Packing Balanced Complete Multipartite Graphs with Hexagons*

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Abstract

In this paper, we completely solve the problem of finding a maximum packing of any balanced complete multipartite graph $K_{m(n)}$ with edge-disjoint 6-cycles, and minimum leaves are explicitly given. Subsequently, we also find a minimum covering of $K_{m(n)}$.

1 Introduction and Preliminaries

A *k-cycle packing* of a graph G is a set of edge-disjoint k-cycles in G. A k-cycle packing C is *maximum* if $|C| \geq |C'|$ for all other k-cycle packings C' of G. The *leave* L of a packing C is the subgraph induced by the set of edges of G that does not occur in any k-cycle of the packing C. The leave L of a maximum packing is referred to as a minimum leave, a leave with minimum number of edges. A packing with empty leave is known as a

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k-cycle system of G.(In terms of graph decomposition, we say $C_k|G$.) And a k-cycle system of K_v is referred to as a k-cycle system of order v.

Clearly, if $C_k|K_v$ then v is odd and $k|\binom{v}{2}$. To determine whether the above necessary condition is also sufficient is commonly referred to as the existence problem of k-cycle systems.

The existence problem for *k-cycle systems* of order v has been studied for more than 35 years. Recently, it has been completely solved by Alspach et al., see [1,2,11]. But, the packing of K_v with k-cycles is not that lucky: only partial results have been obtained so far; see [9]. Mainly, $k \in \{3,4,5,6\}$ is considered.

If we turn to the packing of G where G is a complete multipartite graph, then the problem is getting more difficult. Even in the case $k = 3$, the existence problem is still unsolved; see [8]. Recently, Billington, Fu and Rodger completely solved the case $k = 4$, see [3,4]. And the cases other than $k = 4$ remain unsettled.

In this paper, we consider the packing and covering of a balanced complete multipartite graph $K_{m(n)}$ (m parts of size n) with hexagons and we are able to obtain a maximum packing and minimum covering of $K_{m(n)}$.

The result of Sotteau deserves mention first.

Theorem 1.1. *[12] The complete bipartite graph $K_{m,n}$ can be decomposed into $2k$-cycles if and only if (i) $m, n \geq k$, (ii) m and n are even, and (iii) $2k|mn$.*

Now, consider the packing of K_v with hexagons. The following result was obtained by Kennedy.

Theorem 1.2. *[8] The minimum leaves of the maximum packings of K_v with hexagons are as follows: v is considered to be the number modulo* 12.

v	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
L	F	$\emptyset$	F	C_3	F_4	C_4	F	C_3	F	$\emptyset$	F_4	E_7

F is a 1-factor, C_i is a cycle of length i, F_4 is any odd graph with $v/2+4$ edges and E_7 is any simple even graph with 7 edges.

The following terminology was introduced by Billington and Cavenagh. A graph G is said to be k-sufficient if (i) each vertex in G has even degree and (ii) $k||E(G)|$. Then they proved:

Theorem 1.3. *[6] All 6-sufficient complete multipartite graphs are decomposable into 6-cycles.*

A packing C of G is called maximal if $G - E(C)$ contains no 6-cycles; here $E(C)$ denotes the set of edges in the 6-cycles of C. Note that a maximal packing C' may not have a minimum leave. And by Theorem 1.3, we shall consider only the balanced complete multipartite graphs which are not 6-sufficient in the next section.

2 The maximum packing of $K_{m(n)}$

First, we consider the packing of $K_{n,n}$. The following lemmas are essential for the proof of the main theorem. If the proofs are direct, we omit the details.

Lemma 2.1. *Let $n \equiv 1$ or 3 (mod 6) and $n \geq 3$. Then $K_{n,n} - F$ can be decomposed into 6-cycles where F is a 1-factor of $K_{n,n}$.*

Proof. Let (S,t) be a Steiner triple system (3-cycle system) of order n defined on Z_n. Let the two partite sets of $K_{n,n}$ be $Z_n \times \{0\}$ and $Z_n \times \{1\}$.

Now, for each triple $\{i, j, k\} \in t$, let $((i,0),(j,1),(k,0),(i,1),(j,0),(k,1))$ be a 6-cycle in the packing of $K_{n,n}$, then F is exactly the set $\{\{(i,0),(i,1)\} \mid i \in Z_n\}$. □

Note that the above construction is well-known, we include it here for completeness.

