Major Index for Standard Young Tableaux

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Abstract

For a standard tableau T of shape $\lambda \vdash n$, $\text{maj}(T)$ is the sum of i's such that $(i+1)$ appears in a row strictly below that of i in T. We consider the q-polynomial $f^{\lambda}(q) = \sum_T q^{\text{maj}(T)}$, which appears in many contexts: as a dimension of an irreducible representation of finite general linear group, as a special case of Kostka-Foulkes polynomials and so on. In this article we try to understand 'maj' on a standard tableau T in the relation to 'inv' on a multiset permutation(or a permutation of type λ). We construct an injective map from the set of standard tableaux to the set of permutations of *type* λ(increasing in each block) so that the 'maj' of the tableau is the 'inv' of the corresponding permutation when λ is a two part partition. We believe that this helps to understand irreducible unipotent representations of finite general linear groups.

1 Introduction

For a given partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n, we let $\text{SYT}(\lambda)$ denote the set of standard Young tableaux of shape λ. For a standard Young tableau T, the *major index* of T is given by

$$\text{maj}(T) = \sum_{i \in D(T)} i,$$

where $D(T) = \{i \mid i+1 \text{ is in a row strictly below that of } i \text{ in } T\}$. The following q-polynomial is a well known one that can be interpreted in many

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combinatorial ways [3, 4, 5, 8, 9], for example as a special case of Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$;

$$f^\lambda(q) = \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)} .$$

Moreover, $f^\lambda(q)$ gives the dimension of the unipotent representation S_q^λ of the finite general linear group $GL_n(q)$ (see [1, Theorem 15.16] and [4, Example 14 of Section 1.5]).

Another interesting q-polynomial in the relation to the unipotent representations of $GL_n(q)$ is the q-multinomial coefficient

$$g^\lambda(q) = \begin{bmatrix} n \\ \lambda_1, \lambda_2, \ldots, \lambda_l \end{bmatrix} = \frac{[n]!}{[\lambda_1]![\lambda_2]! \cdots [\lambda_l]!} ,$$

which calculates the dimension of the permutation representation M_q^λ of $GL_n(q)$([1, Chapter 10]), where $[a] = \frac{1-q^a}{1-q} = 1 + q + \cdots + q^{a-1}$ for $a > 0$, and $[a]! = [a][a-1] \cdots [1]$. We let W_λ be the subset of the permutation group S_n on n letters, whose elements satisfy the following condition;

$$\pi(s_i + 1) < \pi(s_i + 2) < \cdots < \pi(s_i + \lambda_{i+1}) \text{ for all } i = 0, 1, \ldots, l-1 ,$$

where $s_0 = 0$, $s_{j+1} = s_j + \lambda_{j+1}$, for $j = 0, 1, \ldots, l-2$. Then it is easy to check that

$$g^\lambda(q) = \sum_{\pi \in W_\lambda} q^{\mathrm{inv}(\pi)} .$$

The following is a well known result on the unipotent representation of $GL_n(q)$. Recall that the Kostka number $K_{\mu\lambda}$ is defined to be the number of semistandard(column strict) tableaux of shape μ and content λ.

Proposition 1 ([1]) *For a partition $\lambda \vdash n$,*

$$dim(M_q^\lambda) = \sum_\mu K_{\mu\lambda}\, dim(S_q^\mu) ,$$

where the sum is over all partitions of n and $K_{\mu\lambda}$ is the Kostka number.

By the above proposition, we can conclude that $g^\lambda(q) - f^\lambda(q)$ is a q-polynomial with nonnegative coefficients. Hence there must be a natural way to assign a permutation $\pi_T \in W_\lambda$ to each standard tableau T so that $\mathrm{maj}(T) = \mathrm{inv}(\pi_T)$. In this article, we construct an injection from SYT(λ) to W_λ so that the major index of the tableau is preserved as the inversion number of its image, when λ is a two part partition.

Remark.

