On Resolving Acyclic Partitions of Graphs

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Abstract

For a vertex v of a connected graph G and a subset S of $V(G)$, the distance between v and S is $d(v, S) = \min\{d(v, x) : x \in S\}$, where $d(v, x)$ is the distance between v and x. For an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$, the code of v with respect to Π is the k-vector $c_\Pi(v) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. The k-partition Π is a resolving partition if the codes $c_\Pi(v)$, $v \in V(G)$, are distinct. A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ is acyclic if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \le i \le k)$ is acyclic in G. The minimum k for which there is a resolving acyclic k-partition of $V(G)$ is the resolving acyclic number $a_r(G)$ of G. We study connected graphs with prescribed order, diameter, vertex-arboricity, and resolving acyclic number. It is shown that, for each triple d, k, n of integers with $2 \le d \le n-2$ and $3 \le (n-d+1)/2 \le k \le n-d+1$, there exists a connected graph of order n having diameter d and resolving acyclic number k. Also, for each pair a, b of integers with $2 \le a \le b-1$, there exists a connected graph with resolving acyclic number a and vertex-arboricity b. We present a sharp lower bound for the resolving acyclic number of a connected graph in terms of its clique number. The resolving acyclic number of the Cartesian product $H \times K_2$ of nontrivial connected graph H and K_2 is studied.

Key Words: distance, resolving partition, acyclic resolving partition.

AMS Subject Classification: 05C12

[1] Research supported in part by a Western Michigan University Faculty Research and Creative Activities Fund

1 Introduction

Let G be a nontrivial connected graph. For a set S of vertices of G and a vertex v of G, the *distance* $d(v,S)$ *between* v *and* S is defined as

$$d(v,S) = \min\{d(v,x) \ : \ x \in S\},$$

where $d(v,x)$ is the distance between v and x. For an ordered k-partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ and a vertex v of G, the *code of* v *with respect to* Π is defined as the k-vector

$$c_\Pi(v) = (d(v,S_1), d(v,S_2), \ldots, d(v,S_k)).$$

The partition Π is defined in [4] as a *resolving partition* for G if the distinct vertices of G have distinct codes with respect to Π. The minimum k for which there is a resolving k-partition of $V(G)$ is the *partition dimension* $\mathrm{pd}(G)$ of G. A resolving partition of $V(G)$ containing $\mathrm{pd}(G)$ elements is called *a minimum resolving partition*. Resolving partitions in graphs were first introduced and studied in [4]. Resolving partitions that satisfy some additional prescribed properties have been studied [5, 6, 10]

A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is *independent* if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \le i \le k)$ is independent in G. This topic was first introduced from the point of view of graph coloring in [5, 6]. If Π is an independent partition of $V(G)$, then, by coloring the vertices in S_i by i $(1 \le i \le k)$, we obtain a proper coloring with k colors that distinguishes all vertices of G in terms of their distances from the color classes. Thus, such a coloring of a graph G is called a *resolving-coloring* (or *locating-coloring*). A *minimum resolving-coloring* uses a minimum number of colors and this number is the *resolving-chromatic number* $\chi_r(G)$ of G. Since every resolving-coloring is a coloring, $\chi(G) \le \chi_r(G)$ for each connected graph G, where $\chi(G)$ is the chromatic number of G.

A resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is defined in [10] to be *acyclic* if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \le i \le k)$ is acyclic in G. The minimum k for which G contains a resolving acyclic k-partition is the *resolving acyclic number* $a_r(G)$ of G. The *vertex-arboricity* $a(G)$ of G is defined in [1, 2] as the minimum k such that $V(G)$ has an acyclic k-partition. Since every resolving acyclic partition is an acyclic partition, $a(G) \le a_r(G)$ for each connected graph G.

It was observed in [10] that

$$2 \le \mathrm{pd}(G) \le a_r(G) \le \chi_r(G) \le n$$

for every nontrivial connected graph G of order n.

To illustrate these concepts, consider the graph G of Figure 1. Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{x\}$, $S_2 = \{u\}$, and $S_3 = \{v, y, z\}$. Then the corresponding codes of the vertices of G are

$$c_\Pi(u) = (1,0,1) \quad c_\Pi(v) = (1,2,0) \quad c_\Pi(x) = (0,1,1)$$
$$c_\Pi(y) = (1,1,0) \quad c_\Pi(z) = (2,1,0).$$

Since the codes of the vertices of G with respect to Π are distinct, Π is a resolving partition of G. Because no 2-partition is a resolving partition of G, it follows that Π is a minimum resolving partition of G and so $\mathrm{pd}(G) = 3$. However, Π is not acyclic since $\langle S_3 \rangle = K_3$. On the other hand, let $\Pi' = \{S_1', S_2', S_3', S_4'\}$, where $S_1' = \{x\}$, $S_2' = \{u\}$, $S_3' = \{v, y\}$, and $S_4' = \{z\}$. It can be verified that Π' is a resolving acyclic partition of G and no 3-partition is a resolving acyclic partition of G. Thus $a_r(G) = 4$. Furthermore, $\chi_r(G) = 5$ (see [6]). Therefore, $\mathrm{pd}(G) < a_r(G) < \chi_r(G)$ for the graph G of Figure 1. The example just described also illustrates an important point. When determining whether a given partition Π is a resolving partition of a connected graph G, we need only verify that vertices of G belonging to the same subset of $V(G)$ in Π have distinct codes since the codes of two vertices in different subsets in Π have 0 in different coordinates.

