Trees with equal average domination and independent domination numbers

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Abstract

For a vertex v of a graph $G = (V, E)$, the domination number $\gamma_v(G)$ of G relative to v is the minimum cardinality of a dominating set in G that contains v. The average domination number of G is $\gamma_{av}(G) = \frac{1}{|V|}\sum_{v\in V}\gamma_v(G)$. The independent domination number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set in G that contains v. The average independent domination number of G is $i_{av}(G) = \frac{1}{|V|}\sum_{v\in V} i_v(G)$. In this paper, we show that a tree T satisfies $\gamma_{av}(T) = i_{av}(T)$ if and only if $A(T) = \emptyset$ or each vertex of $A(T)$ has degree 2 in T, where $A(T)$ is the set of vertices of T that are contained in all its minimum dominating sets.

Keywords: *average domination, average independent domination, trees*

1 Introduction

Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7, 8]. In this paper, we introduce and study the concept of average domination and independence in graphs.

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For notation and graph theory terminology we in general follow [7]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E of size q, and let v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* $N[S] = N(S) \cup S$. The *private neighborhood* $\mathrm{pn(v, S)}$ of $v \in S$ is defined by

$$\mathrm{pn(v, S) = N[v] - N[S - \{v\}]}.$$

If $\mathrm{pn(v, S)} \neq \emptyset$, then every vertex of $\mathrm{pn(v, S)}$ is called a *private neighbor of v with respect to S*. A *leaf* is a vertex of degree one and its neighbor is called a *support vertex*. A *strong support vertex* is adjacent to at least two leaves. A cycle on n vertices is denoted by C_n and a path on n vertices by P_n. We let $x \equiv_\ell y$ mean $x \equiv y \pmod{\ell}$.

A set S is a *dominating set* of G if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to a vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. The *independence number* $\beta(G)$ of G is the maximum cardinality of an independent set in G, while the *independent domination number* (also called the *lower independence number*) $i(G)$ of G is the minimum cardinality of a maximal independent set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set, while a maximal independent set of cardinality $i(G)$ is called an $i(G)$-set.

For any two graph theoretical parameters λ and μ, we define a graph G to be a (λ, μ)-graph if $\lambda(G) = \mu(G)$. It is well-known that $\gamma(G) \leq i(G)$ for all graphs G, and that the class of (γ, i)-graphs is very difficult to characterize. Several classes of (γ, i)-graphs have been found–see, for example, [1, 2, 4, 5, 13].

The class of (γ, i)-trees was first characterized in [6] but this characterization, which involves reducing transformations and forbidden configurations, is rather difficult to use. Recently, Cockayne, Favaron, Mynhardt, and Puech [3] provided a more elegant characterization of (γ, i)-trees which is relatively easy to use. Their characterization is in terms of the set of vertices of the tree which are contained in all its γ-sets and i-sets. These sets were characterized by Mynhardt [9] who used an ingenius tree pruning procedure.

In this paper, we investigate a subclass of (γ, i)-trees. For this purpose, we introduce the concept of average domination and average independence in graphs. Thereafter we characterize trees with equal average domination and average independent domination numbers in terms of the set of vertices of the tree which are contained in all its γ-sets. Using the tree pruning procedure of Mynhardt [9], this set can be found in complexity bounded by $O(n^2)$, where n is the order of the tree. Hence, our characterization of trees with equal average domination and average independent domination numbers is relatively easy to use. We also show that the average domination number of a nontrivial tree is at most half its order, while the average independent domination number of a tree of order $n \geq 2$ is at most $n - 2 + 2/n$.

2 The Average Domination Number

For a vertex v of G, we define the *domination number of G relative to v*, denoted $\gamma_v(G)$, as the minimum cardinality of a dominating set in G that contains v. The *average domination number* of G, denoted $\gamma_{av}(G)$, is defined as $\frac{1}{|V|} \sum_{v \in V} \gamma_v(G)$. A dominating set of cardinality $\gamma_v(G)$ containing v we call a $\gamma_v(G)$-set.