Lemma 2.2. *$K_{4,4}$ can be packed with two 6-cycles with leave a 4-cycle.*

Lemma 2.3. *$K_{5,5}$ can be packed with three 6-cycles with leave a disjoint union of $K_{1,3}$, $K_{1,3}$ and an edge.*

For convenience, we denote an odd graph of order v with $v/2+2$ edges by F_2. Therefore, the leave obtained in $K_{5,5}$ is in fact an F_2.

Lemma 2.4. *$K_{8,8}$ can be packed with ten 6-cycles with leave a 4-cycle.*

Lemma 2.5. *Let $n \equiv 2$ or 4 $(mod\ 6)$ and $n \geq 4$. Then $K_{n,n} - C_4$ can be decomposed into 6-cycles.*

Proof. Let $n = 6k+2$ or $6k+4$, $k \geq 1$. It is not difficult to see that $K_{6k+2,6k+2}$ can be decomposed into the edge disjoint union of $K_{8,8}$, $K_{8,6k-6}$ and $K_{6k-6,6k-6}$; also $K_{6k+4,6k+4}$ can be decomposed into the edge disjoint union of $K_{4,4}$, $K_{4,6k}$ and $K_{6k,6k}$. Therefore, the proof follows by Lemma 2.4, 2.2, and using Sotteau's Theorem. □

Lemma 2.6. *Let $n \equiv 5$ $(mod\ 6)$, then $K_{n,n} - F_2$ can be decomposed into 6-cycles.*

Proof. Since $n \equiv 5 (mod\ 6)$, there exists a pairwise balanced design (PBD) of order n with one block of size 5 and the rest all of size 3 [10]. In fact, we can use the one with exactly one block of size 5. For convenience, let the PBD be defined on Z_n and the two partite sets of $K_{n,n}$ be $Z_n \times \{0\}$ and $Z_n \times \{1\}$. Let $\{1,2,3,4,5\} = B_1$ be the only block of size 5. Then the induced subgraph of $K_{n,n}$ with vertex set $S = \{(1,0),(2,0),(3,0),(4,0),(5,0),(1,1),(2,1),(3,1),(4,1),(5,1)\}$ is isomorphic to $K_{5,5}$. By Lemma 2.3, $K_{5,5}$ can be packed with 6-cycles with leave an F_2 and for each triple $\{i,j,k\} \in B_i$, $i \neq 1$, let $((i,0),(j,1),(k,0),(i,1),(j,0),(k,1))$ be a 6-cycle in the packing of $K_{n,n} \setminus K_{5,5}$. Then the proof follows. □

Lemma 2.7. *[6] Let $C_{6(n)}$ denote the graph with vertex set $Z_n \times Z_6$ and with edge set $E(C_{6(n)})$ where $\{(i_1,j_1),(i_2,j_2)\} \in E(C_{6(n)})$ if and only if $j_2 \equiv j_1 + 1 \pmod 6$. Then $C_{6(n)}$ has a decomposition into 6-cycles with empty leave.*

Lemma 2.8. *If $n \equiv 1 \pmod 2$ and $n \geq 3$, $K_{n,n,n} - C_3$ can be decomposed into 6-cycles.*

