The problem of outer-embeddings in pseudosurfaces

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Abstract

Many results about outer-embeddings (graphs having all their vertices in the same face) have been obtained recently in topological graph theory in the last times. In this paper we deal with some difficulties appearing in the study of such embeddings. Particularly, we propose several problems concerning to outer-embeddings in pseudosurfaces and we prove that two of them are NP-complete.

We also describe some properties about lists of forbidden minors for outer-embeddings in certain kinds of pseudosurfaces.

1 Preliminaries

Topological graph theory refers to graph-embeddings in surfaces or pseudosurfaces (such that no two edges cross one another). In this paper we study finite graphs in particular. So, we will use the usual notations for them (they can be checked in Harary [11], for instance).

A *face* of a graph G is each connected component of the topological complementary of G, supposed that it is embedded in a surface or in a pseudosurface. In this paper we will deal with a special kind of embedding: the *outer-embedding*, in which all vertices are in the same face.

It is usual to call *surfaces* to 2-connected manifolds. With respect to pseudosurfaces (characterized by Lu [14] in 1949), they have points (named *singular points*) in which does not exist a neighbourhood homeomorphic to an open disk $\{x \in \mathbb{R}^2 : d(x,(0,0)) < 1\}$. Formally, a *pseudosurface* is a topological Haussdorf space such that each point has a neighbourhood homeomorphic either to one open disk or to the union of a finite set of open disks joined by one point. We normally consider pseudosurfaces being compact and with a finite set of singular points.

For example, if $n \in \mathbb{N}$, B_n is the pseudosurface obtained when contracting n different meridians of a torus to n points. So, B_n has n singular points: B_0 is the torus, B_1 is the *spindle surface* and B_2 is the *bananas surface*.

In 1930, Kuratowski [13] characterized planar graphs by giving a list of forbidden topological minors (which coincides, in this case, with the forbidden minors list). Chartrand and Harary [8] characterized in 1967 graphs admitting a planar embedding and with all their vertices in the same face (that is, outerplanar graphs).

Later, in 1996, Cáceres [7]and Archdeacon and others [2] characterized independently the outerprojective graphs and there are no results at present related to outer-embeddings in other surfaces.

It is known that if it is possible to characterize a property by using forbidden minors, then it is also possible to obtain a finite list of forbidden topological minors. A way to obtain the list of forbidden topological minors starting from the corresponding list of forbidden minors can be checked in [7].

It was already known in 1980 (see [3]) that the characterization of graphs admitting an embedding in a non-orientable surface could be given by a finite list of forbidden topological minors. Later, this result was introduced by Robertson and Seymour [17] for any compact surface and for B_1.

However, Širáň and Gvozdjak [18] proved in 1992 that the list of forbidden subgraphs is infinite for the pseudosurface B_2 and thus, we have to consider each pseudosurface separately.

In fact, it is easy to obtain the characterization of outer embeddings in a surface starting from the one of embeddings in the same surface. However, there are known characterization theorems of graphs with embeddings in surfaces involving forbidden minors (or topological minors) only in the case of the projective plane. That characterization was by Archdeacon [4] in 1981, who gave a list of 35 forbidden minors and another with 103 forbidden topological minors.

It is also convenient to note that it is useful to characterize by forbidden minors every family of graphs defined by a hereditary property (that is, every minor of a graph verifying a certain property also verifies it). The same thing could be said in the case of topological minors. Moreover, in the case of minors, it can be deduced, starting from the results by Robertson and Seymour [17], that the lists of forbidden minors are always formed by a finite set of graphs.

2 NP-complete problems

Algorithms are deeply related to Graph Theory. To reach this conclusion, it is sufficient to read texts as the classic by Aho, Hopcroft and Ullman [1] or those by Garey and Johnson [10] or by Manber [15], for instance. Not only does graph theory gives lots of examples and problems which allow to develop efficient algorithms suitable of being used in other areas, but lots of graphs problems can be discussed with or simplified by their computational processing.

