figure 2

Otherwise, we consider the following decomposition of $N(z)$ (which is similar to the decomposition of $N(y)$ used above). Define L_1, P_1, Q_1, T_1, T_1', M_1, R_1 and S_1 similar to L, P, Q, T, T', M, R and S. Then $|T_1' \cup T_1| = 3|T_1|$. Also, if we define X_1 similar to X, then $X_1 = \emptyset$ since z is not incident to a cut-edge of H. Note that $y \in Q_1$ and $u \in P_1$.

Lemma 10.4 $|P_1| \geq 2$.

Proof of Lemma 10.4. Since z has degree at least 3 and $X_1 = \emptyset$, it follows (similar to the proof of Lemma 10.3) that z is adjacent to a good vertex y' of F with $y' \notin \{u, y\}$. Since $\deg_H(y) = 3$ and $\deg_H(u) = 2$ and all the neighbours of y and u are already accounted for, z and y' have another common neighbour $u' \notin \{u, y\}$. If y' is good in H, we are done. Otherwise, by Lemma 10.1, y' is incident to a cut-vertex of H, and so by (5), with y' instead of y and with the sets defined in the obvious way, $|P'| = |X'| = 1$, $Q' = \emptyset$ and $R' = \emptyset$. Thus $\deg_H(u') = 2$, $\deg_H(y') = 3$ and $P' = \{u'\}$. But then u' is good in H, that is, $u' \in P_1$, and the result follows. □

The graph $H[V\backslash S_1]$, if it exists, has a total dominating set D_3 and a dominating set D_3' such that $|D_3|+|D_3'| \leq 7(n-|S_1|)/9$. Define D_4, a total dominating set, and D_4', a dominating set, of $H[S_1]$ similar to D_2 and D_2''; note that

$$|D_4| = 1 + |Q_1| + |T_1| + |R_1|$$

and

$$|D_4'| = 1 + |T_1| + |R_1|.$$

Since

$$|S_1| = 1 + |P_1| + |Q_1| + 3(|T_1| + |R_1|)$$

and $|T_1| \geq |Q_1|$, we get

$$\begin{aligned} 7|S_1| - 9(|D_4| + |D_4'|) &= -11 + 7|P_1| - 2|Q_1| + 3(|T_1| + |R_1|) \\ &\geq 3 + |Q_1| + 3|R_1| \\ &> 0. \end{aligned}$$

Hence the sets $D = D_3 \cup D_4$ and $D' = D_3' \cup D_4'$ are respectively a dominating set and a total dominating set of H satisfying $|D|+|D'| < 7n/9$. This completes the proof by induction. ■

Disjoint unions of the graph of Figure 2 show that the bound in Theorem 10 is sharp.

Corollary 11 *Every claw-free graph G of order n and minimum degree at least 3 satisfies $\gamma(G) + \gamma_t(G) \leq 7n/9$.*

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