Proof. First, consider the maximum packing of $K_{3,3,3}$. Let the three partite sets of $K_{3,3,3}$ be $(Z_2 \cup \{\infty\}) \times \{i\}$, $i \in Z_3$. Then $K_{3,3,3} - C_3$ can be decomposed into 6-cycles by $((0,0),(0,1),(1,2),(1,1),(1,0),(0,2))$, $((0,0),(1,1),(0,2),(0,1),(1,0),(1,2))$, $((\infty,0),(1,1),(\infty,2),(1,0),(\infty,1),(1,2))$, $((\infty,0),(0,1),(\infty,2),(0,0),(\infty,1),(0,2))$. Now, let $n = 2t+1$. Let the three partite sets of $K_{n,n,n}$ be $(Z_{2t} \cup \{\infty\}) \times \{i\}$, $i \in Z_3$. Let $M = [m_{i,j}]$ be an idempotent Latin square of order t. If $i \neq j$, for all $1 \leqslant i,\ j \leqslant t$, $K_{2,2,2}$ with vertex set $\{(2i-1,0),(2i,0)\} \cup \{(2j-1,1),(2j,1)\} \cup \{(2m_{i,j}-1,2),(2m_{i,j},2)\}$ is decomposable into 6-cycle system (by Theorem 1.3). If $i = j$, for all $1 \leqslant i \leqslant t$, $K_{3,3,3}$ with vertex set $\{(\infty,0),(2i-1,0),(2i,0)\} \cup \{(\infty,1),(2i-1,1),(2i,1)\} \cup \{(\infty,2),(2m_{i,i}-1,2),(2m_{i,i},2)\}$ can be decomposed into 6-cycles with leave $\{(\infty,0),(\infty,1),(\infty,2)\}$. So, if $n = 2t+1$,

$K_{n,n,n}$ can be packed with 6-cycles having leave a C_3. □

Lemma 2.9. *If n is odd, then $K_{4(n)}$ can be packed with 6-cycles with leave (i)F, if $n \equiv 3 \pmod 6$ and (ii)F_4, if $n \equiv 1$ or $5 \pmod 6$.*

Proof. Let A, B, C, D be the vertex sets of four partite sets of $K_{4(n)}$ i.e. $|A| = |B| = |C| = |D| = n$. Then $|V(A \cup B)| = |V(C \cup D)| = 2n$. First, we consider the bipartite graph $K_{2n,2n}$ with two partite sets as $(A \cup B)$ and $(C \cup D)$. Second we consider the two bipartite graphs $K_{n,n}$, with two partite sets A, B and C, D respectively.
(i) If $n \equiv 1 \pmod 6$, $K_{2n,2n} = K_{12t+2,12t+2}$ can be packed with 6-cycles with leave a 4-cycle (by Lemma 2.5). $K_{n,n}$ can be packed with 6-cycles with leave a 1-factor F (by Lemma 2.1). Combining C_4 and F gives the graph F_4, so we have packed the complete multipartite graph $K_{4(6t+1)}$ with 6-cycles, having leave F_4.
(ii) If $n \equiv 3 \pmod 6$, $K_{2n,2n} = K_{12t+6,12t+6}$ is a 6-cycle system (by Theorem 1.1). $K_{6k+3,6k+3}$ can be packed with 6-cycles with leave 1-factor F (by Lemma 2.1). So we have packed the complete multipartite graph $K_{4(6t+3)}$ with 6-cycles with leave F.
(iii) If $n \equiv 5 \pmod 6$, $K_{2n,2n} = K_{12t+10,12t+10}$ can be packed with 6-cycles with leave a 4-cycle (by Lemma 2.5). $K_{n,n}$ can be packed with 6-cycles with leave F_2 (by Lemma 2.6). Combining C_4 and $2F_2$ as the following graphs, we conclude that the complete multipartite graph $K_{4(6t+5)}$ can be packed with 6-cycles with leave F_2.

□

Definition 2.1. Let $\{a_1,a_2,a_3\}$ be the vertex set of 3-cycle C_3^a, and $\{b_1,b_2,b_3\}$ be the vertex set of 3-cycle C_3^b. If $(a_1,b_3,a_2,b_2,a_3,b_1)$ is a 6-cycle, we define

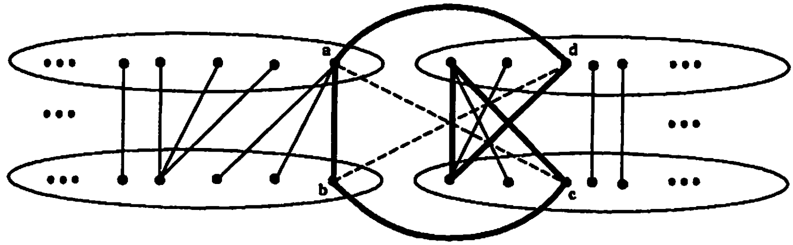

Figure 1: Combination of $C_4 : (a, b, c, d)$ and two F_2

a product: 6-cycle product for $C_3^a \cup C_3^b$ as the following action : pack the two 6-cycles from $C_3^a \cup C_3^b$ as $(a_1, b_3, b_2, a_2, a_3, b_1)$ and $(a_1, a_2, b_3, b_1, b_2, a_3)$. (We denote it as $C_3^a \triangle C_3^b$.)

