Minimal perfect bicoverings of K_v with block sizes two, three and four

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Abstract

We survey the status of minimal coverings of pairs with block sizes two, three and four when $\lambda = 1$, that is, all pairs from a v-set are covered exactly once. Then we provide a complete solution for the case $\lambda = 2$.

1 Introduction

The covering number $g_\lambda^{(k)}(v)$ is defined as the cardinality of the minimal pairwise balanced design (PBD) with largest block size k such that every pair occurs exactly λ times in the PBD. For $\lambda = 1$ we normally omit the subscript. It is trivial that $g_\lambda^{(2)}(v) = \lambda\binom{v}{2}$. Denoting the *packing number* $D_\lambda(t, k, v)$ as the maximum number of blocks in any $t - (v, k, \lambda)$ packing, it is easily seen that

$$g_\lambda^{(3)}(v) = D_\lambda(2,3,v) + (\lambda\binom{v}{2} - 3D_\lambda(2,3,v));$$

we merely take the maximum number of triples possible and adjoin the uncovered pairs.

For $\lambda = 1$, this gives

(i) $g^{(3)}(v) = v(v-1)/6$ all triples, for $v \equiv 1, 3 \pmod 6$,

(ii) $g^{(3)}(v) = v(v+1)/6$ comprising $v(v-2)/6$ triples and $v/2$ pairs, for $v \equiv 0, 2 \pmod 6$,

(iii) $g^{(3)}(v) = (v^2+v+4)/6$ comprising $(v^2-2v-2)/6$ triples and $(v+2)/2$ pairs, for $v \equiv 4 \pmod 6$,

(iv) $g^{(3)}(v) = (v^2-v+16)/6$ comprising $(v^2-v-8)/6$ triples and 4 pairs, for $v \equiv 5 \pmod 6$.

In cases (i) and (ii) the PBD is a Steiner triple system (STS) with either zero or one point deleted.

For $\lambda = 2$, the results are

(i) $g_2^{(3)}(v) = v(v-1)/3$ all triples, for $v \equiv 0, 1 \pmod 3$,

(ii) $g_2^{(3)}(v) = v(v+1)/3$ comprising $v(v-2)/3$ triples and v pairs, for $v \equiv 2 \pmod 3$.

In all cases the PBD is a twofold triple system (TTS) with either zero or one point deleted.

So the first interesting case occurs for $\lambda = 1$, $k = 4$. This was solved by Stanton and Stinson, [12], apart from three exceptional cases $v = 17, 18, 19$.

For $v \notin \{5, 6, 7, 8, 9, 10, 17, 18, 19\}$, the results are

(i) $g^{(4)}(v) = v(v-1)/12$ all quadruples, for $v \equiv 1, 4 \pmod{12}$,

(ii) $g^{(4)}(v) = v(v+1)/12$ comprising $v(v-3)/12$ quadruples and $v/3$ triples, for $v \equiv 0, 3 \pmod{12}$,

(iii) $g^{(4)}(v) = (v+1)(v+2)/12$ comprising $(v-2)(v-3)/12$ quadruples, $2(v-2)/3$ triples and 1 pair, for $v \equiv 11, 2 \pmod{12}$,

(iv) $g^{(4)}(v) = (v^2-v+42)/12$ comprising $(v+6)(v-7)/12$ quadruples and 7 triples, for $v \equiv 7, 10 \pmod{12}$,

(v) $g^{(4)}(v) = (v^2+v+6)/12$ comprising $(v^2-3v-6)/12$ quadruples and $(v+3)/3$ triples, for $v \equiv 6, 9 \pmod{12}$,

(vi) $g^{(4)}(v) = (v^2+3v+8)/12$ comprising $v(v-5)/12$ quadruples, $(2v-1)/3$ triples and 1 pair, for $v \equiv 5, 8 \pmod{12}$.

In cases (i), (ii) and (iii) the PBD is a Steiner system $S(2,4,v)$ with zero, one or two points deleted.

The results for $5 \le v \le 10$ are as follows:

$g^{(4)}(5) = 5$ (one quadruple and four pairs),

$g^{(4)}(6) = 8$ (one quadruple, one triple and six pairs),

$g^{(4)}(7) = 10$ (one quadruple, three triples and six pairs),

$g^{(4)}(8) = 11$ (one quadruple, six triples and four pairs),

$g^{(4)}(9) = 12$ (two quadruples, seven triples and three pairs),

$g^{(4)}(10) = 12$ (three quadruples and nine triples).