1. In [2], K. Kadell considered another major index defined on SYT(λ). Nonnegativity of coefficients of $g^\lambda(q) - f^\lambda(q)$ can be proved by direct calculation or using the relation of Kadell's major index and 'maj', when λ is a two part partition.

2. For each $T \in$ SYT(λ), there exists a way to find a corresponding permutation in S_n so that the descent sets coincide [9, Lemma 7.19.6], and Foata bijection shows that the generating functions of 'inv(σ)' and 'maj(σ)' for $\sigma \in S_n$ coincide, where 'maj(σ)' is the sum of the descents of σ (see [6]). Hence, for each $T \in$ SYT(λ), there must be a way to find a corresponding permutation $\pi \in S_n$ so that maj(T) = inv(π). However, the set of permutations one obtain in this way does not have a nice way to identify.

2 Combinatorics

As we mentioned in Section 1, for any standard Young tableau T of shape λ, there must be a natural way to give a permutation $\pi_T \in W_\lambda$ so that $\mathrm{maj}(T) = \mathrm{inv}(\pi_T)$. In this section, we define a natural injection from SYT(λ) to W_λ with the expected property, when $\lambda = (n-k, k)$. For a given partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, we let λ' denote the *conjugate* of λ: the ith part of λ' is the length of the ith column of λ. We also let

$$b(\lambda) = \sum_{1 \le i \le l} (i-1)\lambda_i .$$

The following proposition is a well known result [7, Theorem 19].

Proposition 2 *$g^\lambda(q)$ is a symmetric(and unimodal) q-polynomial of minimum degree* 0 *and maximum degree* $\binom{n}{2} - b(\lambda')$, *i.e.* $c_i = c_{\binom{n}{2} - b(\lambda') - i}$ *for all i where c_i is the coefficient of q^i in $g^\lambda(q)$.*

From now on we only consider two part partitions $\lambda = (n-k, k)$. Proposition 2 shows that

$$\sum_{\pi \in W_\lambda} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in W_\lambda} q^{\binom{n}{2} - b(\lambda') - \mathrm{inv}(\pi)} . \qquad (1)$$

In the following, we define an injective map ϕ from SYT(λ) to W_λ so that $\mathrm{inv}(\phi(T)) = \binom{n}{2} - b(\lambda') - \mathrm{maj}(T)$ (instead of $\mathrm{inv}(\phi(T)) = \mathrm{maj}(T)$, that is justified by Equation (1)). Note that for $\pi \in W_\lambda$, $q^{\binom{n}{2} - b(\lambda') - \mathrm{inv}(\pi)}$ is the number of left cosets of type π of parabolic subgroup P^λ(upper block diagonal matrices of type λ) of the finite general linear group $GL_n(q)$. Hence we hope that our construction of ϕ would help doing combinatorial work on the unipotent representation of $GL_n(q)$.

We fix a permutation

$$w_\lambda = (k+1)\ (k+2)\ \cdots\ n\ 1\ 2\ \cdots\ k \in W_\lambda,$$

written as a word, then

$$\text{inv}(w_\lambda) = (n-k)k = \binom{n}{2} - b(\lambda').$$

Hence, for a given $T \in \text{SYT}(\lambda)$ we find a way to manipulate w_λ so that the inversion number decreases by $\text{maj}(T)$.

Given permutation $w \in S_n$, written as a word, we call the first $n-k$ numbers the first *block* and next k numbers the second *block*. For $a, b \in \{1, 2, \ldots, n\}$ such that $a \neq b$, $ex_{a,b}$ is an operator defined on W_λ, which exchanges a and b then makes each block increasing so that $ex_{a,b}(w) \in W_\lambda$ for $w \in W_\lambda$. For a permutation $w \in W_\lambda$, we let $R(w)$ be the set of pairs (x, y), $x < y$, where x, y are in the first and the second block of w respectively.