For convenience in packing, we denote the induced subgraph of G as $G[V]$, where V is subset of vertex set $V(G)$ of G, and we also denote the edge set between A and B as $G[A, B]$, where A and B are subsets of $V(G)$.

Lemma 2.10. *$K_{6(2t+1)} - F$ is a 6-cycle system.*

Proof. Let $Z_n \times Z_6$ be the vertex set of $K_{6(n)}$. We decompose the graph $K_{6(n)}$ into complete bipartite graphs with vertex sets B_1 and B_2; let the vertex set of B_1 be $Z_6 \times I_1$, and the vertex set of B_2 be $Z_6 \times I_2$, where $I_1 = \{0, 1, 2\}$, $I_2 = \{3, 4, 5\}$. Then $G[B_1]$, $G[B_2]$ are isomorphic to $K_{n,n,n}$ and $G[B_1, B_2]$ is isomorphic to $K_{n,n}$. Because $n = 2t + 1$, by Theorem 1.1 and Lemma 2.8, $G[B_1]$, $G[B_2]$ have leave two C_3s. By 6-cycle product, then $K_{6(2t+1)}$ can be packed with 6-cycles with leave F. □

Lemma 2.7 provides us with a good idea. In the following discussion, we view each part of $K_{m(n)}$ as a point, and denote the new graph as K'_m and the leave as L' of K'_m.

Lemma 2.11. *$K_{m(n)} - F$ can be decomposed into 6-cycles when $m \equiv 0$ or 2 (mod 6) and $n \equiv 1$ or 3 (mod 6).*

Proof. The lemma follows because K'_m can be packed with 6-cycles which has a leave F' when $m = 6k$ or $6k+2$. For each edge in F' corresponds to a complete bipartite graph $K_{n,n}$ in $K_{m(n)}$. By Lemma 2.1, $K_{n,n}$ has a leave F when $n = 6t+1$ or $6t+3$. So $K_{m(n)} - F$ can be decomposed into 6-cycles when when $m = 6k$ or $m = 6k+2$ and $n = 6t+1$ or $6t+3$. □

Lemma 2.12. *If $m \equiv 3$ or 7 (mod 12), then $K_{m(n)}$ is a 6-cycle system when n is even, $K_{m(n)} - C_3$ is a 6-cycle system when n is odd.*

Proof. If $m \equiv 3$ or 7 (*mod* 12), $K'_m - C'_3$ is a 6-cycle system. Here C'_3 means $C_{3(n)}$ in $K_{m(n)}$. By Theorem 1.3, $C_{3(n)}$ is a 6-cycle system when n is even; by Lemma 2.8 $C_{3(n)}$ can be packed with 6-cycles with leave a C_3 when n is odd. □

Lemma 2.13. *If $m \equiv 4$ (mod 6), then $K_{m(n)}$ can be packed with 6-cycles with the following leaves. (i)$\emptyset$, when n is even; (ii)F_4, when $n \equiv 1$ (mod 6); (iii)F, when $n \equiv 3$ (mod 6), and(iv)F_2, when $n \equiv 5$ (mod 6).*

Proof. We partition K'_m into $k+1$ partite sets $\{A_i\}_{i=1}^{k+1}$, such that $|A_i| = 6$ for i= 1, 2, $\cdots$ k, $|A_{k+1}| = 4$. Then $G[A_i]$ is isomorphic to $K_{6(n)}$ for i= 1, 2, $\cdots$ k, and $G[A_{k+1}]$ is isomorphic to $K_{4(n)}$. By Theorem 1.1, $G[A_i, A_j]$ is a 6-cycle system. We only consider $K_{6(n)}$ and $K_{4(n)}$. By Lemmas 2.9 and 2.10, we have proved this lemma. □

Definition 2.2. Let $Z_3 \times Z_4$ be the vertex sets of three disjoint 4-cycles. If there exists a 6-cycle C_6:((0,0),(1,2),(2,0),(0,2),(1,0), (2,2)), we define a product for these three 4-cycles and C_6 as the following action: pick three 6-cycles as ((0,0),(0,1),(0,2),(1,2),(1,3),(1,0)), ((1,0),(1,1),(1,2),(2,2),(2,1), (2,0)), ((2,0),(2,3),(2,2),(0,0),(0,3),(0,2)). (Denote the product by $3C_4 \triangle C_6$.)