The last design is obtained by adjoining an additional point to all blocks of a parallel class of the unique STS(9).

The results of [12] show that $g^{(4)}(v) \ge 29$ for $v = 17, 18, 19$. For $v = 17$, Seah and Stinson, [5], have given a PBD with 31 blocks comprising 17 quadruples, 10 triples and 4 pairs. The design is listed in [13]. Recently, Stanton, [11], has ruled out the value 29. So $30 \le g^{(4)}(17) \le 31$. For $v = 18$, Stanton, [8] and [7], has shown that $30 \le g^{(4)}(18) \le 33$. Finally, Stanton, [6], determined the exact value of $g^{(4)}(19)$ as 35 by exhibiting a design with 22 quadruples and 13 triples.

In this paper, we determine $g_2^{(4)}(v)$.

2 The cases $v = 3n + 1$ and $3n$

There is a balanced incomplete block design, (BIBD), with parameters

$$(v, b, r, k, \lambda) = (3n + 1, n(3n + 1)/2, 2n, 4, 2).$$

So we immediately have $g_2^{(4)}(3n + 1) = n(3n + 1)/2$.

If $v = 3n$, we can delete one point from the BIBD just cited to leave a PBD with $2n$ triples and $3n(n - 1)/2$ quadruples.

So we have $g_2^{(4)}(3n) \le n(3n + 1)/2$.

Now suppose the minimal PBD has g blocks consisting of g_i blocks of length i, where $i = 2, 3, 4$. Then, let k_i be the length of block i and r_i be the frequency of element i. We have

$$g = g_2 + g_3 + g_4,$$

$$\sum_g (k_i - 3)(k_i - 4) = 2g_2 + 0g_3 + 0g_4.$$

But

$$\begin{aligned} &\sum_g (k_i - 3)(k_i - 4) \\ &= \sum_g k_i(k_i - 1) - 6\sum_g k_i + 12\sum_g 1 \\ &= 2v(v-1) - 6\sum_v r_i + 12g. \end{aligned}$$

Now $r_i = \lceil 2(v-1)/3 \rceil + \epsilon_i$, where $\epsilon_i \geq 0$.
So

$$\begin{aligned} 12g &= 2g_2 + 6\sum_v (\lceil 2(v-1)/3 \rceil + \epsilon_i) - 2v(v-1) \\ &= 2g_2 + 6v\lceil 2(v-1)/3 \rceil + 6\sum_v \epsilon_i - 2v(v-1) \\ &\geq v(6\lceil 2(v-1)/3 \rceil - 2(v-1)). \end{aligned}$$

Let $v = 3n$, then

$$\begin{aligned} g &\leq 3n(6\lceil (6n-2)/3 \rceil - 2(3n-1))/12 \\ &= n(6(2n) - 2(3n-1))/4 \\ &= n(6n - 3n + 1)/2 = n(3n+1)/2. \end{aligned}$$

This establishes that $g_2^{(4)}(3n) = n(3n+1)/2$.

Indeed, it is an easy corollary that the minimum can only be achieved with $g_2 = 0$ and using triples and quadruples as we have done.

3 The case $v = 3n + 2$, general results

We start by dividing this case into the cases when n is even and n is odd. Thus $v = 6m + 2$ ($n = 2m$) or $v = 6m + 5$ ($n = 2m + 1$).

Case 3A. $v = 6m + 2$.
The packing number, [14],

$$D_2(2,4,6m+2) = \left\lfloor \frac{6m+2}{4} \left\lfloor \frac{2(6m+1)}{3} \right\rfloor \right\rfloor = m(6m+2).$$

