On the Detour Number and Geodetic Number of a Graph

Gary Chartrand, Western Michigan University

Garry L. Johns, Saginaw Valley State University

Ping Zhang,[1] Western Michigan University

ABSTRACT

For vertices u and v in a connected graph G with vertex set V, the distance $d(u,v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is called a $u-v$ geodesic. The closed interval $I[u,v]$ consists of u, v, and all vertices that lie in some $u-v$ geodesic of G; while for $S \subseteq V$, $I[S]$ is the union of closed intervals $I[u,v]$ for all $u,v \in S$. A set S of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. For vertices x and y in G, the detour distance $D(x,y)$ is the length of a longest $x-y$ path in G. An $x-y$ path of length $D(x,y)$ is called an $x-y$ detour. The closed detour interval $I_D[x,y]$ consists of x, y, and all vertices in some $x-y$ detour of G. For $S \subseteq V$, $I_D[S]$ is the union of $I_D[x,y]$ for all $x,y \in S$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. We study relationships that can exist between minimum detour sets and minimum geodetic sets in a graph. A graph F is a minimum detour subgraph if there exists a graph G containing F as an induced subgraph such that $V(F)$ is a minimum detour set in G. It is shown that K_3 and P_3 are minimum detour subgraphs. It is also shown that for every pair $a, b \geq 2$ of integers, there exists a connected graph G with $dn(G) = a$ and $g(G) = b$.

Key Words: detour distance, detour set, detour number, geodetic set, geodetic number.

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1 Introduction

For vertices u and v in a connected graph G with vertex set V, the *distance* $d(u,v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is called a *$u-v$ geodesic.* A vertex w is said to *lie in* a $u-v$ geodesic P if w is an internal vertex of P, that is, if w is a vertex of P distinct from u and v. The *closed interval* $I[u,v]$ consists of u, v, and all vertices lying in some $u-v$ geodesic of G; while for $S \subseteq V$,

$$I[S] = \bigcup_{u,v \in S} I[u,v].$$

A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is a *minimum geodetic set* of G. If G is a nontrivial connected graph of order n, then every geodetic set of G contains at least two vertices and at most n vertices. Thus $2 \leq g(G) \leq n$ for every nontrivial connected graph G of order n. The geodetic number has been studied extensively (see [1, 2, 4], for example).

For vertices x and y in a nontrivial connected graph G of order n, the *detour distance* $D(x,y)$ is the length of a longest $x-y$ path in G. The *detour diameter* $\operatorname{diam}_D G$ is $\max\{D(x,y)\}$, where the maximum is taken over all pairs x,y of vertices of G. Thus $\operatorname{diam}_D G \leq n-1$, and $\operatorname{diam}_D G = n-1$ if and only if G has a hamiltonian path. An $x-y$ path of length $D(x,y)$ is called an *$x-y$ detour.* The *closed detour interval* $I_D[x,y]$ consists of x, y, and all vertices lying in some $x-y$ detour of G; while for $S \subseteq V$,

$$I_D[S] = \bigcup_{x,y \in S} I_D[x,y].$$

A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set.* These concepts were introduced in [3]. If G is a nontrivial connected graph of order $n \geq 3$, then G contains a path of order 3 or more. Thus every minimum detour set of G contains at least two vertices and at most $n-1$ vertices. Therefore $2 \leq dn(G) \leq n-1$ for every nontrivial connected graph G of order $n \geq 3$.

To illustrate these concepts, consider the graph G of Figure 1. For vertices u and v in G, $d(u,v) = 1$ and $D(u,v) = 5$, where the hamiltonian path u, z, y, w, x, v is a $u-v$ detour in G. Thus $\{u,v\}$ is a minimum detour set and so $dn(G) = 2$. On the other hand, $\{u,x,w\}$ is a geodetic set of G and since there is no 2-element geodetic set in G, it follows that $g(G) = 3$.

A vertex v in a graph G is a *detour vertex* if v belongs to every minimum detour set of G. Thus if G has a unique minimum detour set S, then every vertex in S is a detour vertex. The following result appeared in [3].