Lemma 2.14. *If $m \equiv 2$ (mod 6), $K_{m(n)}$ can be packed with 6-cycles which has leave (i)C_4, if $n \equiv 2$ or 4 (mod 6); and (ii)F_2, if $n \equiv 5$ (mod 6).*

Proof. Let $m = 6k + 2$. We decompose $K_{m(n)}$ into k partite sets such that $k - 1$ partite sets are isomorphic to $K_{6(n)}$, and the k-th partite set is isomorphic to $K_{8(n)}$.

(i)If $n \equiv 2$ or 4(*mod* 6), then $K_{6(n)}$ is a 6-cycle system, and K' can be packed with 6-cycles with leave F'. To each edge of F', there corresponds a complete bipartite graph $K_{n,n}$ in $K_{8(n)}$. By Lemma 2.5, $K_{n,n}$ can be packed with 6-cycles with leave a C_4. So $K_{m(n)}$ can be packed with 6-cycles which with four C_4. We pack $K_{m(n)} - 4C_4$ with 6-cycles and using $3C_4 \triangle C_6$, we see that $K_{m(n)}$ can be packed with 6-cycles with leave a 4-cycle.

(ii) If $n \equiv 5$ (*mod* 6), then $K_{6(n)}$ can be packed with 6-cycles with leave a 1-factor F. We just consider $K_{8(n)}$. Let the eight partite sets of $K_{8(n)}$ be $\{P_i\}_{i=1}^{8}$. Then $G[P_1,P_2,P_3,P_4] = G[P_5,P_6,P_7,P_8] = K_{4(n)}$ can be packed with 6-cycles with leave F_2. $G[P_1 \cup P_2 \cup P_3 \cup P_4, P_5 \cup P_6 \cup P_7 \cup P_8]$ is isomorphic to $K_{6(4t+3)+2,6(4t+3)+2}$. By Lemma 2.5, $K_{6(4t+3)+2,6(4t+3)+2}$ can be packed with 6-cycles with leave a C_4. Combining $2F_2$ and C_4 as Figure 1 then $K_{8(n)}$ has a packing with leave F_2. So when $n \equiv 5$ (*mod* 6), $m \equiv 2$ (*mod* 6), $K_{m(n)}$ can be packed with 6-cycles with leave F_2. □

Lemma 2.15. *If $m \equiv 5$ (mod 6), $n \equiv 2$ or 4 (mod 6), then $K_{m(n)}$ can be packed with 6-cycles with leave a C_4.*

Proof. If $m \equiv 5$ (*mod* 6), $n \equiv 2$ (*mod* 6), decompose $K_{m(n)}$ into a bipartite graph G with two partite sets A, B such that $A = K_{m_1(n)}$ and $B = K_{m_2(n)}$, where $m_1 = 2$, and $m_2 = 6k+3$. Then $|V(A)| = 2(6t+2) = 6(2t)+4$ and $|V(B)| = (6k+3)(6t+2) = 6(6kt+3t+2k+1)$. The bipartite graph G is isomorphic to $K_{6r_1+4,6r_2}$, where $r_1 = 2t$, and $r_2 = 6kt+3t+2k+1$. By Theorem 1.1, G is a 6-cycle system. We consider the sets A and B. The graph induced by A, $G[A]$ is isomorphic to $K_{(6t+2),(6t+2)}$ which can be packed with 6-cycles with leave a 4-cycle. Similarly $G[B]$ is isomorphic to $K_{m_3(n)}$ where $m_3 = 6k+3$ and $n = 6t+2$ which is a 6-cycle system.(By Lemma 2.5, 2.15 and 2.16). So, when $m \equiv 5$ (*mod* 6) and $n \equiv 2$ (*mod* 6), $K_{m(n)}$ can be packed with 6-cycles having leave a 4-cycle. The proof of the case $n \equiv 4$ (*mod* 6) is similar. □