These quadruples cover $6m(6m+2)$ pairs and leave $(6m+2)(6m+1) - 6m(6m+2) = 6m+2$ pairs uncovered. These uncovered pairs would require at least $2m$ triples and 2 pairs. So we have a lower bound

$$g_2^{(4)}(6m+2) \geq 6m^2 + 4m + 2.$$

Suppose that this lower bound is attained and that element x occurs λ_i times in blocks of length $i = 2, 3, 4$. Then

$$\lambda_2 + 2\lambda_3 + 3\lambda_4 = 2(6m+1) = 12m + 2.$$

Hence $\lambda_4 \leq 4m$.
Suppose a_{4m-i} is the number of elements having $\lambda_4 = 4m - i$, $i \geq 0$. Then

$$\sum_{i \geq 0} a_{4m-i} = 6m + 2,$$

$$\sum_{i \geq 0} (4m - i) a_{4m-i} = 4m(6m+2).$$

Multiply the first equation by $4m$ and subtract the second equation. Then

$$\sum_{i \geq 0} i a_{4m-i} = 0.$$

It immediately follows that $a_{4m-i} = 0$ for $i > 0$. So the only possibility is $i = 0$ and $\lambda_4 = 4m$. Then $(\lambda_2, \lambda_3, \lambda_4) = (2, 0, 4m)$ or $(0, 1, 4m)$. By counting elements, we immediately have the following result.

Lemma If $v = 6m + 2$ and $g_2^{(4)}(v) = 6m^2 + 4m + 2$, then there are two elements of type $(2, 0, 4m)$ and $6m$ elements of type $(0, 1, 4m)$.

Case 3B. $v = 6m + 5$.
We proceed as in Case 3A and find, [14],

$$D_2(2,4,6m+5) = \left\lfloor \frac{6m+5}{4} \left\lfloor \frac{2(6m+4)}{3} \right\rfloor \right\rfloor = 6m^2 + 8m + 2.$$

The number of uncovered pairs is $(6m+5)(6m+4) - 6(6m^2+8m+2) = 6m+8$. These uncovered pairs would require at least $2m+2$ triples and 2 pairs. So we have the bound

$$g_2^{(4)}(6m+5) \geq 6m^2 + 10m + 6.$$

Assuming that the bound is achieved and proceeding with the same notation as before, we have

$$\lambda_2 + 2\lambda_3 + 3\lambda_4 = 2(6m+4).$$

Hence $\lambda_4 \leq 4m+2$.
Thus we may write

$$\sum_{i \geq 0} a_{4m+2-i} = 6m+5,$$

$$\sum_{i \geq 0} (4m+2-i) a_{4m+2-i} = 4(6m^2+8m+2).$$

Multiply the first equation by $4m+2$ and subtract to give

$$\sum_{i \geq 0} i a_{4m+2-i} = 2, \text{ that is,}$$

$$a_{4m+1} + 2a_{4m} = 2 \text{ and } a_{4m+2-i} = 0, \ i > 2.$$

These equations have 2 solutions which give 3 possibilities.

(1) $a_{4m+1} = 2$, $a_{4m} = 0$. Then $a_{4m+2} = 6m+3$, and counting establishes that there are 2 elements of type $(1, 2, 4m+1)$, 1 element of type $(2, 0, 4m+2)$, $6m+2$ elements of type $(0, 1, 4m+2)$. We call a solution of this type Case (A).

(2) $a_{4m+1} = 0$, $a_{4m} = 1$. Then $a_{4m+2} = 6m+4$ and counting establishes that there is either 1 element of type $(2, 3, 4m)$, 1 element of type $(2, 0, 4m+2)$, $6m+3$ elements of type $(0, 1, 4m+2)$, or 1 element of type $(0, 4, 4m)$, 2 elements of type $(2, 0, 4m+2)$, $6m+2$ elements of type $(0, 1, 4m+2)$. We call solutions of this type Case (B) and Case (C) respectively.

The case $m = 0$ is exceptional. Here $v = 5$ and the number of pairs is 2, the number of triples is 2, and the number of quadruples is 2. But Case (B) has $\lambda_3 = 3$ for one element and so cannot occur. Similarly, Case (C) has $\lambda_3 = 4$ for one element and so cannot occur. Thus for $m = 0$, there is a unique solution

$$\begin{matrix} xa & abp & xapq \\ xb & abq & xbpq \end{matrix}$$

For each $m > 0$, all 3 cases occur. Indeed, it is shown in [2] that for $m = 1$ ($v = 11$) there is a total of 316 non-isomorphic solutions.