Particularly, the problem consisting on knowing if a graph admits a surface or pseudosurface embedding in such a way that all its vertices lie in the same face can be seen from an algorithmic point of view.

In algorithm design, a *problem* is a general question to be answered, usually possessing several parameters, or free variables, whose values are left unspecified. And an *algorithm* is a procedure "step by step" to solve a problem. Indeed, all definitions in this Section can be *formally* given (in terms of "languages" and "Turing machines") or *informally* (by using the terms of "problems" and "algorithms", only). We will simplify the following explanations by using [1, 10, 15]:

We describe the *decision problems* by using a name, a generic instance of the problem in the term of various components (which are graphs, numbers, etc.) and a question to be answered. This question is a *yes-no question* asked in terms of the generic instance. In general, we are interested in finding the most *efficient* algorithm for solving a problem (see [10]). To prove that a problem is NP-complete it is sufficient to reduce some other problem which we know that it is NP-complete to it.

The *orientable genus* of a graph G is the minor non-negative integer $\gamma = \gamma(G)$ such that G can be embedded in the orientable surface of genus γ. The genus graph problem is:

PROBLEM: **GRAPH GENUS (GG).**
INSTANCE: A graph G and a non-negative integer k.
QUESTION: Is $\gamma(G) \leq k$?

If $k = 0$, GG consists on knowing if a graph is planar. Hopcroft and Tarjan [12] (and other authors, separately) described a polynomial algorithm to solve this problem.

With respect to outerplanar graphs, Mitchell [16] gave in 1979 a linear algorithm based on the fact that the maximal outerplanar graphs (that is, those not admitting more edges without leaving of being outerplanar) are the triangulation of some polygon.

Apart from that, the NP-completeness of GG was proved by Thomassen in 1989 [19], who also proved in [20] that the analogous problem for cubic graphs is also NP-complete:

PROBLEM: **CUBIC GRAPH GENUS (CGG).**
INSTANCE: A cubic graph G and $k \in \mathbb{N}$.
QUESTION: Is $\gamma(G) \leq k$?

However, it is not always possible to reduce the NP-completeness of a problem for cubic graphs starting from a NP-complete problem which involves graphs with no restrictions.

To represent multigraphs, embeddings, faces, etc, in a combinatorial way, some definitions by Thomassen [19] can be used. Although we will only work with graphs, we must generalize the definition of embedding, in the case of pseudosurfaces:

Let $G = (V, E)$ be a connected graph, with $V = \{v_1, \ldots, v_n\}$. A *rotation system* on G is a set $\Pi = \{\pi_1, \ldots, \pi_n\}$, where π_i is a *cyclic permutation* of vertices of G adjoint to v_i.

We use this concept to give the combinatorial definition of embedding. In this expression, the compact surface where the graph is embedded is not named in an explicit way:

An *embedding* of a connected graph is a pair (G, Π), in which Π is a rotation system on G. Here, we say that Π is an embedding of G and π_i represents the *positive orientation* around v_i.

A Π-*face* is a finite sequence $x_0, \ldots, x_r$ of vertices of G such that $x_r = x_0$ and, if $x_i = v_j$, $x_{i+1} = \pi_j(x_{i-1})$ when $i = 1, 2, \ldots, r-1$ and $x_1 = \pi_j(x_{r-1})$ when $x_0 = v_j$. All the possible sequences will form the set C of Π-faces of (G, Π).

The Π-*genus* of G (or embedding genus Π of G) is the number $\gamma(G, \Pi)$ which verifies *Euler's Formula*:

$$|V| - |E| + |C| = 2 - 2\gamma(G, \Pi).$$

The *genus* of G, $\gamma(G)$, is combinatorially defined as the minimum of all $\gamma(G, \Pi)$, with Π chosen among all possible embeddings of G. Note that in this definition is mentioned the connected surface in which the embedding of G is considered, although in an implicit way.