Lemma 2.16. *If $n \equiv 3$ (mod 6), then $K_{m(n)}$ can be packed with 6-cycles with leave a C_3 when $m \equiv 5$ (mod 12) or a 6-cycle system when $m \equiv 11$ (mod 12). If $n \equiv 1$ or 5 (mod 6), then $K_{m(n)}$ can be packed with 6-cycles having leave a C_4 when $m \equiv 5$ (mod 12) or having leave E_7 when $m \equiv 11$ (mod 12).*

Proof. First, K'_m can be packed with 6-cycles, with leave a C'_4 when $m \equiv 5$ (*mod* 12) and with leave E'_7 when $m \equiv 11$ (*mod* 12). If $m \equiv 5$ (*mod* 12), the graph corresponding to C'_4 in $K_{m(n)}$ is $C_{4(n)}$. Let A_1, A_2, A_3 and A_4 be the four partite sets of $C_{4(n)}$; then $G[A_i, A_{i+1}]$ is a complete bipartite graph $K_{n,n}$. If $n \equiv 1$ or 3 (*mod* 6), $K_{n,n}$ can be packed with 6-cycles, with a 1-factor leave. By packing $C_{4(n)}$ suitably with 6-cycles, then $C_{4(n)}$ can be packed with 6-cycles with n parallel C_4s. For every three 4-cycles (with $Z_3 \times Z_4$ be the vertex sets of three 4-cycles.), we pack $K_{m(n)} - C_{4(n)}$ with 6-cycles such that ((0,0),(1,2),(2,0),(0,2),(1,0),(2,2)) is a 6-cycle C'' in this 6-cycle packing. Then by $3C_4 \bigtriangleup C_6$, $3C_4 \cup C''$ is a 6-cycle system. So, when $m \equiv 5$ (*mod* 12) and $n = 6t+1$, then $K_{m(n)}$ can be packed with

6-cycles with leave a 4-cycle. If $n = 6t + 3$, then $K_{m(n)}$ is a 6-cycle system when $m \equiv 5$ (*mod* 12). If $n = 6t+5$, $G[A_i, A_{i+1}] = K_{6t+5,6t+5}$. By Lemma 2.6, $K_{6t+5,6t+5}$ can be packed with 6-cycles, with leave F_2. By combining four F_2 (see Figure 2), we know that $C_{4(6k+5)}$ can be packed with four 6-cycles and $6t + 1$ parallel C_4s.

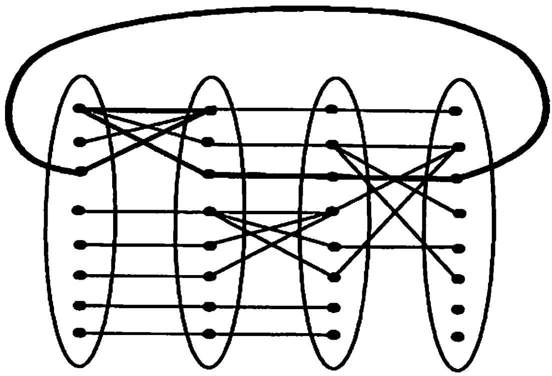

Figure 2: Combine $4F_2$ into 6-cycles.

For $6t+1$ parallel C_4, similar to the above discussion. $K_{m(n)}$ can be packed with 6-cycles with leave a C_4.

If $m \equiv 11$ (*mod* 12) then $K'_{m(n)}$ can be packed with 6-cycles with leave E_7(by definition of E_7, we use $C_3 \cup C_4$ for E_7). Then by the above discussion and Lemma 2.7, $K_{m(n)}$ can be packed with 6-cycles with leave E_7 when $n \equiv 1$ or 5 (*mod* 6) or C_3 when $n \equiv 3$ (*mod* 6) respectively. □

Combining the above lemmas, we are able to prove:

Theorem 2.17. *The minimum leaves of the maximum packings of $K_{m(n)}$ with hexagons are the follows:*

m/n	$6t$	$6t+1$	$6t+2$	$6t+3$	$6t+4$	$6t+5$
$6k$	$\emptyset$	F	$\emptyset$	F	$\emptyset$	F
$6k+1$	$\emptyset$	C_3 (*k is odd*)	$\emptyset$	C_3 (*k is odd*)	$\emptyset$	C_3 (*k is odd*)
		$\emptyset$ (*k is even*)		$\emptyset$ (*k is even*)		$\emptyset$ (*k is even*)
$6k+2$	$\emptyset$	F	C_4	F	C_4	F_2
$6k+3$	$\emptyset$	$\emptyset$ (*k is odd*)	$\emptyset$	$\emptyset$ (*k is odd*)	$\emptyset$	$\emptyset$ (*k is odd*)
		C_3 (*k is even*)		C_3 (*k is even*)		C_3 (*k is even*)
$6k+4$	$\emptyset$	F_4	$\emptyset$	F	$\emptyset$	F_2
$6k+5$	$\emptyset$	E_7 (*k is odd*)	C_4	C_3 (*k is odd*)	C_4	E_7 (*k is odd*)
		C_4 (*k is even*)		$\emptyset$ (*k is even*)		C_4 (*k is even*)

3 Minimum Coverings

Let G denote a graph, and $E(G)$ denote the collection of edges in the graph G. If E and P are collections of edges, then $E+P$ denotes the union of the two collections (so if e occurs in E and occurs y times in P, then it occurs $1+y$ times in $E+P$).

A *covering* of $K_{m(n)}$ with hexagons is a triple (S, C, P), where S is the vertex set of $K_{m(n)}$, $P \subseteq E(G)$ is called the padding, and C is a collection of hexagons that partition $E(G)+P$. The *number* mn is called the order of the covering. So that there is no confusion: an edge $\{a, b\}$ belongs to exactly $x+1$ hexagons of C, where x is the number of times $\{a, b\}$ belongs to the padding P. If $|P|$ is as small as possible, then (S, C, P) is called a minimum covering of $K_{m(n)}$ with hexagons. So, a 6-cycle system is a minimum covering with hexagons, with padding $P = \emptyset$.

The following lemmas are essential to the main result.

Lemma 3.1. *Let L be a leave of a packing of G with hexagons. Then P is a padding of the covering of G with hexagons if $P \cup L$ can be decomposed into hexagons.*

Lemma 3.2. *Let P be a padding of the covering of G with hexagons and $|P| < 6$; then P is a minimum padding.*

Lemma 3.3. *Let G be an odd graph i.e. each vertex of G is of odd degree. Then each padding P of the covering of G with hexagons has at least $|V(G)|/2$ edges. Moreover, if the padding P has at most $|V(G)|/2+5$ edges, then P is a minimum padding of the covering of G with hexagons.*

Lemma 3.4. *For $m \equiv 4 \pmod 6$ and $n \equiv 1 \pmod 6$, the minimum padding of a minimum covering of $K_{m(n)}$ with hexagons is F.*

Proof. Without loss of generality, let $F_4 = K_4 \cup F'$, where F' is the 1-factor with $V(F') = V(G) \setminus \{a_1,a_2,a_3,a_4\}$, $V(K_4) = \{a_1,a_2,a_3,a_4\}$. Let $\{b_1,b_2\}$, $\{c_1,c_2\}$ be two edges in F'. Then adding four edges $\{a_1,c_1\}$, $\{a_4,c_2\}$, $\{a_2,d_1\}$, and $\{a_3,d_2\}$, we can get two 6-cycles $(a_1,a_2,a_3,a_4,c_2,c_1)$ and $(d_1,a_2,a_4,a_1,a_3,d_2)$. And for F', let $\{a'_1,a'_2\}$, $\{b'_1,b'_2\}$, and $\{c'_1,c'_2\}$ be three edges in the leave F'. Then a 6-cycle can be obtained by combining $\{a'_1,b'_1\}$, $\{b'_2,c'_2\}$, and $\{c'_1,a'_2\}$, with the above three edges. So, if $m \equiv 4 \pmod 6$ and $n \equiv 1 \pmod 6$, F is a minimum padding of a minimum covering of $K_{m(n)}$ with hexagons. □

Lemma 3.5. *For $m \equiv 2$ or $4 \pmod 6$ and $n \equiv 5 \pmod 6$, the minimum padding of a minimum covering of $K_{m(n)}$ with hexagons is F.*