In this section, we establish bounds on the average domination number of a graph in terms of its domination number. Clearly, for a vertex v in a graph G, $\gamma(G) \leq \gamma_v(G)$ with equality if and only if v belongs to a $\gamma(G)$-set. Thus for any graph G, $\gamma(G) \leq \gamma_{av}(G)$ with equality if and only if every vertex of G belongs to a $\gamma(G)$-set. Consequently, $\gamma_{av}(K_n) = 1$, while for a cycle C_n on $n \geq 3$ vertices, $\gamma_{av}(C_n) = \gamma(C_n) = \lceil n/3 \rceil$.

The next result establishes an upper bound on the average domination of a graph in terms of its domination number. The proof is straightforward and therefore omitted.

Proposition 1 *For any graph G of order n with domination number γ,*

$$\gamma_{av}(G) \leq \gamma + 1 - \frac{\gamma}{n},$$

with equality if and only if G has a unique $\gamma(G)$-set.

For $n \geq 3$, by Proposition 1, $\gamma_{av}(K_{1,n-1}) = 2 - 1/n$. Next we establish the average domination number of a path.

Proposition 2 *For a path P_n on n vertices,*

$$\gamma_{av}(P_n) = \begin{cases} \dfrac{n+2}{3} - \dfrac{2}{3n} & \text{if } n \equiv_3 2 \\ \dfrac{n+2}{3} & \text{otherwise} \end{cases}$$

Proof. Suppose $n \equiv_3 0$. Then there is a unique $\gamma(P_n)$-set, and so, by Proposition 1, $\gamma_{av}(P_n) = \gamma(P_n) + 1 - \gamma(P_n)/n = n/3 + 1 - 1/3 = (n+2)/3$. Suppose $n \equiv_3 1$. Then every vertex of P_n belongs to a $\gamma(P_n)$-set, and so $\gamma_{av}(P_n) = \gamma(P_n) = (n+2)/3$. Suppose $n \equiv_3 2$. Let P_n be given by $v_1, v_2, \ldots, v_n$. Then, $\gamma_{v_i}(P_n) = \gamma(P_n) + 1 = (n+4)/3$ if $i = 3k$, $1 \leq k \leq (n-2)/3$ and $\gamma_{v_i}(P_n) = \gamma(P_n) = (n+1)/3$ otherwise. Hence,

$$\sum_{v \in V(P_n)} \gamma_v(P_n) = \left(\frac{n-2}{3}\right) \cdot \left(\frac{n+4}{3}\right) + \left(\frac{2n+2}{3}\right) \cdot \left(\frac{n+1}{3}\right) = \frac{n^2+2n-2}{3},$$

and so, $\gamma_{av}(P_n) = (n+2)/3 - 2/3n$. □

Note that $\gamma_{av}(P_3) = 5/3 > 3/2$. We show next that P_3 is the only nontrivial tree whose average domination exceeds half its order. Recall that the corona $H \circ K_1$ of a graph H is the graph constructed from H by adding for each vertex v of H, a new vertex v' and a pendant edge vv'.

For any forest H, let $\mathcal{S}(H)$ denote the set of trees, each of which can be formed from $H \circ K_1$ by adding a new vertex x and edges joining x to one or more vertices of H. Then define

$$\mathcal{T} = \bigcup_H \mathcal{S}(H)$$

where the union is taken over all forests H. As a consequence of a characterization of connected graphs G of order n with $\gamma(G) = (n-1)/2$ (see Theorem 2.6 in [7] or [12]), we have the following result.

Corollary 3 ([7, 12]) *A tree T of order $n \geq 3$ satisfies $\gamma(T) = (n-1)/2$ if and only if $T \in \mathcal{T}$.*

Theorem 4 *If T is a tree of order $n \geq 4$, then $\gamma_{av}(T) \leq n/2$ with equality if and only if T is the corona of a tree.*

Proof. The sufficiency follow immediately from the observation that if T is the corona of a tree, then $\gamma(T) = n/2$ and every vertex of T belongs to a $\gamma(T)$-set. To prove the necessity, we recall (see Theorem 2.1 in [7]) that $\gamma(T) \leq n/2$. If $\gamma(T) \leq (n-2)/2$, then by Proposition 1, $\gamma_{av}(T) < n/2$. If $\gamma(T) = (n-1)/2$, then, by Corollary 3, $T \in \mathcal{T}$, and so $\gamma_{av}(T) \leq (n-1)/2 + 2/n < n/2$ (since here $n \geq 5$). Finally, if $\gamma(T) = n/2$, then T is the corona $H \circ K_1$ of a tree H (see Payan and Xuong [10]), and so $\gamma_{av}(T) = n/2$. □