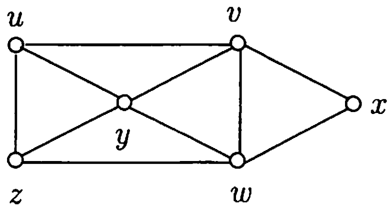

Figure 1: A graph G with $dn(G) = 2$ and $g(G) = 3$

Theorem A *Every end-vertex of a nontrivial connected graph G is a detour vertex of G. Moreover, if the set S of all end-vertices of G is a detour set, then S is the unique minimum detour set for G.*

A vertex in a graph G is a *complete vertex* if the subgraph induced by its neighborhood is a complete subgraph of G. In particular, every end-vertex is a complete vertex. The following two results appeared in [2] and [3], respectively.

Theorem B *Let G be a nontrivial connected graph. Then every minimum geodetic set of G contains every complete vertex of G but no cut-vertex of G.*

Theorem C *Let G be a nontrivial connected graph. Then every minimum detour set of G contains no cut-vertex of G.*

Combining results from both [2] and [3] gives us the following result.

Theorem D *If T is a tree with a end-vertices, then $g(T) = dn(T) = a$.*

2 Minimum Detour Sets and Minimum Geodetic Sets

In this section, we explore relationships that can exist between minimum detour sets and minimum geodetic sets in a graph. First, we show the existence of a graph G and sets of vertices of G that can be both of these kinds of sets or exactly one of them.

Proposition 2.1 *There exists a connected graph G containing*

(1) *a minimum geodetic set that is also a minimum detour set,*

(2) *a minimum geodetic set that is not a minimum detour set, and*

(3) *a minimum detour set that is not a minimum geodetic set.*

Proof. Consider the graph G in Figure 2, where $g(G) = dn(G) = 2$. Since $\{x_1, x_4\}$ is a minimum geodetic set that is also a minimum detour set, $\{x_2, x_5\}$ is a minimum geodetic set that is not a minimum detour set, and $\{x_2, x_6\}$ is a minimum detour set that is not a minimum geodetic set, this graph has the desired properties. ■

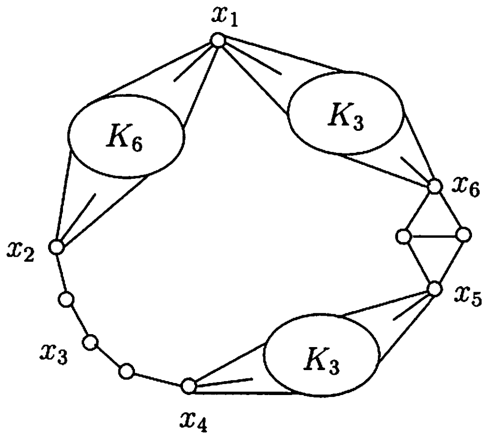

Figure 2: The graph G in Proposition 2.1

For an even integer $n \geq 4$, every minimum geodetic set of C_n consists of two antipodal vertices of C_n. On the other hand, a minimum detour set consists either of two antipodal vertices or two adjacent vertices of C_n. This establishes the following.

Proposition 2.2 *There exists a connected graph G such that every minimum geodetic set is a minimum detour set but some minimum detour set is not a minimum geodetic set.*

Question: Does there exist a connected graph in which every minimum detour set is a minimum geodetic set but some minimum geodetic set is not a minimum detour set?

Next, we illustrate the existence of two graphs, where in the first every minimum geodetic set is a proper subset of some minimum detour set, and in the second, the reverse property holds.

Proposition 2.3 *For each integer $k \geq 2$, there exists a connected graph G with $g(G) = k$ and $dn(G) = k+1$ such that the unique minimum geodetic set of G is a proper subset of every minimum detour set.*

Proof. Let $F = (K_2 \cup K_1) + \overline{K}_2$, where $V(\overline{K}_2) = \{x, y\}$, $V(K_2) = \{u, v\}$, and $V(K_1) = \{w\}$. Then the graph G is obtained from F by adding the k new vertices $v_1, v_2, \ldots, v_k$ and joining (1) v_1 to x and (2) each of v_i $(2 \leq i \leq k)$ to y. The graph G is shown in Figure 3. Since the set $S = \{v_1, v_2, \ldots, v_k\}$ of end-vertices of G is a geodetic set of G, it follows that S is the minimum geodetic set of G by Theorem B. Therefore, $g(G) = |S| = k$.