Lemma 3 *Let $A = \{k+1, k+2, \ldots, n\}$ and $B = \{1, 2, \ldots, k\}$, then For each $(a, b) \in A \times B$, $ex_{a,b}$ decreases the inversion number of w_λ by $a-b$ i.e. $inv(ex_{a,b}(w_\lambda)) = inv(w_\lambda) - (a-b)$.*

Proof. Note that there is no inversion inside each block of $ex_{a,b}(w_\lambda)$ since $ex_{a,b}$ makes each block increasing. Hence, to determine the number of inversions of $ex_{a,b}(w_\lambda)$, we only need to count the pairs (x, y) in $R(ex_{a,b}(w_\lambda))$. It is easy to find them as

$$\{(m, a)\,|\, m = k+1, \ldots, a-1\} \cup \{(b, m)\,|\, m = b+1, \ldots, k\} \cup \{(b, a)\}$$

whose size is $(a-k-1) + (k-b) + 1 = a-b$. This completes the proof. ∎

For T a standard Young tableau of shape λ, we let $T_{i,j}$ be the (i,j)th entry of the tableau T. Then we define $\phi(T)$ as follows:

<u>**Algorithm**</u>

1. Let $w = w_\lambda$.
2. For $l = 1$ to $n-k$ do
   if $T_{1,l} \in D(T)$ then
   let $a = k+l$, $b = k+l-T_{1,l}$ and $w = ex_{a,b}(w)$
   end do
3. $\phi(T) = w$. ∎

We have to show that the algorithm is well defined and $\phi(T)$ defined as above has the expected property.

Lemma 4 *1. If $1 \le l \le n-k$ and $T_{1,l} \in D(T)$, then $1 \le k+l-T_{1,l} \le k$.*

2. Let $D(T) = \{T_{1,l_1}, T_{1,l_2}, \ldots, T_{1,l_d}\}$ for $1 \le l_1 < l_2 < \cdots < l_d \le n-k$ and let $a_i = k+l_i$, $b_i = k+l_i-T_{1,l_i}$ for $i = 1, \ldots, d$. Then $f_\sigma(w_\lambda) = f(w_\lambda)$ for any $\sigma \in S_d$, where $f = ex_{a_d,b_d} ex_{a_{d-1},b_{d-1}} \cdots ex_{a_1,b_1}$ and $f_\sigma = ex_{a_{\sigma(d)},b_{\sigma(d)}} ex_{a_{\sigma(d-1)},b_{\sigma(d-1)}} \cdots ex_{a_{\sigma(1)},b_{\sigma(1)}}$.

3. $inv(f(w_\lambda)) = \binom{n}{2} - b(\lambda') - \sum_{i=1}^{d}(a_i - b_i)$

Proof. Since T is standard, $T_{1,l}$ should be at least l. Since T is standard and $T_{1,l} \in D(T)$, there are at least $(n-k-l)+1$ positions in T that should have bigger numbers than $T_{1,l}$, hence $T_{1,l}$ must be at most $k+l-1$.

For the second part, note that $b_d < \cdots < b_1 \le k < a_1 < \cdots < a_d$. Therefore, it is easy to see that $f_\sigma(w_\lambda) = f(w_\lambda)$ for any $\sigma \in S_d$.

As we showed in Lemma 3,

$$\begin{aligned} R(ex_{a_1,b_1}(w_\lambda)) &= \{(m, a_1) \mid m = k+1, \ldots, a_1 - 1\} \cup \\ &\quad \{(b_1, m) \mid m = b_1+1, \ldots, k\} \cup \{(b_1, a_1)\}. \end{aligned}$$

Note that a_2 and b_2 do not appear as components of pairs in $R(ex_{a_1,b_2}(w_\lambda))$. Therefore, $R(ex_{a_1,b_1}(w_\lambda)) \subseteq R(ex_{a_2,b_2}(ex_{a_1,b_1}(w_\lambda)))$. Moreover, by applying ex_{a_2,b_2} to $ex_{a_1,b_1}(w_\lambda)$, we obtain more pairs:

$$\tau_1(R(ex_{a_2,b_2}(w_\lambda))) = \{(\tau_1(x), \tau_1(y)) \mid (x,y) \in R(ex_{a_2,b_2}(w_\lambda))\},$$

where τ_1 is the transposition (a_1, b_1).