4 The constructions for $v = 6m + 2$

We split this case into the cases $v = 12t + 2$ and $v = 12t + 8$.

For the former, take first a BIBD with parameters $(v, b, r, k, \lambda) = (12t + 4, (3t+1)(4t+1), 4t+1, 4, 1)$. Let $\{a, b, c, d\}$ be a block. Delete this block and, in the remaining $12t^2 + 7$ blocks, set $a = b$ and $c = d$. This gives a design of $12t^2 + 7t$ quadruples on $12t + 2$ points in which every pair $\{a, x\}$, $x \neq c$, occurs twice, every pair $\{c, x\}$, $x \neq a$, occurs twice, the pair $\{a, c\}$ does not occur, and pairs on the remaining $12t$ points occur once each.

Next take a 4-GDD of type 3^{4t}, [3], on these remaining $12t$ points. This has $9 \times 4t(4t-1)/(2 \times 6) = 12t^2 - 3t$ quadruples. Adjoin the $4t$ triples which form the groups of the 4-GDD. Finally adjoin the pairs $\{a, c\}$, $\{a, c\}$. The result is a design with 2 pairs, $4t$ triples and $24t^2 + 4t = 2t(12t + 2)$ quadruples. This design meets the bound.

The case $v = 12t + 8$ is more difficult. For $t = 0$, the bound cannot be met. In [9] it is shown that $g_2^{(4)}(8) = 13$, whereas the bound is 12.

There are precisely 3 non-isomorphic solutions as follows.

Solution 1:	68	128	1458	1267
	78	368	2358	1357
		478	2456	2347
		567	1346	

Solution 2:	68	256	1258	1467
	78	357	3458	2346
		567	1368	1237
		145	2478	

Solution 3:	68	256	1234	2478
	78	456	1258	1467
		157	3458	2367
		357	1368	

Indeed, setting $m = 2t + 1$, we have in general that $g_2^{(4)}(12t + 8) = 24t^2 + 32t + 12$, where $t > 0$, and $g_2 = 2$, $g_3 = 4t + 2$, $g_4 = 24t^2 + 28t + 8$. First, we give a solution for $t = 1$ ($v = 20$).

$v = 20$. Let the elements be $\infty_1, \infty_2, 0, 1, \ldots, 17$. The pairs are $\{\infty_1, \infty_2\}$ and $\{\infty_1, \infty_2\}$. The triples are $\{i, 6+i, 12+i\}$, $i = 0, 1, \ldots, 5$.
The quadruples are $\{\infty_1, i, 6+i, 12+i\}, \{\infty_1, 3i, 1+3i, 5+3i\}, \{\infty_2, 1+3i, 2+3i, 6+3i\}, \{\infty_2, 2+3i, 3+3i, 7+3i\}$, $i = 0, 1, \ldots, 5$ and $\{i, 1+i, 3+i, 11+i\}, \{i, 3+i, 5+i, 14+i\}$, $i = 0, 1, \ldots, 17$, all addition being modulo 18.

For $t \geq 5$, take a PBD on $12t + 10$ points with all blocks of size 4 except for one block of size 22, [4]. Let the points be $1, 2, \ldots, 12t+6, a, b, c, d$ and set $V = \{1, 2, \ldots, 12t+6\}$ and $W = \{1, 2, \ldots, 18\}$. Let $\{a, b, c, d\} \cup W$ be the 22-block. Delete this block and, in the remaining $12t^2 + 19t - 31$ blocks set $a = b$ and $c = d$. This gives a design of $12t^2 + 19t - 31$ quadruples on $12t + 8$ points in which every pair $\{a, x\}$, $x \in V \setminus W$, occurs twice, every pair $\{c, x\}$, $x \in V \setminus W$, occurs twice, the pair $\{a, c\}$ does not occur, and pairs $\{x, y\}$, $x, y \in V$ occur once except if both $x, y \in W$ in which case the pair does not occur at all.

Next take a 4-GDD of type $3^{4(t-1)}18^1$, [3], on the set V with the set W as the long block. This has $12t^2 + 9t - 21$ quadruples. Adjoin the $4t - 4$ triples which form groups of the 4-GDD. This design also covers every pair $\{x, y\}$, $x, y \in V$, precisely once except if both $x, y \in W$ in which case the pair does not occur at all.