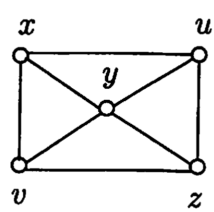

Figure 1: A graph G with $\mathrm{pd}(G) < a_r(G) < \chi_r(G)$

The concept of resolvability in graphs has appeared in the literature. Slater introduced and studied these ideas with different terminology in [11, 12]. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [7] discovered these concepts independently. Recently, these concepts were rediscovered by Johnson [8, 9] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. We refer to the book [3] for graph theory notation and terminology not described here.

The resolving acyclic numbers of some familiar classes of graphs have been determined in [10].

Theorem A *Let $n \geq 2$ and let G be a connected graph of order n. Then*

(*a*) $a_r(G) = 2$ *if and only if* $G = P_n$;

(*b*) $a_r(G) = n$ *if and only if* $G = K_n$*;*

(*c*) $a_r(G) = n - 1$ *if and only if*

$$G \in \{C_4 + K_1, K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-2})\}$$

for $n \geq 5$.

Theorem B *Let* r, s *be positive integers. If* G *is a connected bipartite graph with partite sets of cardinalities* r *and* s*, then*

$$a_r(G) \leq \begin{cases} r+1 & \text{if } r = s \\ \max\{r,s\} & \text{if } r \neq s. \end{cases}$$

Moreover, equality holds if G *is a complete bipartite graph.*

2 Graphs with Prescribed Resolving Acyclic Number and Other Parameters

In this section, we study connected graphs with prescribed resolving acyclic number and other parameters, such as diameter, order, vertex-arboricity, clique number, and etc. We begin with the connected graphs with prescribed resolving acyclic number, order, and diameter. The *diameter* of a connected graph G is the largest distance between two vertices in G. Bounds for the resolving acyclic number of a connected graph was established in terms of its order and diameter in [10].

Theorem C *If* G *is a connected graph of order* $n \geq 3$ *and diameter* $d \geq 2$*, then*

$$\log_{d+1} n \leq a_r(G) \leq n - d + 1.$$

The lower bound in Theorem C is not sharp. In order to show this, we present a result that gives an upper bound for the order of a connected graph in terms of its diameter and resolving acyclic number.

Proposition 2.1 *If* G *is a nontrivial connected graph of order* n*, diameter* d*, and resolving acyclic number* k*, then* $n \leq kd^{k-1}$.

Proof. If $d = 1$, then $G = K_n$. By Theorem A, $a_r(G) = n = k$, and so the result holds for $d = 1$. Thus we may assume that $d \geq 2$ and so $n \geq 3$. Let Π be a resolving acyclic k-partition of $V(G)$. Since (1) exactly one coordinate of the code of a vertex in G with respect to Π is 0 and there are k choices for the zero coordinates in the code of a vertex, (2) each of the $k - 1$ nonzero coordinate of the code of a vertex is a positive integer not exceeding d, and (3) all codes of the n vertices of G are distinct, it follows that $kd^{k-1} \geq n$. ■

Certainly, for each pair d, k of positive integers, $(d+1)^k > kd^{k-1}$. Thus, if G is a nontrivial connected graph of order n, diameter d, and resolving acyclic number k, then, by Proposition 2.1, $(d+1)^k > kd^{k-1} \geq n$, and so $a_r(G) > \log_{d+1} n$. Thus, the following corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2 *If G is a connected graph of order $n \geq 3$ and diameter $d \geq 2$, then*

$$a_r(G) \geq 1 + \log_{d+1} n.$$

Notice that if $G = P_n$, then $d = n-1$ and $\log_{d+1} n = 1$. By Theorem A, the equality in Corollary 2.2 holds for $G = P_n$. The upper bound in Theorem C is sharp. In fact, more can be said, as we will see later. First, we present a useful lemma, whose proof is straightforward and is therefore omitted.

Lemma 2.3 *Let Π be a resolving partition in a connected graph G. If u and v are distinct vertices of G such that $d(u,w) = d(v,w)$ for all $w \in V(G) - \{u,v\}$, then u and v belong to distinct elements of Π. In particular, if G is a connected graph containing a vertex that is adjacent to k end-vertices of G, then $a_r(G) \geq k$.*

We now study those triples of positive integers that are realizable as the diameter, resolving acyclic number, and order of some connected graph.