3 The Average Independent Domination Number

In this section, we consider the concept of average independence in graphs, a concept closely related to the problem of finding large independent sets in graphs. The independent domination number $i(G)$ of a graph G can be viewed as a worst case bound on the performance of the 'naive' greedy-algorithm for approximating a maximum independent set of G: choose a vertex v, let $S = \{v\}$, and add vertices to S, one at a time, which are not adjacent to any vertex already in S. The algorithm stops when S is a maximal independent set. The class of those graphs for which this 'naive' greedy-algorithm always yields a maximum independent set is exactly the class of well-covered graphs. A graph is *well-covered* if every maximal independent set of vertices is also a maximum independent set. The study of well-covered graphs was proposed by Plummer [11].

The lower bound $i(G)$ on the cardinality of an independent set obtained by the 'naive' greedy-algorithm can be improved if one takes into account that the first vertex is chosen randomly. For a vertex v of G, we define the *independent domination number of* G *relative to* v, denoted $i_v(G)$, as the minimum cardinality of a maximal independent set in G that contains v. Then, $i_v(G)$ is a worst case bound on the cardinality of an independent set obtained by the 'naive' greedy-algorithm if we use v as a start vertex. A maximal independent set of cardinality $i_v(G)$ containing v we call an $i_v(G)$-set. We define the *average independent domination number* $i_{av}(G)$ of G, or the *lower average independence number* of G, by $\frac{1}{|V|}\sum_{v\in V} i_v(G)$. Then, $i_{av}(G)$ is a lower bound on the expected value of the cardinality of the independent set obtained by the 'naive' greedy-algorithm if the first vertex is chosen randomly.

Since $i(G) \leq i_v(G) \leq \beta(G)$ for every vertex v in a graph G, we observe that $i(G) \leq i_{av}(G) \leq \beta(G)$ for any graph. Furthermore, equality holds in the lower bound if and only if every vertex of G belongs to an $i(G)$-set, while equality holds in the upper bound if and only if G is well-covered. If G is not a well-covered graph, then the upper bound can be improved.

Theorem 5 *For any graph G of order n with independent domination number i and independence number β,*

$$i_{av}(G) \leq \beta - \frac{i \cdot (\beta - i)}{n}$$

with equality if and only if G is well-covered or G has a unique $i(G)$-set and for each vertex not in the $i(G)$-set, every maximal independent set containing it has cardinality $\beta(G)$.

Proof. Let S be an $i(G)$-set of G. If $v \notin S$, then, clearly, $i_v(G) \leq \beta$. On the other hand, for each vertex $v \in S$, $i_v(G) = i$. Hence,

$$i_{av}(G) \leq \frac{1}{n}\left(i^2 + (n-i)\cdot\beta\right) = \beta - \frac{1}{n}\left(i\cdot(\beta - i)\right).$$

This establishes the upper bound. Furthermore, if S is not the unique minimum independent dominating set of G, then there exists a vertex $w \notin S$ such that $i_w(G) = i$, and so

$$\begin{aligned} i_{av}(G) &\leq \tfrac{1}{n}\left((i+1)\cdot i + (n-i-1)\cdot\beta\right) \\ &= \beta - \tfrac{1}{n}((i+1)\cdot(\beta - i)) \\ &< \beta - \tfrac{1}{n}(i\cdot(\beta - i)). \end{aligned}$$

Moreover, if some vertex not in the S belongs to a maximal independent set of cardinality less than $\beta(G)$, then

$$\begin{aligned} i_{av}(G) &\leq \tfrac{1}{n}\left(i^2 + (\beta - 1) + (n-i-1)\cdot\beta\right) \\ &= \beta - \tfrac{1}{n}(i\cdot(\beta - i) + 1) \\ &< \beta - \tfrac{1}{n}(i\cdot(\beta - i)). \end{aligned}$$

Hence, if $i_{av}(G) = \beta - (i\cdot(\beta - i))/n$, then G must have a unique $i(G)$-set and each vertex not in the $i(G)$-set is such that every maximal independent set containing it has cardinality $\beta(G)$. On the other hand, if G has a unique $\gamma(G)$-set and for each vertex not in the $i(G)$-set, every maximal independent set containing it has cardinality $\beta(G)$, then it is straightforward to verify that $i_{av}(G) = \beta - (i\cdot(\beta - i))/n$. □

As an immediate consequence of Theorem 5, we can establish an upper bound on the average independent domination number of a tree in terms of its order.