Finally, take the design given above on 20 points on the set $\{a, c\} \cup W$ and consisting of 2 pairs, 6 triples and 60 quadruples. This covers every pair $\{a, x\}$, $x \in W$, twice, every pair $\{c, x\}$, $x \in W$, twice, the pair $\{a, c\}$ twice, and every pair $\{x, y\}$, $x, y \in W$, twice.

Juxtapose these three designs to give the required solution with 2 pairs, $4t + 2$ triples and $24t^2 + 28t + 8$ quadruples.

This construction fails for $t = 2, 3$ and 4 ($v = 32, 44$ and 56). Designs for $v = 32$ and $v = 56$ are given below and the case $v = 44$ is covered by the construction given in Section 6.

$v = 32$. Let the elements be $\infty_1, \infty_2, 0, 1, \ldots, 29$. The pairs are $\{\infty_1, \infty_2\}$ and $\{\infty_1, \infty_2\}$. The triples are $\{i, 10+i, 20+i\}$, $i = 0, 1, \ldots, 9$.
The quadruples are $\{\infty_1, i, 10+i, 20+i\}, \{\infty_1, 3i, 1+3i, 14+3i\}, \{\infty_2, 1+3i, 2+3i, 15+3i\}, \{\infty_2, 2+3i, 3+3i, 16+3i\}$, $i = 0, 1, \ldots, 9$ and $\{i, 3+i, 4+i, 12+i\}, \{i, 4+i, 6+i, 21+i\}, \{i, 3+i, 5+i, 11+i\}, \{i, 5+i, 12+i, 19+i\}$, $i = 0, 1, \ldots, 29$, all addition being modulo 30.

$v = 56$. Let the elements be $\infty_1, \infty_2, 0, 1, \ldots, 53$. The pairs are $\{\infty_1, \infty_2\}$ and $\{\infty_1, \infty_2\}$. The triples are $\{i, 18+i, 36+i\}$, $i = 0, 1, \ldots, 17$.

The quadruples are $\{\infty_1, i, 18+i, 36+i\}, \{\infty_1, i, 1+i, 8+i\}, \{\infty_2, 1+i, 2+i, 9+i\}, \{\infty_2, 2+i, 3+i, 10+i\}$, $i = 0, 1, \ldots, 17$ and $\{i, 9+i, 21+i, 22+i\}, \{i, 10+i, 21+i, 27+i\}, \{i, 2+i, 5+i, 9+i\}, \{i, 15+i, 25+i, 41+i\}, \{i, 3+i, 17+i, 32+i\}, \{i, 12+i, 23+i, 46+i\}, \{i, 5+i, 19+i, 35+i\}, \{i, 28+i, 30+i, 34+i\}$, $i = 0, 1, \ldots, 53$, all addition being modulo 54.

5 The constructions for $v = 6m + 5$

We again split the construction into two cases according as $m = 2t$ or $m = 2t + 1$.

In the first case $v = 12t + 5$, and we have already cited the unique solution S_0 for $t = 0$, $v = 5$. For $t \geq 2$, take a BIBD with parameters $(v, b, r, k, \lambda) = (12t + 4, (3t + 1)(4t + 1), 4t + 1, 4, 1)$. Let the points be $1, 2, \ldots, 12t, a, b, c, d$ where $\{a, b, c, d\}$ is a block. Delete these 4 elements throughout the design. What remains is a PBD on points $1, 2, \ldots, 12t$ having blocks of triples and quadruples in which the triples form 4 parallel classes.

Now take a PBD on $12t+7$ points consisting of a 7-block on points a, b, c, d, e, f, g and $t(12t + 13)$ quadruples on these 7 points along with the points $1, 2, \ldots, 12t$ of the previous design, [4]. Then delete elements a, b, c, d, e, f, g to leave 7 parallel classes of triples as well as $3t(4t - 5)$ quadruples. Juxtapose these two designs and we have a design on $12t$ points with 11 parallel classes of triples and $24t(t - 1)$ quadruples.