Theorem 2.4 *For each triple d, k, n of integers with $2 \leq d \leq n-2$ and $3 \leq (n-d+1)/2 \leq k \leq n-d+1$, there exists a connected graph of order n having diameter d and resolving acyclic number k.*

Proof. First, assume that $k = n-d+1$. Let G be the graph obtained from K_{n-d+1} and the path $P_{d-1} : v_1, v_2, \ldots, v_{d-1}$ by joining v_1 to a vertex u in K_{n-d+1}. Then the order of G is n and the diameter of G is d. We show that $a_r(G) = n-d+1$. Since every element of any resolving acyclic partition of $V(G)$ contains at most one vertex of $V(K_{n-d+1}) - \{u\}$ by Lemma 2.3, it follows that $a_r(G) \geq n-d$. Assume, to the contrary, that $a_r(G) = n-d$. Let $V(K_{n-d+1}) - \{u\} = \{u_1, u_2, \ldots, u_{n-d}\}$ and let $\Pi = \{S_1, S_2, \ldots, S_{n-d}\}$ be a resolving acyclic partition of $V(G)$. Assume, without loss of generality, that $u_i \in S_i$ and $u \in S_1$. However, then, $c_\Pi(u) = (0,1,1,\ldots,1) = c_\Pi(u_1)$, which is a contradiction. Thus, $a_r(G) \geq n-d+1 = k$. On the other hand, $\Pi^* = \{S_1^*, S_2^*, \ldots, S_{n-d+1}^*\}$, where $S_i^* = \{u_i\}$ $(1 \leq i \leq n-d)$ and $S_{n-d+1}^* = \{u\} \cup V(P_{d-1})$, is a resolving acyclic partition of $V(G)$ and so $a_r(G) \leq |\Pi^*| = n-d+1$. Therefore, $a_r(G) = n-d+1 = k$.

Next, assume that $(n-d+1)/2 \le k \le n-d$. We consider two cases.

Case 1. $d = 2$. Then $(n-1)/2 \le k \le n-2$. If $(n+1)/2 \le k \le n-2$, then $k > n-k$ and $G = K_{k,n-k}$ has the desired property by Theorem B. If $(n-1)/2 \le k < (n+1)/2$, then $k = n/2$ if n is even and $k = (n-1)/2$ if n is odd. We consider these two subcases.

Subcase 1.1. $k = n/2$. Let $G = (K_{k-1} \cup kK_1) + K_1$. Let $V(K_1) = \{u_1\}$, $V(K_{k-1}) = \{u_2, u_3, \ldots, u_k\}$, and $V(kK_1) = \{v_1, v_2, \ldots, v_k\}$. Since u_1 is adjacent to k end-vertices, $a_r(G) \ge k$ by Lemma 2.3. On the other hand, $\Pi = \{S_1, S_2, \ldots, S_k\}$, where $S_i = \{u_i, v_i\}$ $(1 \le i \le k)$, is a resolving acyclic partition of $V(G)$ and so $a_r(G) = k$.

Subcase 1.2. $k = (n-1)/2$. Let $G = (P_k \cup kK_1) + K_1$. Let $V(K_1) = \{w\}$, $V(P_k) = \{u_1, u_2, \ldots, u_k\}$, and $V(kK_1) = \{v_1, v_2, \ldots, v_k\}$. Since w is adjacent to k end-vertices, $a_r(G) \ge k$ by Lemma 2.3. On the other hand, $\Pi = \{S_1, S_2, \ldots, S_k\}$, where $S_1 = \{u_1, v_1, w\}$ and $S_i = \{u_i, v_i\}$ $(2 \le i \le k)$, is a resolving acyclic partition of $V(G)$ and so $a_r(G) = k$.

Case 2. $3 \le d \le n-2$. Again, we consider two subcases.

Subcase 2.1. $k = n-d$. Let G be the graph obtained from the path $P_d : u_1, u_2, \ldots, u_d$ by adding the $n-d$ pendant edges u_1v_i $(1 \le i \le n-d)$. Then the order of G is n and the diameter of G is d. Since u_1 is adjacent to $n-d$ end-vertices, it then follows by Lemma 2.3 that $a_r(G) \ge n-d$. On the other hand, let $\Pi = \{S_1, S_2, \ldots, S_{n-d}\}$ be a partition of $V(G)$ where $S_1 = \{u_3, v_1\}$, $S_2 = \{v_2\} \cup (V(P_{n-d} - \{u_3\})$, and $S_i = \{v_i\}$ for $3 \le i \le n-d$. Since Π is a resolving acyclic partition of G, it follows that $a_r(G) \le |\Pi| = n-d$. Thus $a_r(G) = n-d$.