Similarly, the graph G admits an outer-embedding (G, Π) if there exists a Π-face in C with all the vertices of G. Note that this definition allows us to answer, in a combinatorial way, the question: if S is an orientable connected surface of genus $g = g(S)$, does G admit an outer-embedding in S?

Given and fixed a connected graph $G = (V, E)$, with $V = \{v_1, \ldots, v_n\}$, a *rotation pseudosystem* on G is a set $\Pi = \{\pi_1, \ldots, \pi_n\}$, where π_i is a permutation of the vertices of G adjoint to v_i. π_i can be decomposed in a unique way as the product of disjoint cycles. The number of these cycles is denoted by ζ_i.

According to the previous definitions, an embedding of G in a pseudosurface is a pair (G, Π) in which Π is a rotation pseudosystem. Note that this concept contains and generalizes the corresponding to embedding in a surface and also allows to define Π-face and outer-embeddable graph, although several difficulties can now appear (it is similar to deal with cellular embeddings). The main difference with embeddings on surfaces is that it is not possible to determine a *genus* for pseudosurfaces, due to each embedding can correspond to different pseudosurfaces and it is really very difficult to give a suitable total order among them.

The ζ_i above defined coincides with the one that later, and in a more intuitive way, will be named as the minimum number of cells to which the singular point v_i belongs to. Equivalently, it corresponds with the number of disks which can be joined by one point to obtain a set homeomorphic to a neighbourhood of the singular point v_i in a pseudosurface admitting the embedding (G, Π). From now on and for this reason, we can obtain some results by omitting the formal definitions, as we will do next.

It is natural to consider in the first place the following problem (which will be NP-complete, as we will prove):

> PROBLEM: **GRAPH OUTER-PSEUDOGENUS (GOP).**
> INSTANCE: A graph G and a pseudosurface S.
> QUESTION: Does G admit an embedding on S having all its vertices in a same face?

Note that the multiple kinds of pseudosurfaces allows an easy proof of the following result:

Theorem 2.1 *GOP is NP-complete.*

Proof. We will prove the result by reducing GG to GOP:

Let us consider a finite graph $G = (V, E)$ and a non-negative integer k. Starting from S' (which will be the oriented surface of genus k) and G, we construct a pseudosurface S as follows:

We choose $|V|$ points of S' which will be denoted by $P_1, \ldots, P_{|V|}$ and we glue a sphere to S' for each of them. Finally, we choose a point belonging to each sphere (different from the P_i points) and we identify all of them as P_0. We will call S to the 2-pseudomanifold so constructed.

It is immediate that the construction of S is made in a polynomial time, starting from input data. To finish the proof, we will prove that G admits an embedding on S' if and only if G is outer-S:

[$\Rightarrow$] Given and fixed an embedding of G in S', it can be supposed that each vertex of G belongs to one of the spheres joined to S' (to generate S). P_0 is in the same face as all the vertices of G and thus, G is outer-S.

[⇐] Let us call C to one face of S in which all the vertices of G are and denote $C' = C \cap S'$. With no loss of generality it can be supposed that there are not either planar connected components or planar blocks. Let us consider an outer-embedding of G in S. We will study, separately, the possible cases:

1. If none of the P_i belongs to the embedding of G, G would be embedded in S'.
2. If P_0 is an inner point of an edge of G, such an edge is the only element of G being out of S' and such an edge can be represented in S' due to there exists an arc of curve between both singular points within C'. So, an embedding of G in S' is obtained.
3. If $P_0 \in V$, P_0 can be incident with edges such that all the singular points of G belongs to these edges except one point (because each singular point out of P_0 belongs to a unique edge and P_0 has to be in the same face as the rest of vertices).

   So, G is represented in S' with all its vertices in the same face C', except P_0 and the edges from P_0 to vertices of G. But P_0 can be represented in the inner of the face C' and thus, an embedding of G in S' can be given. □

As an immediate consequence of the previous proof we can reduce CGG to the following CGO:

PROBLEM: **CUBIC GRAPH OUTER-PSEUDOGENUS (CGO).** INSTANCE: A cubic graph G and a pseudosurface S. QUESTION: Does G admit an embedding in S having all its vertices in a same face?