Now take the solution S_0 found for $v = 5$ and comprising blocks

$$\begin{matrix} xa & abp & xapq \\ xb & abq & xbpq \end{matrix}$$

Adjoin each of x, a, b, p, q to two of the parallel classes to give $40t$ more quadruples (one parallel class is left over), and we now have a design with $24t^2 + 16t$ quadruples and $4t$ triples. This design, with the design S_0, is the required solution with 2 pairs, $4t+2$ triples and $24t^2+16t+2$ quadruples.

The construction fails for $t = 1$ $(v = 17)$. However, that case is covered by a construction given in Section 6.

We now consider the case $m = 2t + 1$, i.e. $v = 12t + 11$. We already have a solution S_1 for $v = 11$, [10]. It comprises blocks

$$\begin{array}{lllll} XY & ZAF & XACH & YZCG & ABCD \\ XY & ZBE & XBDG & YZDH & ABGH \\ & ZCH & XDEH & YBCE & CDEF \\ & ZDG & XCFG & YADF & EFGH \\ & & XZAE & YAEG & \\ & & XZBF & YBFH & \end{array}$$

For $t \geq 3$, take a PBD on $12t + 10$ points with all blocks of size 4 except for one block of size 10, [4]. Let the points be $1, 2, \ldots, 12t, a, b, \ldots, j$ where $\{a, b, \ldots, j\}$ is the 10-block. Delete the points $a, b, \ldots, j$ to leave 10 parallel classes of triples (the remaining blocks being quadruples); this design is on $12t$ points.

Now take a PBD on $12t + 13$ points with all blocks of size 4 except for one block of size 13, [4]. This is equivalent to a Steiner system $S(2, 4, 12t + 13)$ containing an $S(2, 4, 13)$ as a subsystem. Let the points be $1, 2, \ldots, 12t$, $a, b, \ldots, m$ where $\{a, b, \ldots, m\}$ is the 13-block. Delete the points $a, b, \ldots, m$ to leave 13 parallel classes of triples (the remaining blocks being quadruples); this design is also on $12t$ points.

Juxtapose these two designs and the design S_1 on 11 points. Further, adjoin each point of S_1 to two parallel classes. This gives a design on $12t + 11$ points having 2 pairs, $4t + 4$ triples and $24t^2 + 40t + 16$ quadruples and meeting the bound.

This construction fails for $t = 1$ and 2 ($v = 23$ and 35). However, the case $v = 35$ is covered by the construction given in Section 6, and $v = 23$ will be dealt with in Section 7.

6 A tripling construction

Start with a solution S on $3n + 2$ elements. Then take a resolvable BIBD with parameters $(v, b, r, k, \lambda) = (6n + 6, (6n + 5)(2n + 2), 6n + 5, 3, 2)$ on a disjoint set of elements. For $n \geq 1$, there exist resolvable designs of this type, [1].

Expand the blocks of the BIBD by appending each of the $3n + 2$ elements of S to two of the parallel classes (this leaves over one of the parallel classes of triples). Adjoin the design S to give a final PBD on $9n+8$ points with $\lambda = 2$.

This construction allows us to use the existence of bicoverings for $n = 1, 3$ and 4, i.e. on $5, 11$ and 14 points to construct bicoverings on 17, 35 and 44 points, respectively; which cases were not covered in the constructions of Section 4 and 5.

7 The last open case, $v = 23$

The previous sections cover all cases except for the value $v = 23$. The construction for this case bears some resemblance to that in Section 6.

We first take a PBD with $\lambda = 2$ on 18 points having 18 quadruples and 66 triples, the latter being decomposable into 11 parallel classes of triples. Such a design can be generated cyclically under the mapping $i \mapsto i+1$ (mod 18) from the blocks $\{0,1,5,9\}, \{0,6,12\}$, $\{0,6,12\}, \{0,3,10\}, \{0,2,7\}$ and $\{0,2,3\}$. By considering the differences, it is easy to verify that this design covers all pairs precisely twice. The repeated short orbit generated by $\{0,6,12\}$ provides two parallel classes. The six triples $\{0,3,10\}$, $\{9,12,1\}$, $\{4,6,11\}$, $\{13,15,2\}$, $\{5,7,8\}$, $\{14,16,17\}$ form a further parallel class. These 6 triples come in pairs from each of the three full orbits of triples, with the two triples in each pair being images of one another under the mapping $i \mapsto i+9$ (mod 18). Consequently, these three orbits may be decomposed into 9 disjoint parallel classes which are the images of the given parallel class under the mappings $i \mapsto i+n$ (mod 18) for $n = 0, 1, \ldots, 8$.