Subcase 2.2. $k \le n-d-1$. Since $k \le n-d-1$, it follows that $n-d-k+1 \ge 2$. Let G be the graph obtained from the path $P_{d-1} : u_1, u_2, \ldots, u_{d-1}$ by adding the $n-d+1$ pendant edges u_1v_i $(1 \le i \le k)$, u_2w_1, and $u_{d-1}w_j$ $(2 \le j \le n-d-k+1)$. Then the order of G is n and the diameter of G is d. We show that $a_r(G) = k$. Since $k \ge (n-d+1)/2$, it follows that $k \ge n-d-k+1 > n-d-k$. Thus $a_r(G) \ge k$ by Lemma 2.3. On the other hand, let $\Pi = \{S_1, S_2, \ldots, S_k\}$, where $S_1 = \{v_1, w_1\}$, $S_2 = \{u_2, v_2, w_2\}$, $S_3 = (V(P_{d-1}) - \{u_2\}) \cup \{v_3, w_3\}$, $S_i = \{v_i, w_i\}$ for $4 \le i \le n-d-k+1$, and $S_i = \{v_i\}$ for $n-d-k+2 \le i \le k$. Since G is a tree, it suffices to show that Π is a resolving partition of G. Observe that (1) $c_\Pi(v_1) = (0, 2, \ldots)$ and $c_\Pi(w_1) = (0, 1, \ldots)$, (2) $c_\Pi(u_2) = (1, 0, 1, \ldots)$, $c_\Pi(v_2) = (2, 0, 1, \ldots)$, and $c_\Pi(w_2) = (2, 0, 2, \ldots)$ if $d = 3$ and $c_\Pi(w_2) = (d-1, 0, 1, \ldots)$ if $d \ge 4$, (3) $c_\Pi(v_3) = (2, 2, 0, \ldots)$, and $c_\Pi(w_3) = (2, 1, 0, \ldots)$ if $d = 3$ and $c_\Pi(w_3) = (d-1, 2, 0, \ldots)$ if $d \ge 4$, $c_\Pi(u_1) = (1, *, 0, \ldots)$ and $c_\Pi(u_i) = (i-1, *, 0, \ldots)$ for $3 \le i \le d-1$, (4) $c_\Pi(v_i) = (2, *, 1, \ldots)$, $c_\Pi(w_i) = (2, *, 2, \ldots)$ if $d = 3$ and $c_\Pi(w_i) = (d-1, *, 1, \ldots)$ if $d \ge 4$, where $*$ represents an irrelevant coordinate. Thus all codes of the vertices of G are distinct, and so Π is resolving, implying that $a_r(G) \leq |\Pi| = k$. Therefore, $a_r(G) = k$. ∎

We have seen that if G is a connected graph with $a(G) = a$ and $a_r(G) = b$, then $1 \leq a \leq b$ and $b \geq 2$. Next we study those pairs a, b of integers with $1 \leq a \leq b$ and $b \geq 2$ that are realizable as vertex-arboricity and resolving acyclic number of some connected graph. Since trees are the only connected graphs having vertex-arboricity 1, it follows that $a(T) \neq a_r(T)$ for all trees T. Thus we may assume that $a \geq 2$. It is known that if G is a connected graph of order n, then $a(G) \leq \lceil n/2 \rceil$. Thus if $a_r(G) > \lceil n/2 \rceil$, then $a(G) \neq a_r(G)$. On the other hand, we show next that each pair a, b of integers with $2 \leq a \leq b-1$ is realizable as vertex-arboricity and resolving acyclic number of some connected graph.

Theorem 2.5 *For each pair a, b of integers with $2 \leq a \leq b-1$, there exists a connected graph G with $a(G) = a$ and $a_r(G) = b$.*

Proof. For $a = 2$, let $G = K_{a,b}$. It is known that $a(G) = 2$. Since $a < b$, it follows by Theorem B that $a_r(G) = b$ and so G has the desired properties. For $a \geq 3$, let G be the graph obtained from K_{2a} with $V(K_{2a}) = \{u_1, u_2, \ldots, u_{2a}\}$ by adding the $b + a - 1$ new vertices $v_1, v_2, \ldots, v_a$ and $w_1, w_2, \ldots, w_{b-1}$, joining each vertex v_i to u_i $(1 \leq i \leq a)$ and joining each vertex w_j $(1 \leq j \leq b-1)$ to u_a. Thus $a(G) = a$.

Next, we show that $a_r(G) = b$. Since u_a is adjacent to b end-vertices, namely, v_a and w_j $(1 \leq j \leq b-1)$, it then follows by Lemma 2.3 that $a_r(G) \geq b$. On the other hand, let $\Pi = \{S_1, S_2, \ldots, S_b\}$ be a partition of $V(G)$, where $S_i = \{u_i, u_{a+i}, w_i\}$, $1 \leq i \leq a$, $S_i = \{w_i\}$, $a + 1 \leq i \leq b-1$, and $S_b = \{v_1, v_2, \ldots, v_a\}$. Thus Π is acyclic. Notice that (1) $d(u_i, S_j) = 1$ for $1 \leq i \neq j \leq a$ and $d(u_i, S_b) = 1$, (2) $d(u_{a+i}, S_j) = 1$ for $1 \leq i \neq j \leq a$ and $d(u_{a+i}, S_b) = 2$, and (3) $d(w_i, S_j) = 2$ for $1 \leq i \neq j \leq a$ and $j \neq a$ and $d(w_i, S_a) = 1$ for $1 \leq i \leq a$. Since $a \geq 3$, it follows that $c_\Pi(u_i)$, $c_\Pi(u_{a+i})$, and $c_\Pi(w_i)$ are distinct for $1 \leq i \leq a$. Furthermore, the ith coordinate of $c_\Pi(v_i)$ is 1, the bth coordinate of $c_\Pi(v_i)$ is 0, and the remaining coordinates of $c_\Pi(v_i)$ are 2 or 3, implying that the codes $c_\Pi(v_i)$, $1 \leq i \leq a$, are distinct. Hence Π is a resolving acyclic partition of $V(G)$ and so $a_r(G) \leq |\Pi| = b$. Therefore, $a_r(G) = b$. ∎

The *clique number* of a graph is the maximum order among the complete subgraphs of the graph. We state a lower bound for the resolving acyclic number of a connected graph in terms of its clique number.