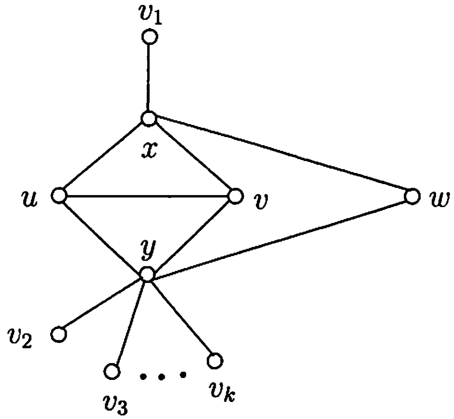

Figure 3: The graph G in Proposition 2.3

By Theorem A, the minimum geodetic set S belongs to every minimum detour set of G. Since $I_D[S] = V(G) - \{w\} \neq V(G)$, it follows that S is not a detour set of G. Thus $dn(G) \geq |S| + 1 = k + 1$. On the other hand, $S \cup \{w\}$ is a detour set of G and so $dn(G) \leq |S| + 1 = k + 1$. Thus $dn(G) = k + 1$. The minimum geodetic set S of G is a proper subset of every minimum detour set of G and so G has the desired properties. ∎

Proposition 2.4 *For each pair a, b of integers with $2 \leq a < b$, there exists a connected graph G with $dn(G) = a$ and $g(G) = b$ such that the unique minimum detour set is a proper subset of every minimum geodetic set.*

Proof. Let a, b be integers with $2 \leq a < b$ and let

$$P_{2(b-a)+1} : u_1, u_2, \cdots, u_{2(b-a)+1}$$

be a path of order $2(b-a)+1 \geq 3$. Then the graph G is obtained from $P_{2(b-a)+1}$ by (1) adding the edges $u_i u_{i+2}$ for each odd integer i with $1 \leq i \leq 2(b-a) - 1$ and (2) adding a new vertices $v_1, v_2, \ldots, v_a$ and joining v_1

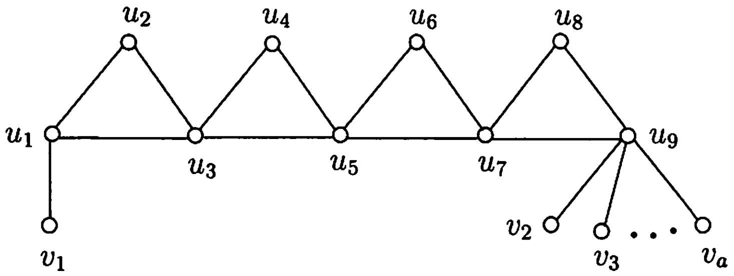

Figure 4: The graph G in Proposition 2.4

to u_1 and each of v_i $(2 \le i \le a)$ to $u_{2(b-a)+1}$. The graph G is shown in Figure 4 for $2(b-a)+1 = 9$.

Observe that $S = \{v_1, v_2, \ldots, v_a\}$ is the set of end-vertices of G, and $S' = \{u_2, u_4, \ldots, u_{2(b-a)}\} \cup S$ is the set of all complete vertices of G. Since $I_D[S] = V(G)$, it follows that S is a detour set of G. Thus S is the unique minimum detour set of G and so $dn(G) = a$. On the other hand, by Theorem B, S' belongs to every minimum geodetic set of G. Since $I[S'] = V(G)$, it follows that S' is the minimum geodetic set of G and so $g(G) = |S'| = (b-a) + a = b$. Furthermore, S is a proper subset of S' and the graph G has the desired properties. ■

Let G be a connected graph and let S and T be two subsets of $V(G)$. The ***distance between*** S ***and*** T is defined as

$$d(S,T) = \min\{d(s,t) : s \in S, t \in T\}.$$

Proposition 2.5 *For every pair a, b of integers with $2 \le a \le b$, there exists a connected graph G with $dn(G) = a$ and $g(G) = b$. Moreover, every minimum geodetic set of G is arbitrarily far apart from every minimum detour set of G.*