Proof. Let a_1, a_2 be the vertices with degree 3 in F_2, $\{a_1,a_2\}$ be an edge in F_2, b_1, b_2 be adjacent to a_1, and let b_3, b_4 be adjacent to a_2. Let

c_1, c_2 be two vertices with $\{c_1, c_2\}$ an edge in F_2, and $\{c_1, c_2\}$ not adjacent to a_i, b_j for all $1 \le i \le 2$, $1 \le j \le 4$. Add two edges $\{c_1, b_1\}$ and $\{c_2, b_3\}$ in F_2; then we can get a 6-cycle $(a_1, a_2, b_3, c_2, c_1, b_1)$. And $F_2 \setminus \{\{a_1, b_1\}, \{c_1, c_2\}, \{a_2, b_3\}, \{a_1, a_2\}\}$ is isomorphic to F. Then, similar to Lemma 3.5, the proof follows. □

Lemma 3.6. *If G is packed with 6-cycles with leave F, then G has a covering (i)F, if $|F| = 0$ (mod 3), (ii) F_4, if $|F| = 1$ (mod 3) (iii) F_2, if $|F| = 2$ (mod 3).*

Proof. By Lemma 3.4 and Lemma 3.5, this lemma is easy to see. □

Lemma 3.7. *If $K_{m(n)}$ is packed with 6-cycles with leave a C_3, then $K_{m(n)}$ has a padding C_3.*

Proof. By Lemmas 2.12, and 2.16, there exists a packing with 6-cycles for $K_{m(n)}$ such that C_3^a:(a_1, a_2, a_3) is the leave, and C_6^b:$(a_1, b_1, a_2, b_2, a_3, b_3)$ is a 6-cycle in the packing. Let $P = (b_1, b_2, b_3)$, then by 6-cycle product, $C_3^a \cup C_6^b \cup P$ can be decomposed into two 6-cycles. □

Lemma 3.8. *If $K_{m(n)}$ is packed with 6-cycles with leave a C_4, then $K_{m(n)}$ has a padding D, where $D = \{\{v_1, v_2\}, \{v_1, v_2\}\}$, v_1, $v_2 \in V(D)$.*

Proof. By Lemmas 2.14, 2.15 and 2.16, there exists a packing with 6-cycles for $K_{m(n)}$ such that C_4^a:(a_1, a_2, a_3, a_4) is the leave, and C_6^b : $(a_1, b_1, b_2, a_2, b_3, b_4)$ is a 6-cycle in the packing. Let $P = \{\{b_1, b_4\}, \{b_1, b_4\}\}$; then $C_4^a \cup C_6^b \cup P$ can be decomposed into two 6-cycles:$(a_1, b_1, b_4, b_3, a_3, a_4)$ and $(a_1, b_5, b_4, b_1, b_2, b_3)$. □

Now, we are ready for the main result. It is obtained by combining Lemma 3.1 to 3.8.

Theorem 3.9. *The minimum covering of $K_{m(n)}$ with hexagons are the following:*

m/n	$6t$	$6t+1$	$6t+2$	$6t+3$	$6t+4$	$6t+5$
$6k$	$\emptyset$	F	$\emptyset$	F	$\emptyset$	F
$6k+1$	$\emptyset$	C_3 (*k is odd*)	$\emptyset$	C_3 (*k is odd*)	$\emptyset$	C_3 (*k is odd*)
		$\emptyset$ (*k is even*)		$\emptyset$ (*k is even*)		$\emptyset$ (*k is even*)
$6k+2$	$\emptyset$	F_4	D	F	D	F
$6k+3$	$\emptyset$	$\emptyset$ (*k is odd*)	$\emptyset$	$\emptyset$ (*k is odd*)	$\emptyset$	$\emptyset$ (*k is odd*)
		C_3 (*k is even*)		C_3 (*k is even*)		C_3 (*k is even*)
$6k+4$	$\emptyset$	F	$\emptyset$	F	$\emptyset$	F
$6k+5$	$\emptyset$	C_5 (*k is odd*)	D	C_3 (*k is odd*)	D	C_5 (*k is odd*)
		D (*k is even*)		$\emptyset$ (*k is even*)		D (*k is even*)

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