Proposition 2.6 *If the clique number of a connected graph G is ω, then*

$$a_r(G) \geq \left\lceil \frac{\omega}{2} \right\rceil + 1.$$

Moreover, for each integer $\omega \geq 2$, there exists a connected graph G_ω having clique number ω such that $a_r(G_\omega) = \lceil \omega/2 \rceil + 1$.

3 Cartesian Products

We show next that the resolving acyclic number of the Cartesian product of K_2 and a nontrivial connected graph H is bounded above by $a_r(H) + a(H)$.

Theorem 3.1 *For every nontrivial connected graph H,*

$$a_r(H \times K_2) \leq a_r(H) + a(H)$$

Proof. Let $G = H \times K_2$, where H_1 and H_2 are the two copies of H in the construction of G. Suppose that $a_r(H) = k$ and $a(H) = a$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving acyclic partition of $V(H_1)$, and let $\{W_1, W_2, \ldots, W_a\}$ be an acyclic partition of $V(H_2)$. Then

$$\Pi^* = \{S_1, S_2, \ldots, S_k, W_1, W_2, \ldots W_a\}$$

is an acyclic partition of $V(G)$. We show that Π^* is a resolving partition of $V(G)$. Let x and y be vertices of G such that $c_{\Pi^*}(x) = c_{\Pi^*}(y)$. We show that $x = y$. Assume, to the contrary, that $x \neq y$. We consider three cases.

Case 1. *Both x and y belong to H_1.* Then $d_G(x, S_i) = d_{H_1}(x, S_i)$ and $d_G(y, S_i) = d_{H_1}(y, S_i)$ $(1 \leq i \leq k)$. Since $d_G(x, S_i) = d_G(y, S_i)$ for all $1 \leq i \leq k$, it follows that $d_{H_1}(x, S_i) = d_{H_1}(y, S_i)$ for $1 \leq i \leq k$ and so $c_\Pi(x) = c_\Pi(y)$, contradicting the fact that Π is a resolving acyclic partition of H_1.

Case 2. *Both x and y belong to H_2.* Let x' and y' be the vertices of H_1 that correspond to x and y in H_2, respectively. Since $x \neq y$, we have $x' \neq y'$. Notice that $d_G(x, S_i) = d_G(x', S_i) + 1 = d_{H_1}(x', S_i) + 1$ and $d_G(y, S_i) = d_G(y', S_i) + 1 = d_{H_1}(y', S_i) + 1$ for $1 \leq i \leq k$. Thus $d_{H_1}(x', S_i) = d_G(x, S_i) - 1$ and $d_{H_1}(y', S_i) = d_G(y, S_i) - 1$ for $1 \leq i \leq k$. Since $d_G(x, S_i) = d_G(y, S_i)$ for $1 \leq i \leq k$, it follows that $d_{H_1}(x', S_i) = d_{H_1}(y', S_i)$ for $1 \leq i \leq k$ and so $c_\Pi(x') = c_\Pi(y')$, a contradiction.

Case 3. *Either $x \in V(H_1)$ and $y \in V(H_2)$, or $x \in V(H_2)$ and $y \in V(H_1)$, say the former.* Suppose that $y \in W_i$, where $1 \leq i \leq a$ and so $d_G(y, W_i) = 0$. However, $x \notin V(H_2)$ and so $x \notin W_i$, which implies that $d_G(x, W_i) > 0$. Thus $c_{\Pi^*}(x) \neq c_{\Pi^*}(y)$, contradicting the assumption.

Therefore, Π^* is a resolving acyclic partition of $V(G)$ and so $a_r(G) \leq |\Pi^*| = a_r(H) + a(H)$, as desired. ∎

Equality in Theorem 3.1 can hold. For example, if $H = P_n$, where $n \geq 2$, then $a_r(P_n) = 2$ and $a(P_n) = 1$. By Theorem A, $a_r(P_n \times K_2) \geq 3$. On the other hand, let $H_1 : u_1, u_2, \ldots, u_n$ and $H_2 : v_1, v_2, \ldots, v_n$ be two copies of P_n in $P_n \times K_2$. Since $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = V(H_1)$, $S_2 = \{v_1\}$, and $S_3 = V(H_2) - \{v_1\}$, is a resolving acyclic partition of $V(P_n \times K_2)$, it follows that $a_r(P_n \times K_2) = 3$. Therefore, $a_r(P_n \times K_2) = 3 = a_r(P_n) + a(P_n)$ for $n \geq 2$.

Strict inequality in Theorem 3.1 can also hold. As an example, we study $a_r(K_n \times K_2)$ for $n \geq 3$, beginning with $n = 3, 4$.