Proof. Let $N \ge 2$ be an integer. For each i with $1 \le i \le a$, let

$$F_i\colon v_{i,1}, v_{i,2}, \ldots, v_{i,2N}, v_{i,1}$$

be a copy of the cycle C_{2N} of order $2N$, and let F be the graph obtained from the graphs F_i $(1 \le i \le a)$ by identifying the a vertices $v_{i,1}$ $(1 \le i \le a)$ and labeling the identified vertex by v. Then the graph G is obtained from F by (1) replacing the vertex $v_{1,N+1}$ of F by the complete graph K_{b-a+1} such that every vertex of K_{b-a+1} is adjacent to $v_{1,N}$ and $v_{1,N+2}$ and (2) adding the edge $v_{1,N}v_{1,N+2}$. The graph G is shown in Figure 5 for $a = 3$, $b = 5$, and $N = 3$. Then $V(K_{b-a+1})$ is the set of complete vertices of G.

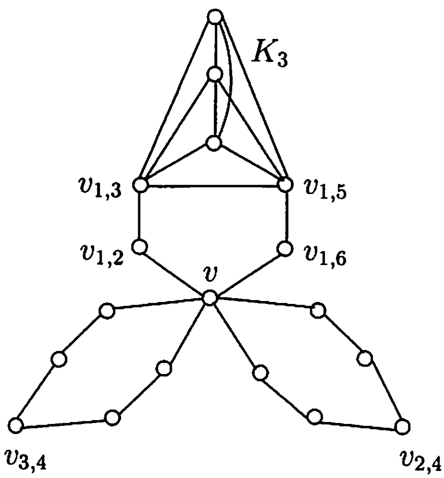

Figure 5: The graph G in Proposition 2.5 for $a = 3$, $b = 5$, and $N = 3$

Observe that, in general, every minimum detour set and every minimum geodetic set of a graph G must contain at least one vertex (that is not a cut-vertex) from each end-block of G. Since $S = \{v_{i,2} : 1 \leq i \leq a\}$ is a minimum detour set of G, it follows that $dn(G) = |S| = a$. On the other hand, $S' = V(K_{b-a+1}) \cup \{v_{i,N+1} : 2 \leq i \leq a\}$ is the unique minimum geodetic set of G and so $g(G) = |S'| = (a-1)+(b-a+1) = b$. Furthermore, it can be verified that every minimum detour set of G consists of exactly one vertex from each set $\{v_{i,2}, v_{i,2N}\}$ for all i with $1 \leq i \leq a$. Moreover, $d(v_{i,2}, v_{i,N+1}) = d(v_{i,2N}, v_{i,N+1}) = N - 1$ for all i with $2 \leq i \leq a$. Thus if S^* is a minimum detour set of G, then $d(S^*, S') = d(S, S') = N - 1$, as desired. ∎

3 Minimum Detour Subgraphs

A graph F is called a *minimum geodetic subgraph* if there exists a graph G containing F as an induced subgraph such that $V(F)$ is a minimum geodetic set in G. It was shown in [2] that a nontrivial graph F is a minimum geodetic subgraph if and only if every vertex of F has eccentricity 1 or no vertex of F has eccentricity 1. Similarly, we define a graph F as a *minimum detour subgraph* if there exists a graph G containing F as an induced subgraph such that $V(F)$ is a minimum detour set in G. Determining graphs that are minimum detour subgraphs appears to be far more complicated. The largest connected minimum detour subgraph that we have seen has order 2. We now show that order 3 is also possible.

Proposition 3.1 *The graph K_3 is a minimum detour subgraph.*

Proof. Let $F = K_3$ with vertex set $V(F) = \{v_1, v_2, v_3\}$. For each j with $1 \le j \le 3$, let Q_j be a copy of K_4, let R_j be a copy of K_3, and let S_j be a copy of K_2. The graph G is obtained from the graphs F, Q_j, R_j, S_j $(1 \le j \le 3)$ by (1) joining every vertex in Q_j to v_1 and v_2, (2) joining every vertex in R_j to v_1 and v_3, and (3) joining every vertex in S_j to v_2 and v_3. The graph G is shown in Figure 6.