Theorem 2.2 *CGO is NP-complete.* □

Note that we can even reduce GOP to certain pseudosurfaces and the problem continues being NP-complete. However, if we consider compact surfaces instead of pseudosurfaces, the answers are not usually easy. In this way, we conjecture that the following problem is NP-complete:

PROBLEM: **GRAPH OUTER-GENUS.** INSTANCE: A graph G and a non-negative integer k. QUESTION: Does G admit an embedding in a surface of genus k, having all its vertices in a same face?

3 Outer-embeddings in some pseudosurfaces

Although its theoretical interest, the knowledge that a problem is NP-complete does not provide a productive line of approach. It is usually more appropriate to concentrate on solving various special cases of the general problem. So, the following result allows to characterize outer-embeddings starting from graphs which can be embedded in surfaces or pseudosurfaces with a unique singular point.

Theorem 3.1 *Let S be a surface or a pseudosurface with an unique singular point. G is outer-S if and only if $G + K_1$ admits an embedding in S.*

Proof. If there exists, let P be the only singular point of S.
$\boxed{\Rightarrow}$ Let us consider an outer-S embedding of G. We distinguish two possibilities:

1. If S is a surface or P is in G, every face in the embedding is homeomorphic to one open disk and thus, the result follows for the same reason as in a connected surface: the outer-S embedding gives another embedding of $G + K_1$ in S.

2. If S is a pseudosurface and P is not in G, there is a face which is not homeomorphic to one open disk. If such a face is the unique containing all the vertices (on the opposite case, there would be no difficulty), it is sufficient to choose the point P to add K_1. By placing a vertex in P, all the faces of the graph are homeomorphic to one open disk and we can continue as in the previous case.

$\boxed{\Leftarrow}$ Given an embedding of $G + K_1$ in S, when suppressing the edges from K_1 to vertices of G, it is obtained an embedding of G in S in which K_1 belong to the same face as the rest of vertices. □

Next, we will deal with another kind of pseudosurfaces, having more than one singular point. Some of them have been already studied in other papers. Indeed, outer-embeddings in unions of several spheres for two points were characterized in [6], some results of [5] are related to embeddings in two surfaces joined by n points and other characterizations can be found in [9].

Although it could be thought that we are going to study other graphs in a quite small family, because we consider pseudosurfaces formed by spheres (in most cases, in a way that each of them has two singular points only), we will see next that this study has some interest:

Lemma 3.2 *Every graph admits an outer-embedding in some pseudosurface formed by spheres and having two singular points only in each of them.*

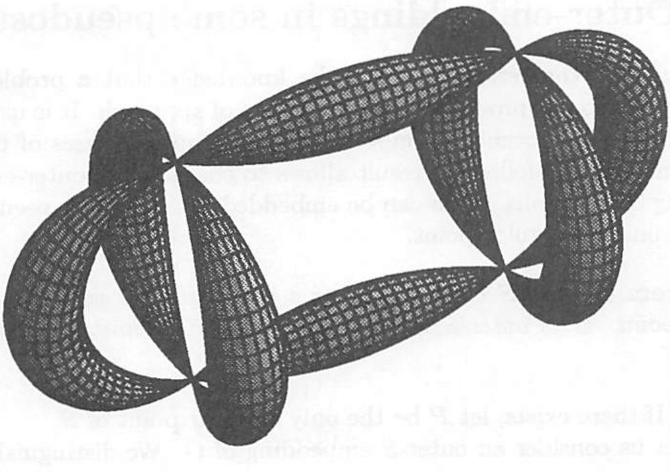

Figure 1: Pseudosurface named S_1.