Note that $R(ex_{a_1,b_1}(w_\lambda))$ and $\tau_1(R(ex_{a_2,b_2}(w_\lambda)))$ are disjoint. Hence $R(ex_{a_2,b_2}(ex_{a_1,b_1}(w_\lambda))) = R(ex_{a_1,b_1}(w_\lambda)) \cup \tau(R(ex_{a_2,b_2}(w_\lambda)))$ is of size $(a_1 - b_1) + (a_2 - b_2)$.

If we keep doing this procedure, then we can show that

$$R(f(w_\lambda)) = R(ex_{a_1,b_1}(w_\lambda)) \cup \left(\bigcup_{i=2}^{d} \tau_{i-1} \cdots \tau_1(R(ex_{a_1,b_1}(w_\lambda))) \right),$$

where $\tau_i = (a_i, b_i)$ is the transposition of a_i and b_i, and $|R(f(w_l))| = \sum_{i=1}^{d}(a_i - b_i)$. This completes the proof. ∎

Previous lemma shows that each time we apply the Step 2 of the algorithm, we decrease the inversion number exactly as we expect: the process does not depend on the previous step. Hence we have the following theorem.

Theorem 5 *For $T \in STY(\lambda)$, $\phi(T) \in W_\lambda$ is well defined and $inv(\phi(T)) = \binom{n}{2} - b(\lambda') - maj(T)$.*

Example. Let $\lambda = (4,2)$ then $w_\lambda = 345612 \in W_\lambda$. If

$$T = \begin{matrix} 1 & 2 & 4 & 5 \\ 3 & 6 & & \end{matrix},$$

then $D(T) = \{T_{1,2}, T_{1,4}\} = \{2,5\}$. Moreover, $a_1 = k+2 = 2+2 = 4$, $b_1 = k+2-T_{1,2} = 2+2-2 = 2$ and $a_2 = 6$, $b_2 = 1$. Hence $\phi(T) = ex_{6,1}(ex_{4,2}(w_\lambda)) = ex_{6,1}(235614) = 123546$.

We can check that

$$\begin{aligned} R(\phi(T)) &= \{(1,4),(1,6),(2,4),(2,6),(3,4),(3,6),(5,6)\} \\ &= R(ex_{4,2}(w_\lambda)) \cup (4,2)(R(ex_{6,1}(w_\lambda))) \\ &= \{(2,4),(3,4)\} \cup (4,2)\{(1,2),(1,6),(3,6),(4,6),(5,6)\} \\ &= \{(2,4),(3,4)\} \cup \{(1,4),(1,6),(3,6),(2,6),(5,6)\} \end{aligned}$$

and $\mathrm{inv}(\phi(T)) = 1$, $\binom{6}{2} - b(\lambda') - \mathrm{maj}(T) = 15 - 7 - 7 = 1$.

The representatives of type $\phi(T)$ of the left cosets of $P^{(4,2)}$ in $GL_6(q)$ are of the following forms:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & * & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ * & * & * & * & 0 & 1 \end{pmatrix}$$

where each $*$ represents an element of $\mathbb{F}_q$. Hence there are

$$q^{\mathrm{maj}(T)} = q^{\binom{6}{2} - b(\lambda') - \mathrm{inv}(\phi(T))} = q^7$$

many representatives of type $\phi(T)$. ∎

Remark. $g^\lambda(q) - f^\lambda(q)$ has nonnegative coefficients for all λ, hence there must be a natural injection ϕ from SYT(λ) to W_λ so that $\mathrm{inv}(\phi(T)) = \binom{n}{2} - b(\lambda') - \mathrm{maj}(T)$, we, however were not able to do that work for general λ.

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