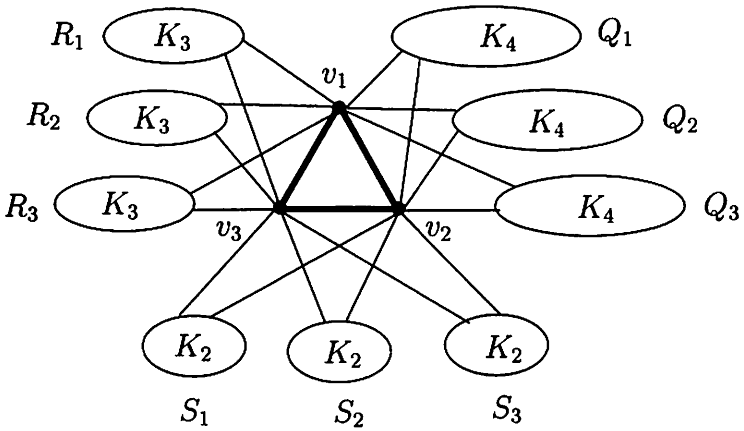

Figure 6: The graph G containing K_3 as a minimum detour subgraph

We first show that $dn(G) = 3$. Observe that

(a) $D(v_1, v_2) = 7$ and $I_D[v_1, v_2] = V(G) - (V(Q_1) \cup V(Q_2) \cup V(Q_3))$,

(b) $D(v_1, v_3) = 8$ and $I_D[v_1, v_3] = V(G) - (V(R_1) \cup V(R_2) \cup V(R_3))$,

(c) $D(v_2, v_3) = 9$ and $I_D[v_2, v_3] = V(G) - (V(S_1) \cup V(S_2) \cup V(S_3))$.

Thus $V(F)$ is a detour set in G and so $dn(G) \le 3$. To show that $dn(G) > 2$, assume, to the contrary, that $dn(G) = 2$. Let $\{x, y\}$ be a detour set of G. It follows by (a)-(c) that at least one of x and y does not belong to $V(F)$. Assume, without loss of generality, that $y \notin V(F)$. We consider two cases, according to whether $x \in V(F)$ or $x \notin V(F)$.

Case 1. *$x \in V(F)$, say $x = v_1$.* There are three subcases.

Subcase 1.1. $y \in V(Q_1)$. Then $D(x, y) = 11$. Since every $x - y$ path containing vertices of Q_2 has length at most 9, it follows that $V(Q_2) \cap I_D[x, y] = \emptyset$, a contradiction.

Subcase 1.2. $y \in V(R_1)$. Then $D(x,y) = 11$. Since every $x - y$ path containing vertices of R_2 has length at most 7, it follows that $V(R_2) \cap I_D[x,y] = \emptyset$, a contradiction.

Subcase 1.3. $y \in V(S_1)$. Then $D(x,y) = 10$. Since every $x - y$ path containing vertices of R_1 has length at most 9, it follows that $V(R_1) \cap I_D[x,y] = \emptyset$, a contradiction.

Case 2. $x, y \notin V(F)$. There are six subcases.

Subcase 2.1. $\{x,y\} \subseteq V(Q_1) \cup V(Q_2) \cup V(Q_3)$. Suppose first that $\{x,y\} \subseteq V(Q_j)$ for some j $(1 \leq j \leq 3)$, say $x, y \in V(Q_1)$. Then $D(x,y) = 11$. Since every $x - y$ path containing vertices of Q_2 has length at most 9, it follows that $V(Q_2) \cap I_D[x,y] = \emptyset$, a contradiction. Suppose next that $x \in V(Q_p)$ and $y \in V(Q_q)$, where $1 \leq p < q \leq 3$, say $x \in V(Q_1)$ and $y \in V(Q_2)$. Then $D(x,y) = 15$. Since every $x - y$ path containing vertices of Q_3 has length at most 13, it follows that $V(Q_3) \cap I_D[x,y] = \emptyset$, a contradiction.

Subcase 2.2. $\{x,y\} \subseteq V(R_1) \cup V(R_2) \cup V(R_3)$. Suppose first that $\{x,y\} \subseteq V(R_j)$ for some j $(1 \leq j \leq 3)$, say $x, y \in V(R_1)$. Then $D(x,y) = 11$. Since every $x - y$ path containing vertices of R_2 has length at most 7, it follows that $V(R_2) \cap I_D[x,y] = \emptyset$, a contradiction. Suppose next that $x \in V(R_p)$ and $y \in V(R_q)$, where $1 \leq p < q \leq 3$, say $x \in V(R_1)$ and $y \in V(R_2)$. Then $D(x,y) = 14$. Since every $x - y$ path containing vertices of R_3 has length at most 10, it follows that $V(R_3) \cap I_D[x,y] = \emptyset$, a contradiction.