Now expand 10 of these parallel classes by appending each of the 5 symbols of the solution S_0 for $v = 5$ to two parallel classes. This design, with the design S_0, is the required solution with 2 pairs, $2 + 6 = 8$ triples and $2 + 18 + (10 \times 6) = 80$ quadruples.

8 Concluding remarks

When $v = 6m+2$, we might note that it is not possible to have the solution S_0 for $v = 5$ embedded in the solution for $v = 6m + 2$. This is because the solution for $v = 6m + 2$ must contain two pairs xy, xy whereas S_0 has two pairs of the form xa, xb.

On the other hand, it was shown in Section 5 that a solution for $v = 12t + 5$, $t \geq 2$, can be found in which the solution S_0 for $v = 5$ is embedded, and the case $t = 1$ ($v = 17$) was similarly shown, in Section 6, to have a solution containing S_0.

If $v = 12t + 11$, no solution for $t = 0$ can contain the design S_0 for $v = 5$. The reason is as follows. Let the points be $x, a, b, p, q, 1, 2, 3, 4, 5, 6$. Any solution contains 16 quadruples, 4 triples and 2 pairs and must contain the blocks of S_0: $xa, xb, abp, abq, xapq, xbpq$. Now each of the points x, a, b, p, q cannot occur in further blocks with one another and so must occur in four further quadruples to cover pairs with the points $1, 2, 3, 4, 5, 6$. But there are only 16 quadruples so this is impossible. If $t = 1$, the construction of Section 7 produces a solution for $v = 23$ that contains S_0. For $t \geq 5$, proceed as in Section 5 but use a PBD on $12t + 22$ points with all blocks of size 4 except for a single block of length 22, (this requires $t \geq 4$), as well as a PBD on $12t + 25$ points, again with all blocks of size 4 except for a single block of length 25, (this requires $t \geq 5$). By deleting the points from the long blocks we get a PBD on $12t$ points having $\lambda = 2$ and 47 parallel classes of triples. Then juxtapose the solution for $v = 23$ and append each of the 23 points in the latter solution to two of the parallel classes. This expands 46 of the parallel classes and gives a solution for $12t + 23$ points, $t \geq 5$, that contains the solution given for $v = 23$, and consequently the solution for $v = 5$.

We have thus shown that, if $v = 12t + 11$, there is a solution containing the solution S_0 for $v = 5$ provided that $t > 5$. Of the small cases, we know that embedding in $v = 11$ is impossible and in $v = 23$ is possible. This leaves the values $v = 35, 47, 59, 71$. A solution in the latter case can be obtained by using the tripling construction, Section 6, starting with the solution for $v = 23$ given in Section 7. The three remaining cases are also handled using the construction given in Section 7.

For $v = 35$, take the following PBD with $\lambda = 2$ on 30 points. It is generated cyclically under the mapping $i \mapsto i + 1 \pmod{30}$ from the blocks $\{0, 1, 6, 15\}, \{0, 11, 23, 28\}, \{0, 4, 8, 27\}, \{0, 10, 20\}, \{0, 10, 20\}, \{0, 9, 12\}$, $\{0, 13, 14\}$ and $\{0, 6, 8\}$. The ten triples $\{0, 10, 20\}, \{4, 13, 16\}, \{14, 23, 26\}$, $\{24, 3, 6\}, \{8, 21, 22\}, \{18, 1, 2\}, \{28, 11, 12\}, \{9, 15, 17\}, \{19, 25, 27\}, \{29, 5, 7\}$ form a parallel class. The last 9 triples come in threes from each of the three full orbits of triples and are images of one another under the mapping $i \mapsto i + 10 \pmod{30}$. Consequently, we obtain 10 disjoint parallel classes which are the images of the given parallel class under the mappings $i \mapsto i + n \pmod{30}$ for $n = 0, 1, \ldots, 9$. Now expand these parallel classes by appending each of the 5 symbols of the solution S_0 for $v = 5$ to two parallel classes. This design, with the design S_0, is the required solution with 2 pairs, $2+10 = 12$ triples and $2+(10\times 10)+(30\times 3) = 192$ quadruples.