Corollary 6 *If T is a tree of order $n \geq 2$, then*

$$i_{av}(T) \leq n - 2 + \frac{2}{n}.$$

Proof. Let T have independence number β. If $\beta \leq n-2$, then by Theorem 5, $i_{av}(T) \leq \beta \leq n-2 < n-2+2/n$. On the other hand, suppose $\beta > n-2$. Then, $T \cong K_{1,n-1}$ and $\beta = n-1$. If $n = 2$, then $i_{av}(T) = 1 = n-2+2/n$. Suppose that $n \geq 3$. Then $i(T) = 1$ and T has a unique $i(T)$-set which consist of the central vertex. Every leaf of T belongs to a unique maximal independent set consisting of the $\beta = n-1$ leaves of T. Hence, by Theorem 5, $i_{av}(T) = \beta - (i \cdot (\beta - i))/n = n-2+2/n$. Thus, if $\beta > n-2$, then $i_{av}(T) = i_{av}(K_{1,n-1}) = n-2+2/n$. □

4 A Characterization of (γ_{av}, i_{av})–Trees

Since $\gamma_v(G) \leq i_v(G)$ for every vertex v in a graph G, we observe that $\gamma_{av}(G) \leq i_{av}(G)$ for any graph G and this bound is sharp. For example, if G is a path or a cycle, then $\gamma_v(G) = i_v(G)$ for every $v \in V(G)$, and so $\gamma_{av}(G) = i_{av}(G)$.

We observe next that every (γ_{av}, i_{av})–tree is also a (γ, i)–tree. For suppose T is a tree satisfying $\gamma_{av}(T) = i_{av}(T)$. Then, $\gamma_v(T) = i_v(T)$ for every vertex v of T. Let v belong to a $\gamma(T)$-set. Then, $\gamma_v(T) = \gamma(T) \leq i(T) \leq i_v(T) = \gamma_v(T)$. Hence we must have equality throughout this inequality chain. In particular, $\gamma(T) = i(T)$.

Our aim is to provide a characterization of trees T satisfying $\gamma_{av}(T) = i_{av}(T)$ in terms of the set of vertices of the tree which are contained in all its γ-sets. For this purpose, we define the vertex subsets $\mathcal{A}(G)$, $\mathcal{A}_i(G)$, $\mathcal{N}(G)$ and $\mathcal{N}_i(G)$ of a graph G by

$$\begin{aligned}
\mathcal{A}(G) &= \{v \in V(G) \mid v \text{ is in every } \gamma(G)\text{–set}\},\\
\mathcal{A}_i(G) &= \{v \in V(G) \mid v \text{ is in every } i(G)\text{–set}\},\\
\mathcal{N}(G) &= \{v \in V(G) \mid v \text{ is in no } \gamma(G)\text{–set}\},\\
\mathcal{N}_i(G) &= \{v \in V(G) \mid v \text{ is in no } i(G)\text{–set}\}.
\end{aligned}$$

Using an ingenius tree pruning technique of Mynhardt [9], the set $\mathcal{A}(T)$ of a tree T can be found in complexity bounded by $O(n^2)$.