Proof. Let $G = (V, E)$ be a finite graph. For each edge of E, we consider a sphere in which this edge is going to be represented. The vertices of the edge will be, obviously, the points which will be used to join the spheres themselves and to form a pseudosurface in which G admits an embedding with all its vertices in singular points. To obtain a spherical pseudosurface in which G admits an outer-embedding, it is sufficient to join one sphere for each singular point (that is, for each vertex of V) and to identify in an unique point, a non-singular point of each of these last spheres. As a result, we have an outer-embedding of G in a *simple spherical pseudosurface* with $|E| + |V|$ cells and $|V| + 1$ singular points. □

The remainder of this section will be devoted to the study of some properties about these outer-embeddings. First we use the pseudosurface S_1 from Figure 1 (which is obtained when joining eight spheres in a suitable way) to show an outer-S_1 graph with a non-outer-S_1 minor. In Figure 2 we show an outer-S_1 graph embedding, from which we can obtain a non-outer-S_1 minor (four K_4s joined by a vertex).

It is obvious that outer-embeddability in pseudosurfaces is a hereditary property for subgraphs. However, it is not for minors, as we have just seen.

Hence, a list of forbidden minors for outer-embeddings in S_1 does not exist.

By choosing two suitable closed curves of S_1 and by contracting to points, we obtain the pseudosurface S_2 of Figure 3, formed by ten spheres. We will use it to show an outer graph which loses this property when duplicating an edge.

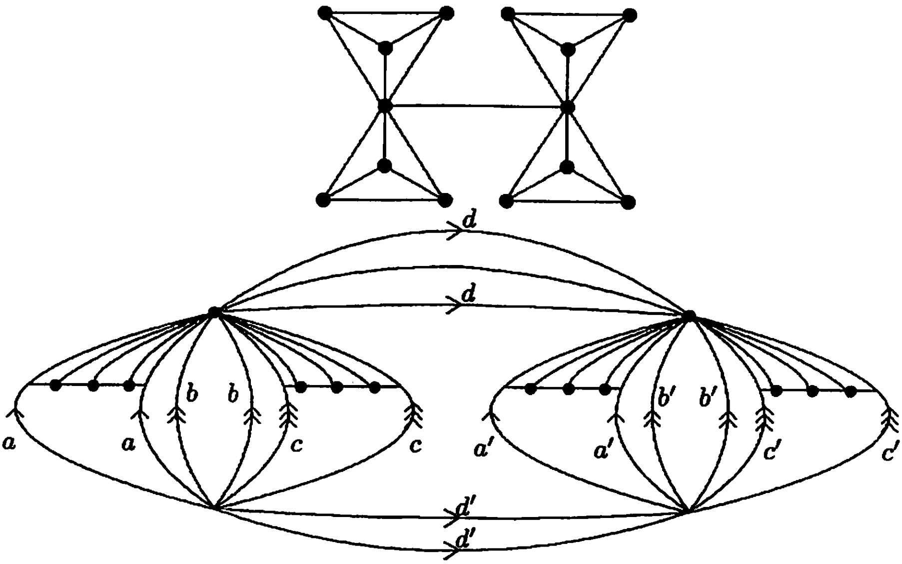

Figure 2: Outer-S_1 embedding.

When defining *multigraphs*, the difference with graphs appears for being allowed several different edges to have the same vertices. When there is more than one edge between two vertices, we say that there is a *multiple edge*.

Outer-embeddings in pseudosurfaces are not always conserved when duplicating an edge in a graph, as it can be checked in examples such as the following:

It is easy to prove that the multigraph of Figure 4 is not outer-S_2. However, the graph obtained from this one deleting one edge is outer-S_2.

It is possible to obtain lots of results related to the outer-embeddings of multigraphs in pseudosurfaces, as it can be seen in [9].

We have just given a pseudosurface without a list of forbidden-outer-minors. Now, we can assure something more: there exist pseudosurfaces without finite lists of forbidden-outer-topological-minors.

For instance, the pseudosurface S_3 of Figure 5 does not have a finite list of forbidden-outer-topological-minors. To prove this, it is sufficient to give an infinite family of graphs such that none of them is outer in such a pseudosurface, but verifying that every subgraph of any of them is.