Proposition 3.2 *$a_r(K_3 \times K_2) = 3$ and $a_r(K_4 \times K_2) = 4$.*

Proof. For $n = 3, 4$, let H_1 and H_2 be two copies of K_n in G, where $V(H_1) = \{u_1, u_2, \ldots, u_n\}$, $V(H_2) = \{v_1, v_2, \ldots, v_n\}$, and $u_i v_i \in E(G)$ for $1 \leq i \leq n$. For $n = 3$, let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{u_1, u_2, v_1\}$, $S_2 = \{v_2, v_3\}$, and $S_3 = \{u_3\}$. Since Π is a resolving acyclic partition, it follows by Theorem A that $a_r(K_3 \times K_2) = 3$.

For $n = 4$, let $\Pi^* = \{S_1^*, S_2^*, S_3^*, S_4^*\}$, be a partition of $V(G)$, where $S_1^* = \{u_1, u_3\}$, $S_2^* = \{u_2, v_2, v_3\}$, $S_3^* = \{v_1, v_4\}$, and $S_4^* = \{u_4\}$. Since Π^* is a resolving acyclic partition, $a_r(G) \leq |\Pi^*| = 4$. Assume, to the contrary, that $a_r(G) = 3$. Let $\Pi = \{S_1, S_2, S_3\}$ be a minimum resolving partition of G. Since $\langle S_i \rangle$, $1 \leq i \leq 3$, is acyclic, S_i contains at most two vertices of H_1. Thus at least two of S_1, S_2, and S_3 contain vertices of H_1. If each of S_1, S_2, and S_3 contains some vertex of H_1, then assume, without loss of generality, that $u_i \in S_i$ for $1 \leq i \leq 3$. Since Π is a partition of $V(G)$, it follows that $u_4 \in S_i$ for some i with $1 \leq i \leq 3$, say $u_4 \in S_1$. Then $c_\Pi(u_1) = c_\Pi(u_4) = (0, 1, 1)$, a contradiction. Thus exactly two of S_1, S_2, and S_3 contains vertices of H_1, say S_1 and S_2 contain vertices of H_1. Moreover, each of S_1 and S_2 contains exactly two vertices of H_1. We may assume that $u_1, u_2 \in S_1$ and $u_3, u_4 \in S_2$. Similarly, exactly two of S_1, S_2, and S_3 contains vertices of H_2, each of which contains exactly two vertices of H_2. Since $S_3 \neq \emptyset$ and $S_3 \cap V(H_1) = \emptyset$, it follows that S_3 consists of exactly two vertices of H_2. This implies that exactly one of S_1 and S_2 contains the two vertices in $V(H_2) - S_3$, say $V(H_2) - S_3 \subseteq S_1$ and so $S_2 = \{u_3, u_4\}$. Since $\langle S_1 \rangle$ is acyclic and $u_1, u_2 \in S_1$, it follows that S_1 contains at most one of v_1 and v_2. We consider two cases.

Case 1. *S_1 contains exactly one of v_1 and v_2, say $v_1 \in S_1$.* Thus $S_1 = \{u_1, u_2, v_1, v_j\}$, where $j = 3$ or $j = 4$. If $S_1 = \{u_1, u_2, v_1, v_3\}$, then $S_2 = \{u_3, u_4\}$, and $S_3 = \{v_2, v_4\}$. Thus $c_\Pi(u_2) = c_\Pi(v_3) = (0, 1, 1)$, a contradiction. If $S_1 = \{u_1, u_2, v_1, v_4\}$, then $S_2 = \{u_3, u_4\}$, and $S_3 = \{v_2, v_3\}$. Thus $c_\Pi(u_2) = c_\Pi(v_4) = (0, 1, 1)$, a contradiction.

Case 2. *S_1 contains neither v_1 nor v_2.* Thus $S_1 = \{u_1, u_2, v_3, v_4\}$, $S_2 = \{u_3, u_4\}$, and $S_3 = \{v_1, v_2\}$. Then $c_\Pi(u_1) = c_\Pi(u_2) = (0, 1, 1)$, a contradiction. ∎

Thus $a_r(K_n \times K_2) = a_r(K_n) = n$ for $n = 3, 4$. However, $a_r(K_n \times K_2) < n$ for $n \geq 5$. We present an upper bound for $a_r(K_n \times K_2)$ in terms of n without a proof.

Proposition 3.3 *If $n \geq 5$, then*

$$a_r(K_n \times K_2) \leq \begin{cases} \frac{3n}{4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{3(n-1)}{4} + 2 & \text{if } n \equiv 1 \pmod 4 \\ \frac{3(n-2)}{4} + 2 & \text{if } n \equiv 2 \pmod 4 \\ \frac{3(n-3)}{4} + 3 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Since $a(K_n) = \lceil n/2 \rceil$ for $n \geq 3$, it follows by Proposition 3.3 that strict inequality in Theorem 3.1 can hold.