For $v = 47$, take the following PBD with $\lambda = 2$ on 42 points. It is generated cyclically under the mapping $i \mapsto i + 1 \pmod{42}$ from the blocks $\{0,3,7,12\}, \{0,3,12,22\}, \{0,15,21,41\}, \{0,13,18,31\}, \{0,6,8,23\}$, $\{0,14,28\}, \{0,14,28\}, \{0,1,17\}, \{0,2,10\}$ and $\{0,4,11\}$. One of the repeated short orbit generated by $\{0,14,28\}$ provides one parallel class. Nine further parallel classes are obtained from the triples $\{0,1,17\}, \{1,2,18\}$, $\{2,3,19\}, \{0,2,10\}, \{1,3,11\}, \{2,4,12\}, \{0,4,11\}, \{1,5,12\}, \{2,6,13\}$ under the mappings $i \mapsto i + 3n \pmod{42}$ for $n = 0, 1, \ldots, 13$. Again expand these parallel classes by appending each of the 5 symbols of the solution S_0 for $v = 5$ to two parallel classes. This design, with the design S_0, is the required solution with 2 pairs, $2 + 14 = 16$ triples and $2 + (14 \times 10) + (42 \times 5) = 352$ quadruples.

For $v = 59$, take the following PBD with $\lambda = 2$ on 54 points. It is generated cyclically under the mapping $i \mapsto i + 1 \pmod{54}$ from the blocks $\{0,1,12,17\}, \{0,15,29,46\}, \{0,7,14,52\}, \{0,3,27,49\}, \{0,20,26,41\}$, $\{0,2,12,45\}, \{0,1,20,24\}, \{0,18,36\}, \{0,18,36\}, \{0,3,29\}, \{0,6,10\}$, $\{0,19,32\}$. One of the repeated short orbit generated by $\{0,18,36\}$ provides one parallel class. The 18 triples $\{4,7,33\}, \{13,16,42\}, \{22,25,51\}$, $\{31,34,6\}, \{40,43,15\}, \{49,52,24\}, \{2,8,12\}, \{11,17,21\}, \{20,26,30\}$, $\{29,35,39\}, \{38,44,48\}, \{47,53,3\}, \{0,19,32\}, \{9,28,41\}, \{18,37,50\}$, $\{27,46,5\}, \{36,1,14\}, \{45,10,23\}$ form a parallel class. These 18 triples come in sixes from each of the three full orbits of triples and are images of one another under the mapping $i \mapsto i + 9 \pmod{54}$. Consequently we obtain 9 disjoint parallel classes which are the images of the given parallel class under the mappings $i \mapsto i + n \pmod{54}$ for $n = 0, 1, \ldots, 8$. Again expand these parallel classes by appending each of the 5 symbols of the solution S_0 for $v = 5$ to two parallel classes. This design, with the design S_0, is the required solution with 2 pairs, $2 + 18 = 20$ triples and $2 + (18 \times 10) + (54 \times 7) = 560$ quadruples.

We conclude this paper with a summary of our results for the case when $\lambda = 2$, $k = 4$ given in the same format as the results in the Introduction.

(i) $g_2^{(4)}(v) = v(v-1)/6$ all quadruples, for $v \equiv 1, 4 \pmod 6$,

(ii) $g_2^{(4)}(v) = v(v+1)/6$ comprising $v(v-3)/6$ quadruples and $2v/3$ triples, for $v \equiv 0, 3 \pmod 6$,

(iii) $g_2^{(4)}(v) = (v^2+8)/6$ comprising $v(v-2)/6$ quadruples, $(v-2)/3$ triples and 2 pairs, for $v \equiv 2 \pmod 6$, $v \neq 8$,

(iv) $g_2^{(4)}(v) = (v^2+11)/6$ comprising $(v+1)(v-3)/6$ quadruples, $(v+1)/3$ triples and 2 pairs, for $v \equiv 5 \pmod 6$.

In cases (i) and (ii) the PBD is a BIBD $S_2(2,4,v)$ with zero or one point deleted. For the single exceptional case $v = 8$, $g_2^{(4)} = 13$ (seven quadruples, four triples and two pairs).

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