Let $V_2(T) = \{v \in V(T) \mid \deg v = 2\}$. We shall prove:

Theorem 7 *A tree T satisfies $\gamma_{av}(T) = i_{av}(T)$ if and only if $\mathcal{A}(T) \subseteq V_2(T)$.*

4.1 Proof of Theorem 7

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We define a *branch vertex* as a vertex of degree at least 3. The set of branch vertices of T is denoted by $B(T)$. A path P in T is said to be a u–L path, if P joins a vertex u to a leaf of T. The length of P is denoted by $\ell(P)$, and for $j = 0, 1, 2$, we define

$$C^j(v) = \{u \in C(v) \mid T_u \text{ contains a } u\text{–}L \text{ path } P \text{ with } \ell(P) \equiv_3 j\}.$$

We sometime write $C^j_T(v)$ to emphasize the tree (or subtree) concerned. We next describe a technique, introduced by Mynhardt in [9], called *tree pruning*. For any vertex u of a rooted tree T, denote the set of all u–L paths in T_u by $\Pi(u)$ (or $\Pi_{T_u}(u)$ if necessary). For $j = 0, 1, 2$, define

$$\Pi^j(u) = \{P \in \Pi(u) \mid \ell(P) \equiv_3 j\}.$$

The pruning of T is performed with respect to the root. Hence suppose T is rooted at v, i.e., $T = T_v$. Let u be a branch vertex at maximum distance from v; note that $|C(u)| \geq 2$ and $\deg x \leq 2$ for each $x \in D(u)$. For each $w \in C(u)$, allocate a *priority* to w or, equivalently, to the unique path $P \in \Pi(w)$, where $w^0 \in C^0(u)$ and $P^0 \in \Pi^0(u)$ have higher priority than $w^1 \in C^1(u)$ and $P^1 \in \Pi^1(u)$, which again have higher priority than $w^2 \in C^2(u)$ and $P^2 \in \Pi^2(u)$. Let z be a child of u of highest priority. For each $w \in C(u) - \{z\}$, delete $D[w]$. This step of the pruning process, where all but one child of u together with their descendants are deleted to give a tree in which u has degree 2, is called a *pruning of T_v at u*. Repeat the above process until a tree $\overline{T}_v$ is obtained with $\deg u \leq 2$ for each $u \in V(\overline{T}_v) - \{v\}$. Then $\overline{T}_v$ is called a *pruning* (here used as a noun) of T_v. The tree $\overline{T}_v$ need not be unique; however, if $\overline{T}_v$ and $\overline{T}'_v$ are two prunings of T_v, then $C^j_{\overline{T}_v}(v) = C^j_{\overline{T}'_v}(v)$ for each $j = 0, 1, 2$. Thus, to simplify notation, we write $\overline{C}^j(v)$ instead of $C^j_{\overline{T}_v}(v)$.

Using the technique of tree pruning, the following characterization of $\mathcal{A}(T)$ and $\mathcal{N}(T)$ were obtained by Mynhardt [9].

Theorem 8 (Mynhardt [9]) *For any tree T and any vertex v of T, $v \in \mathcal{A}(T)$ if and only if $|\overrightarrow{C}^0(v)| \geq 2$, and $v \in \mathcal{N}(T)$ if and only if $\overrightarrow{C}^0(v) = \emptyset$ and $\overrightarrow{C}^1(v) \neq \emptyset$.*

Mynhardt [9] also showed that if a tree T has no vertices which appear in all $\gamma(T)$-sets, then it also has no vertices which appear in all $i(T)$-sets. However, in this case $\mathcal{N}(T)$ and $\mathcal{N}_i(T)$, although also equal, may be nonempty.

Theorem 9 (Mynhardt [9]) *If $\mathcal{A}(T) = \emptyset$, then $\mathcal{A}_i(T) = \emptyset$ and $\mathcal{N}(T) = \mathcal{N}_i(T)$.*

The proof of the following result is identical to the proofs used in [9] to prove Theorem 8 and 9, and is therefore omitted.

Theorem 10 *If $\mathcal{A}(T) = \emptyset$ or if each vertex of $\mathcal{A}(T)$ has degree 2 in T, then $\mathcal{N}(T) = \mathcal{N}_i(T)$.*

In order to prove Theorem 7, we shall also need some results of Cockayne et al. in [3]. Let v be a vertex in a rooted tree T, and let $x \in N(v)$. For notational convenience, we may assume $x \in C(v)$. Cockayne et al. [3] defined x to be *v–noble* if there exists a $\gamma(T_x)$-set S such that $\mathrm{pn(s,S)} = \{\mathrm{x}\}$ for some $s \in S$, and they defined x to be *v–grand* if there exists an $i(T_x)$-set I such that $x \in I$ and $\mathrm{pn(x,I)} = \{\mathrm{x}\}$. Cockayne et al. [3] provided the following characterizations of v–noble vertices and v–grand vertices.