Let us consider two copies of the graph $F = K_1 + (K_3 \cup K_3)$ and $n \geq 2$. We can define a graph H_n (see Figure 6), "connecting" the central vertices of the Fs by a "chain of K_3s".

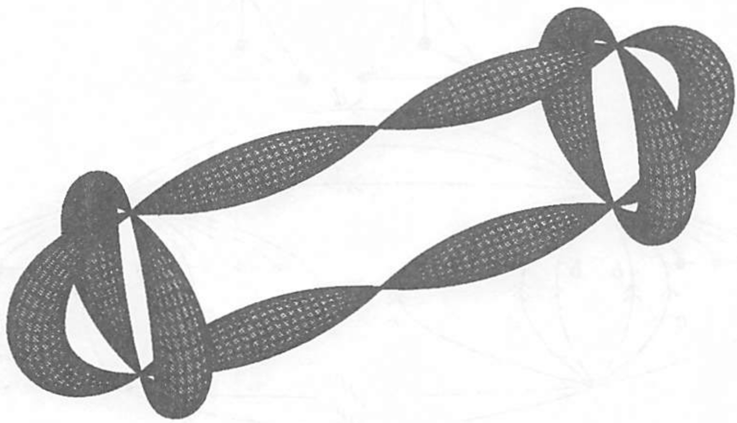

Figure 3: Pseudosurface named S_2.

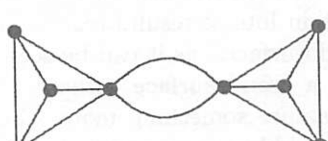

Figure 4: Non-outer-S_2 multigraph.

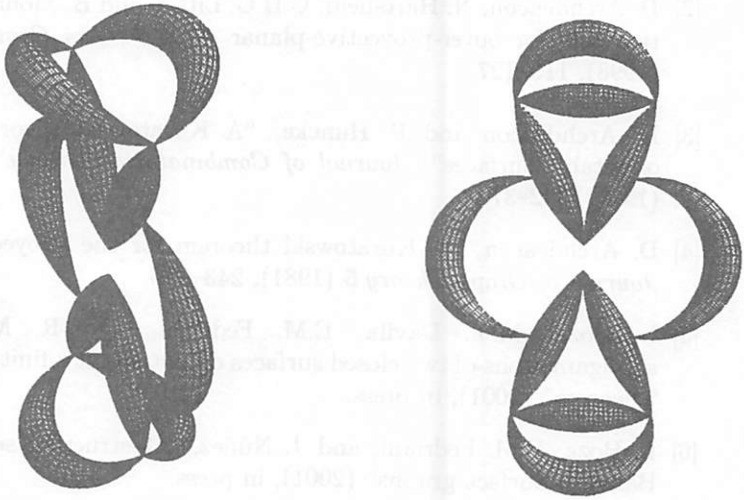

Figure 5: Two representations of S_3.

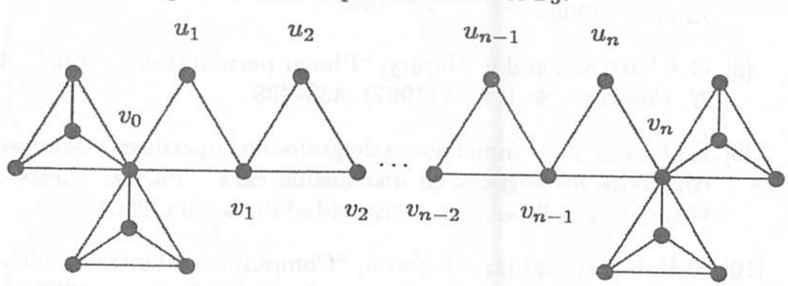

Figure 6: Infinite family of non-outer-S_3 graphs.

It is sufficient to analyze the possible cases (according to what singular points correspond to vertices) to prove that none of the generated H_n are outer-S_3 in the pseudosurface S_3 from Figure 5, in spite of their subgraphs being.

We think all these examples can be useful when dealing outer-embeddings in other pseudosurfaces.

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