Subcase 2.3. $\{x,y\} \subseteq V(S_1) \cup V(S_2) \cup V(S_3)$. Suppose first that $\{x,y\} \subseteq V(S_j)$ for some j $(1 \leq j \leq 3)$, say $x, y \in V(S_1)$. Then $D(x,y) = 11$. Since every $x - y$ path containing vertices of S_2 has length at most 5, it follows that $V(S_2) \cap I_D[x,y] = \emptyset$, a contradiction. Suppose next that $x \in V(S_p)$ and $y \in V(S_q)$, where $1 \leq p < q \leq 3$, say $x \in V(S_1)$ and $y \in V(S_2)$. Then $D(x,y) = 13$. Since every $x - y$ path containing vertices of S_3 has length at most 7, it follows that $V(S_3) \cap I_D[x,y] = \emptyset$, a contradiction.

Subcase 2.4. $x \in V(Q_1)$ *and* $y \in V(R_1)$. Then $D(x,y) = 16$. Since every $x - y$ path containing vertices of S_1 has length at most 15, it follows that $V(S_1) \cap I_D[x,y] = \emptyset$, a contradiction.

Subcase 2.5. $x \in V(Q_1)$ *and* $y \in V(S_1)$. Then $D(x,y) = 15$. Since every $x - y$ path containing vertices of S_2 has length at most 14, it follows that $V(S_2) \cap I_D[x,y] = \emptyset$, a contradiction.

Subcase 2.6. $x \in V(R_1)$ *and* $y \in V(S_1)$. Then $D(x,y) = 14$. Since every $x - y$ path containing vertices of S_2 has length at most 13, it follows that $V(S_2) \cap I_D[x,y] = \emptyset$, a contradiction.

Since G does not contain any detour set of cardinality 2, it follows that $dn(G) = 3$. Therefore, $V(F)$ is a minimum detour set. ∎

In the proof of Theorem 3.1, none of the edges of F occur in any longest path considered. Consequently, it follows that P_3 is also a minimum detour subgraph. Whether there exist graphs having a connected minimum detour subgraph of arbitrarily large order is not known.

4 Graphs With Prescribed Detour Number and Geodetic Number

In this section, we show that the geodetic number and detour number are independent parameters, that is, knowing the value of one of these parameters provides no information about the value of the other.

Theorem 4.1 *For every pair $a, b \geq 2$ of integers, there exists a connected graph G with $dn(G) = a$ and $g(G) = b$.*

Proof. If $2 \leq a \leq b$, then the result follows by Proposition 2.5. Thus we may assume that $2 \leq b < a$. There are three cases.

Case 1. $b = 2$. Let $F = 3K_2 \cup aK_1$ with

$$V(3K_2) = \{u_1, u_2, u_3, w_1, w_2, w_3\},$$

where u_i is adjacent to w_i for $1 \leq i \leq 3$, and $V(aK_1) = \{v_1, v_2, \cdots, v_a\}$. The graph G is obtained from F by adding two new vertices x and y and joining each of x and y to every vertex in F, that is, $G = \overline{K}_2 + F$. The graph G is shown in Figure 7. Since $\{x, y\}$ is a geodetic set of G, it follows that $g(G) = 2$.

We now show that $dn(G) = a$. Let $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, \cdots, v_a\}$ and $W = \{w_1, w_2, w_2\}$. First, we show that every vertex in V is a detour vertex of G. Assume, to the contrary, that this is not the case. We may assume then that v_1 is not a detour vertex of G. Then v_1 lies in some $s - t$ detour P in G, where s and t belong to a minimum detour set, say $P : s = z_0, z_1, \ldots, x, v_1, y, \ldots, z_k = t$. Necessarily, there is some i $(1 \leq i \leq 3)$ such that both u_i and w_i do not belong to P, say $i = 1$. If we replace x, v_i, y in P by x, u_1, w_1, y, then we obtain an $s - t$ path P', whose length exceeds that of P, contradicting the fact that P is an $s - t$ detour. Therefore, every vertex in V is a detour vertex of G, as claimed. Since V is a subset of every minimum detour set of G, it follows that $dn(G) \geq |V| = a$. On the other hand, V is a detour set of G. Therefore, $dn(G) = |V| = a$.