Theorem 11 (Cockayne et al. [3]) *Let v be a vertex of a rooted tree T and let $x \in C(v)$. Then*
(1) *x is v–noble if and only if for each $y \in N(x) - \{v\}$, $y \in N(T_y)$, and*
(2) *x is v–grand if and only if for each $y \in N(x) - \{v\}$, $y \in N_i(T_y)$.*

Cockayne et al. [3] also characterized the set $\mathcal{A}(T)$ in terms of v–noble vertices.

Theorem 12 (Cockayne et al. [3]) *A vertex v of a tree T is in $\mathcal{A}(T)$ if and if $N(v)$ contains at least two v–noble vertices.*

This characterization of $\mathcal{A}(T)$ corresponds with the characterization in Theorem 8. Cockayne et al. [3] also showed that if $\mathcal{A}(T) = \emptyset$ and if T contains both a v–noble vertex and a v–grand vertex, then every v–grand vertex is also a v–noble vertex. In order to extend this result, we shall need the following result of Cockayne et al. [3].

Lemma 13 (Cockayne et al. [3]) *Let v be a vertex of a rooted tree T such that $C(v)$ contains a v–noble vertex x and a v–grand vertex z. Then*

(a) *For $y \in C(z)$, if $y \notin \mathcal{N}(T_y)$ and $\mathcal{A}(T_y) \nsubseteq \mathcal{A}(T)$, then $y \in \overline{C}^1(z)$.*

(b) *If $\overline{C}^0(z) = \emptyset$, then $\mathcal{A}(T_y) \subseteq \mathcal{A}(T)$ for each $y \in C(z)$ such that $y \notin \mathcal{N}(T_y)$.*

Lemma 14 *Let v be a vertex of a rooted tree T such that $C(v)$ contains a v–noble vertex and a v–grand vertex z. If $\mathcal{A}(T) = \emptyset$ or if each vertex of $\mathcal{A}(T)$ has degree 2 in T, then z is also a v–noble vertex.*

Proof. Suppose $\mathcal{A}(T) = \emptyset$ or each vertex of $\mathcal{A}(T)$ has degree 2 in T. Let $y \in C(z)$. If $y \in \mathcal{N}(T_y)$, then by Theorem 8 applied to T_y, $\overline{C}^0(y) = \emptyset$ and $\overline{C}^1(y) \neq \emptyset$. Hence in T, $y \in \overline{C}^2(z)$. Suppose that $y \notin \mathcal{N}(T_y)$. If $y \in \overline{C}^0(z) \cup \overline{C}^2(z)$, then, by Lemma 13($a$), $\mathcal{A}(T_y) \subseteq \mathcal{A}(T)$. Hence, $\mathcal{A}(T_y) = \emptyset$ or each vertex of $\mathcal{A}(T_y)$ has degree 2 in T_y. Thus, by Theorem 10, $\mathcal{N}(T_y) = \mathcal{N}_i(T_y)$, and so $y \notin \mathcal{N}_i(T_y)$. This, however, contradicts Theorem 11 since z is v–grand. Hence, $y \in \overline{C}^1(z)$. In particular, we notice that $\overline{C}^0(z) = \emptyset$. Thus applying Lemma 13(b), $\mathcal{A}(T_y) \subseteq \mathcal{A}(T)$. Hence, $\mathcal{A}(T_y) = \emptyset$ or each vertex of $\mathcal{A}(T_y)$ has degree 2 in T_y. By Theorem 10, $\mathcal{N}(T_y) = \mathcal{N}_i(T_y)$ and we again contradict the fact that z is v–grand. We deduce, therefore, that $y \in \mathcal{N}(T_y)$ for each $y \in C(z)$. Thus, by Theorem 11, z is v–noble. □

We are now in a position to prove Theorem 7.