4 Acyclic Partition Ratios

If H is an induced subgraph of a graph G, then it is known that $a(H) \leq a(G)$. Thus

$$0 < \frac{a(H)}{a(G)} \leq 1$$

for each induced subgraph H of a graph G. However, it is not true, in general, for the resolving acyclic numbers. Let G be a nontrivial connected graph and H a connected induced subgraph of G. We defined *the acyclic partition ratio* of G and H, respectively, by

$$r_a(H, G) = \frac{a_r(H)}{a_r(G)}$$

By Theorem B $a_r(K_{1,m}) = m$ for all $m \geq 2$. Hence for $G = K_{1,m}$ and $H = K_2$, we can make the ratio $r_a(H, G)$ as small as we wish by choosing m arbitrarily large. Although this may not be surprising, it may be unexpected that, in fact, we can make $r_a(H, G)$ as large as we wish. We now establish the truth of this statement.

For $n \geq 3$, we label the vertices of the star $K_{1,2^{n+1}}$ with v_0, v_1, v_2, $\ldots$, v_{2^n}, v'_1, v'_2, $\ldots, v'_{2^n}$, where v_0 is the central vertex. Then we add two new vertices x and x' and 2^{n+1} edges xv_i and $x'v'_i$ for $1 \leq i \leq 2^n$. Next, we add two sets $W = \{w_1, w_2, \ldots, w_n\}$ and $W' = \{w'_1, w'_2, \ldots, w'_n\}$ of vertices, together with the edges $w_i x$ and $w'_i x'$ for $1 \leq i \leq n$. Finally, we add edges between W and $\{v_0, v_1, v_2, \ldots, v_{2^n}\}$ so that each of the 2^n possible n-tuples of 1s and 2s appears exactly once such that the representations $(d(v_i, w_1), d(v_i, w_2), \ldots, d(v_i, w_n))$ are distinct for $1 \leq i \leq$

2^n. Similarly, edges are added between W' and $\{v'_1, v'_2, \ldots, v'_{2^n}\}$ so that $(d(v'_i, w'_1), d(v'_i, w'_2), \ldots, d(v'_i, w'_n))$ are distinct for $1 \le i \le 2^n$. Denote the resulting graph by G. The graph G for $n = 3$ is shown in Figure 2.

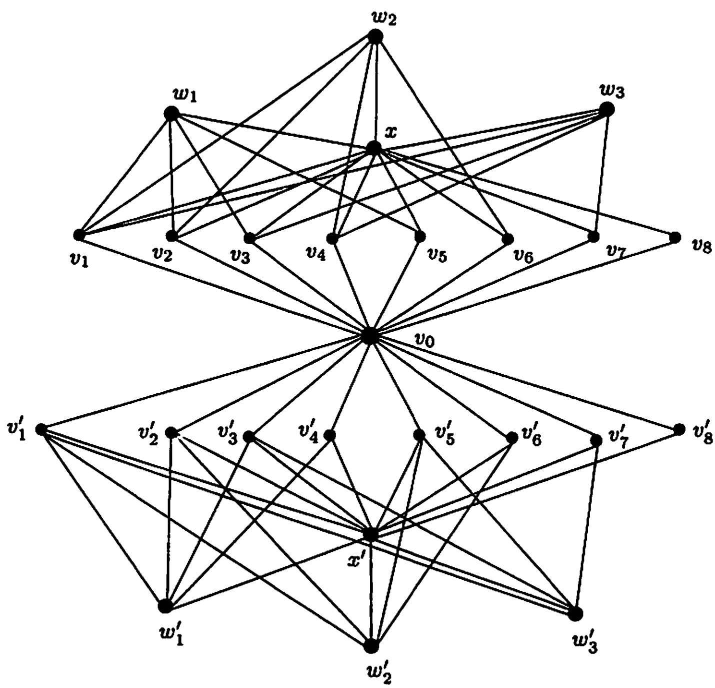

Figure 2: The graph G for $n = 3$

Let $\Pi = \{S_1, S_2, \ldots, S_{2n+2}\}$ be the partition of $V(G)$ where $S_i = \{w_i\}$ $(1 \le i \le n)$, $S_{n+j} = \{w'_j\}$ $(1 \le j \le n)$, and $S_{2n+1} = \{x, x'\}$ and $S_{2n+2} = V(K_{1,2^{n+1}})$. Since $\langle S_i \rangle$ is acyclic for $1 \le i \le 2n+2$, it follows that Π is acyclic. Next we show that Π is a resolving partition of $V(G)$. By construction, $c_\Pi(v_i) = c_\Pi(v_j)$ implies that $i = j$ and $c_\Pi(v'_i) = c_\Pi(v'_j)$ implies that $i = j$. Moreover,

$$\begin{aligned}
c_\Pi(x) &= (1, 1, \ldots, 1, 4, 4, \ldots, 4, 0, 1), \\
c_\Pi(v_i) &= (*, *, \ldots, *, 3, 3, \ldots, 3, 1, 0), \ 1 \le i \le 2^n, \\
c_\Pi(v_0) &= (2, 2, \ldots, 2, 2, 2, \ldots, 2, 2, 0), \\
c_\Pi(v'_i) &= (3, 3, \ldots, 3, *, *, \ldots, *, 1, 0), \ 1 \le i \le 2^n, \\
c_\Pi(x') &= (4, 4, \ldots, 4, 1, 1, \ldots, 1, 0, 1),
\end{aligned}$$