Case 2. $b = 3$. Let $F_i = K_2$ and $H_i = K_2$ for $i = 1, 2$, where $V(F_i) = \{w_i, w_i'\}$ and $V(H_i) = \{u_i, u_i'\}$. Also, let

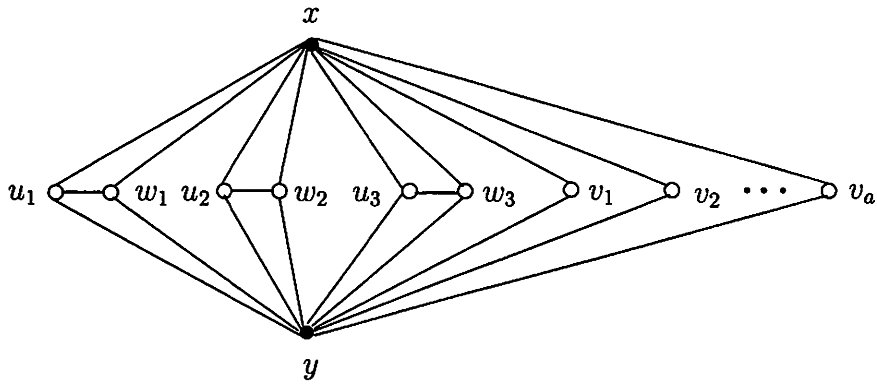

Figure 7: The graph G in Case 1

$$V = V(\overline{K}_{a-1}) = \{v_1, v_2, \ldots, v_{a-1}\}.$$

Let $F = \overline{K}_2 + (F_1 \cup F_2 \cup \overline{K}_{a-1})$, where $V(\overline{K}_2) = \{x_1, x_2\}$. The graph G is obtained from H_1, H_2, and F by adding the vertex x_3 and (1) joining x_1 and x_3 to the vertices of H_1 and (2) joining x_2 and x_3 to the vertices of H_2. The graph G is shown in Figure 8. Since $\{x_1, x_2, x_3\}$ is a geodetic set and G contains no 2-element geodetic set, it follows that $g(G) = 3$.

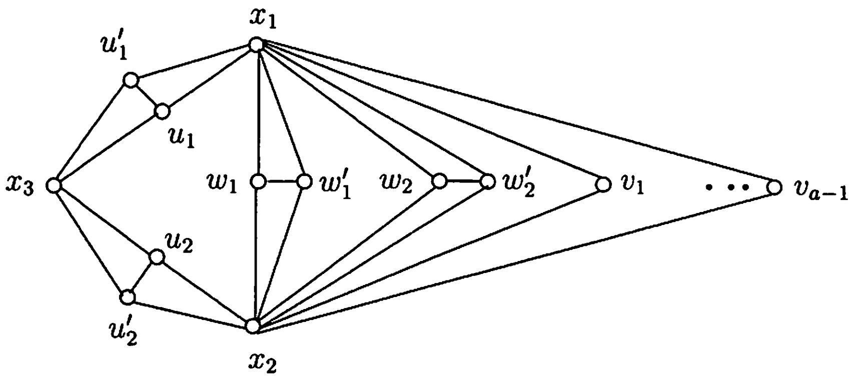

Figure 8: The graph G in Case 2

Next, we show that $dn(G) = a$. By an argument that is similar to that used in Case 1, one can show every vertex in $V = \{v_1, v_2, \ldots, v_{a-1}\}$ is a detour vertex of G. Thus V is a subset of every minimum detour set of G. Since

$$I_D[V] = V - \{V(F_1) \cup V(F_2)\} \neq V(G),$$

it follows that V is not a detour set of G and so $dn(G) \geq |V|+1 = a$. Since there is a $v_1 - x_3$ detour containing each vertex of $V(G) - (V - \{v_1\})$, it follows that $S = V \cup \{x_3\}$ is a detour set of G and so $dn(G) \leq |S| = a$. Therefore, $dn(G) = a$.