4.1.1 Proof of Necessity

Lemma 15 *Let T be a tree rooted at a vertex v where $v \in \mathcal{A}(T)$ and* $\deg_T v \geq 3$. *Then, $\gamma_{av}(T) < i_{av}(T)$.*

Proof. By Theorem 12, $N(v)$ contains at least two v–noble vertices. Let v_1 and v_2 be two v–noble vertices. Let $x \in N(v) - \{v_1, v_2\}$ (note that the vertex x exists since $\deg v \geq 3$). Let $T' = T - V(T_x)$; that is, T' is the component of $T - vx$ containing v. Then, by Theorem 11, each of v_1 and v_2 is a v–noble vertex in T'. Thus, by Theorem 12, $v \in \mathcal{A}(T')$. Let X' be a $\gamma(T')$-set (and so, $v \in X'$) and let X be a $\gamma_x(T)$-set. Let $X_x = X \cap D[x]$.

Claim 1 $v \in X$.

Proof. Suppose, to the contrary, that $v \notin X$. Since $v \in \mathcal{A}(T')$, $X - X_x$ cannot be a $\gamma(T')$-set. Thus, if $X - X_x$ is a dominating set of T', then

$|X'| = \gamma(T') < |X - X_x|$. But then $X' \cup X_x$ is a dominating set of T containing x with $|X' \cup X_x| < |X| = \gamma_x(T)$, which is impossible. Hence, $X - X_x$ cannot be a dominating set of T'. However, X is a dominating set of T, and so $X - X_x$ must be a dominating set of $T' - v$. Let $X_1 = X \cap D[v_1]$. Since $v \notin X$, X_1 is a dominating set of T_{v_1}. Since v_1 is a v–noble vertex, there exists a $\gamma(T_{v_1})$-set S such that pn(s, S) $=$ {v$_1$} for some $s \in S$. Without loss of generality, we may assume that $s = v_1$, for otherwise we consider $(S - \{s\}) \cup \{v_1\}$. Since S is a $\gamma(T_{v_1})$-set, $|S| \le |X_1|$. Thus, $X^* = (X - X_x - X_1) \cup S$ is a dominating set of T' that does not contain v. Since $v \notin X^*$, X^* cannot be a $\gamma(T')$-set. Hence, $|X'| = \gamma(T') < |X^*| = |X| - |X_x| - |X_1| + |S| \le |X| - |X_x|$. But then $X' \cup X_x$ is a dominating set of T containing x with $|X' \cup X_x| < |X| = \gamma_x(T)$, which is impossible. Hence, $v \in X$. □

By Claim 1, every $\gamma_x(T)$-set must contain both v and x and is therefore not independent. Consequently, $\gamma_x(T) < i_x(T)$. This implies that $\gamma_{av}(T) < i_{av}(T)$. □

Note that, by Theorem 12, no leaf of a tree T belongs to $\mathcal{A}(T)$. As an immediate consequence of Lemma 15, we have that if T is a tree satisfying $\gamma_{av}(T) = i_{av}(T)$, then $\mathcal{A}(T) = \emptyset$ or each vertex of $\mathcal{A}(T)$ has degree 2 in T. This establishes the necessity.

4.1.2 Proof of Sufficiency

Before proceeding with a proof of the sufficiency of Theorem 7, we introduce some additional notation. If X is a dominating set in T, then we write $X = X_Y \cup X_Z$, where $X_Z = \{x \in X \mid x$ is isolated in the subgraph induced by $X\}$ and $X_Y = X - Z_Y$. (Possibly, $X_Y = \emptyset$, in which case X is an independent dominating set of T.) We are now in a position to prove our sufficient condition for a tree T to satisfy $\gamma_{av}(T) = i_{av}(T)$.

Suppose that T is a tree such that $\mathcal{A}(T) = \emptyset$ or each vertex of $\mathcal{A}(T)$ has degree 2 in T but $\gamma_{av}(T) < i_{av}(T)$. Then there exists $x \in V(T)$ such that $\gamma_x(T) < i_x(T)$. Among all $\gamma_x(T)$-sets, let X be one such that X_Y is a minimum. Since $\gamma_x(T) < i_x(T)$, we know that $X_Y \neq \emptyset$, i.e, $|X_Y| \ge 2$.

Claim 2 *Each vertex of $X_Y - \{x\}$ has degree at least* 3.