where $*$ represents an irrelevant coordinate. Thus Π is a resolving acyclic $(2n+2)$-partition of $V(G)$. Observe that G contains $H = K_{1,2^{n+1}}$ as an induced subgraph and

$$\frac{a_r(H)}{a_r(G)} \geq \frac{2^{n+1}}{2n+2}.$$

Since

$$\lim_{n\to\infty} \frac{2^{n+1}}{2n+2} = \infty,$$

there exists a connected graph G and an induced subgraph H of G such that $r_a(H,G) = a_r(H)/a_r(G)$ is arbitrary large.

5 Open Questions and Topics for Study

The lower and upper bounds for the resolving acyclic number of a connected graph in terms of its resolving-chromatic number were established in [10], which we state next.

Theorem D *For every nontrivial connected graph G,*

$$\frac{\chi_r(G)}{2} \leq a_r(G) \leq \chi_r(G).$$

Thus, if G is a connected graph G with $a_r(G) = a$ and $\chi_r(G) = b$, then $a \geq 2$ and $b/2 \leq a \leq b$. On the other hand, it is easy to show that, for each pair a, b of integers with $a \geq 2$ and $b/2 < a \leq b$, there exists a connected graph G such that $a_r(G) = a$ and $\chi_r(G) = b$. It was shown in [5] that if G is a nontrivial complete graph or complete bipartite graph, then $\chi_r(G) = n$, where n is the order of G. Thus, if $a = b \geq 2$, then the graph K_a has the property that $a_r(G) = \chi_r(G) = a$ by Theorem A. If $a \geq 2$ and $b/2 < a < b$, then $K_{a,b-a}$ has the property that $a_r(G) = a$ by Theorem B and $\chi_r(G) = b$. These observations yield the following.

Proposition 5.1 *For every pair a, b of integers with $a \geq 2$ and $b/2 < a \leq b$, there exist a connected graph G with $a_r(G) = a$ and $\chi_r(G) = b$.*

However, the following question remains open.

Problem 5.2 *Does there exist a connected graph G such that $\chi_r(G) = 2a_r(G)$?*

We have seen in Theorem 2.4 that, for each triple d, k, n of integers with $2 \leq d \leq n-2$ and $3 \leq (n-d+1)/2 \leq k \leq n-d+1$, there exists a connected graph of order n having diameter d and resolving acyclic number k. On the other hand, for those triples d, k, n of integers with $2 \leq d \leq n-2$ and $3 \leq k \leq (n-d-1)/2$, the following question is open.

Problem 5.3 *For which triples d, k, n of integers with $2 \leq d \leq n-2$ and $3 \leq k \leq (n-d-1)/2$, does there exist a connected graph of order n having diameter d and resolving acyclic number k?*

Theorem 2.5 shows, for each pair a, b of integers with $2 \leq a \leq b-1$, that there exists a connected graph G with $a(G) = a$ and $a_r(G) = b$. However, we don't know any connected graph G with $a(G) = a_r(G)$. This suggests the following question.

Problem 5.4 *Does there exist a connected graph G such that $a(G) = a_r(G)$?*

We have seen in Theorem 3.1 that if H is a nontrivial connected graph, then $a_r(H \times K_2) \leq a_r(H) + a(H)$, which suggests the following question.

Problem 5.5 *For which triples a, b, c of positive integers with $a \geq 2$ and $3 \leq c \leq a + b$, does there exist a connected graph H such that $a_r(H) = a$ and $a(H) = b$, and $a_r(H \times K_2) = c$?*

We conclude this paper by describing some topics for further study. If G is a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the ordered partition $\Pi = \{S_1, S_2, \ldots, S_n\}$, where $S_i = \{v_i\}$ for $1 \leq i \leq n$, into singleton subsets of $V(G)$ is always a resolving partition of $V(G)$. Since $\langle S_i \rangle$ is trivially acyclic for each i $(1 \leq i \leq n)$, it follows that Π is a resolving acyclic partition of $V(G)$ as well, and, consequently, $a_r(G)$ is defined. A similar argument shows that $\chi_r(G)$ is defined for every connected graph G.

This suggests a variety of concepts to study. If P is any graphical property possessed by a trivial subgraph of a connected graph G, then the ordered partition Π of $V(G)$ described above is said to satisfy property P and the P-partition number is defined. Among the various properties P, in addition to the property of being independent or being acyclic, are (1) the property of being a specified graph, say path, star, cycle, a linear forest (every component is a path), or a galaxy (every component is a star), (2) the property of being planar or hamiltonian, and (3) the property of having maximum degree or girth at most k for a fixed nonnegative integer k.

6 Acknowledgments

We are grateful to Professor Gary Chartrand for suggesting the concept of acyclic partition to us and kindly providing useful information on this topic. Also, we thank Professor Peter Slater for the useful conversation.

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