Proof. Suppose $v \in X_Y - \{x\}$. If v is a leaf, then $X - \{v\}$ is a dominating set of T containing x of cardinality $\gamma_x(T) - 1$, which is impossible. Hence,

$\deg_T v \geq 2$. Suppose $\deg_T v = 2$. Let u be the vertex of X_Y adjacent to v (possibly, $u = x$), and let $N(v) = \{u, w\}$. If $\text{pn}(v, X) = \emptyset$, then $X - \{v\}$ is a dominating set containing x, a contradiction. Hence, $\text{pn}(v, X) = \{w\}$, and so $X^* = (X - \{v\}) \cup \{w\}$ is a $\gamma_x(T)$-set with $|X^*_Y| < |X_Y|$, a contradiction. Thus, $\deg_T v \geq 3$. □

We now return to the proof of the sufficiency. Consider T to be rooted at x, and let v be a vertex of X_Y furthest from x. Since $|X_Y| \geq 2$, $v \neq x$. By our choice of v, the parent of v (which may possibly be x) belongs to X. By Claim 2, $\deg_T v \geq 3$, and so, by hypothesis, $v \notin \mathcal{A}(T)$. The proofs of the following two claims are similar to those found in [3], but we include them for completeness. For each vertex y of T, let $X_y = X \cap D[y]$.

Claim 3 *There exists a v-noble vertex $w \in \text{pn}(v, X)$.*

Proof. Suppose, to the contrary, that no vertex in $\text{pn}(v, X)$ is v-noble. Then, by Theorem 11, each $z \in \text{pn}(v, X)$ has a child z' that belongs to some $\gamma(T_{z'})$-set, say S_z. Since $z \notin X$, $X_{z'}$ is a dominating set of $T_{z'}$, and so $|X_{z'}| \geq \gamma(T_{z'}) = |S_z|$. Let

$$X^* = \left(X - \{v\} - \bigcup_{z \in \text{pn}(v,X)} X_{z'} \right) \cup \left(\bigcup_{z \in \text{pn}(v,X)} S_z \right).$$

Then, X^* is a dominating set of T containing the vertex x with $|X^*| \leq |X| - 1 = \gamma_x(T) - 1$, which is impossible. Hence, at least one vertex in $\text{pn}(v, X)$ is v-noble. □

Claim 4 *There exists a v-grand vertex $z \in \text{pn}(v, X) - \{w\}$.*

Proof. Suppose, to the contrary, that no vertex in $\text{pn}(v, X) - \{w\}$ is v-grand. Then, by Theorem 11, each $z \in \text{pn}(v, X)$ has a child z' that belongs to some $i(T_{z'})$-set, say I_z. Since $z \notin X$, $X_{z'}$ is a dominating set of $T_{z'}$. By our choice of v, the set $X_{z'}$ is independent and is therefore an independent dominating set of $T_{z'}$. Thus, $|X_{z'}| \geq i(T_{z'}) = |I_z|$. Let

$$X^* = \left(X - \{v\} - \bigcup_{z \in \text{pn}(v,X)} X_{z'} \right) \cup \{w\} \cup \left(\bigcup_{z \in \text{pn}(v,X)} I_z \right).$$

Then, X^* is a dominating set of T containing the vertex x with $|X^*| \leq |X| = \gamma_x(T)$. Consequently, X^* is a $\gamma_x(T)$-set. However, since $w \in \text{pn}(v, X)$ and $v \notin X^*$, the vertex w is isolated in X^*. Hence, by construction of the set X^*, $|X_Y^*| < |X_Y|$, which contradicts our choice of the $\gamma_x(T)$-set X. Hence, at least one vertex in $\text{pn}(v, X) - \{w\}$ is v-grand. □

By Claims 3 and 4, there exists a v-noble vertex $w \in \text{pn}(v, X)$ and a v-grand vertex $z \in \text{pn}(v, X) - \{w\}$. By Lemma 14, z is also a v–noble vertex. But then v has two v-noble vertices, namely w and z, and so, by Theorem 12, $v \in \mathcal{A}(T)$. This, however, contradicts the hypothesis that each vertex of $\mathcal{A}(T)$ has degree 2 in T. Hence if T is a tree such that $\mathcal{A}(T) = \emptyset$ or each vertex of $\mathcal{A}(T)$ has degree 2 in T, then $\gamma_{av}(T) = i_{av}(T)$. This establishes the sufficiency.

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