Case 3. $b \geq 4$. Let $C_{12} : v_1, v_2, \cdots, v_{12}, v_1$ be a cycle of order 12 and $P_2 : u_1, u_2$ be a path of order 2. Then the graph G is obtained from C_{12} and P_2 by (1) adding $a - b + 2$ new vertices $x_1, x_2, \cdots, x_{a-b+2}$ and joining each vertex x_i $(1 \leq i \leq a-b+2)$ to both v_1 and v_7, (2) adding the two new edges u_1v_1 and u_2v_7, and (3) adding $b - 3$ new vertices $w_1, w_2, \cdots, w_{b-3}$ and joining each of these vertices to v_2. The graph G is shown in Figure 9. Let $V = \{v_1, v_2, \cdots, v_{12}\}$, $U = \{u_1, u_2\}$, $W = \{w_1, w_2, \cdots, w_{b-3}\}$, and $X = \{x_1, x_2, \cdots, x_{a-b+2}\}$. We show that $g(G) = b$ and $dn(G) = a$.

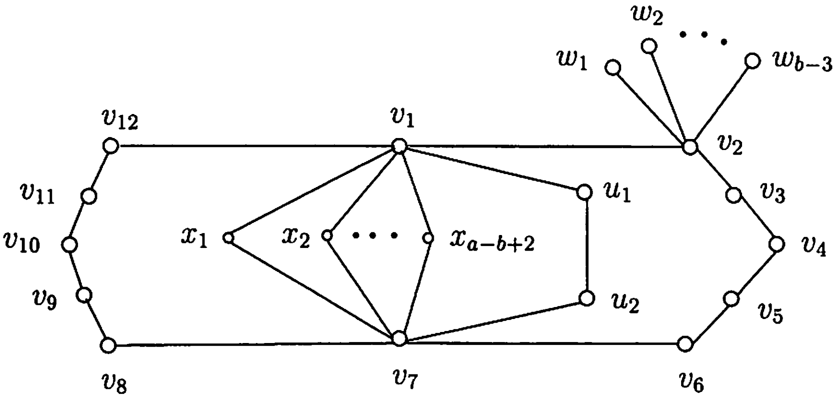

Figure 9: The graph G in Case 3

We first show that $g(G) = b$. Since $W \cup \{u_2, v_6, v_9\}$ is a geodetic set of G, it follows that $g(G) \leq |W| + 3 = b$. To see that there is no geodetic set in G of cardinality $b - 1$, let S_0 be any subset of $V(G)$ with $|S_0| = b - 1$. Then $S_0 = W \cup \{s, t\}$, where $s, t \in V(G) - W$. If $s, t \in V(C_{12}) \cup X$, then $u_1, u_2 \notin I[S_0]$. If $s \in V(C_{12})$ and $t \in \{u_1, u_2\}$, then (1) $X \not\subseteq I[S_0]$ if $s = v_i$ $(1 \leq i \leq 5)$, (2) $\{v_8, v_9, \ldots, v_{12}\} \not\subseteq I[S_0]$ if $s = v_6$ or $s = v_7$, and (3) $\{v_3, v_4, \ldots, v_6\} \not\subseteq I[S_0]$ if $s = v_i$ $(8 \leq i \leq 12)$. If $s, t \in X \cup \{u_1, u_2\}$, then $v_{10} \notin I[S_0]$. Thus S_0 is not a geodetic set of G. This implies that $g(G) \geq b$. Therefore, $g(G) = b$.

Next we show that $dn(G) = a$. By an argument that is similar to that used in Case 1, one can show that every vertex in X is a detour vertex of G and so X is a subset of every minimum detour set of G. Also, every detour set of G contains W by Theorem A. Since $I_D[W \cup X] = V(G) - \{u_1, u_2\}$, it follows that $W \cup X$ is not a detour set and so $dn(G) \geq |W| + |X| + 1 = (b-3) + (a-b+2) + 1 = a$. On the other hand, let $S = W \cup X \cup \{v_8\}$. Observe that $D(v_8, w_1) = 14$ and the path $v_8, v_9, \cdots, v_{12}, v_1, u_1, u_2, v_7, v_6, v_5, \cdots, v_2, w_1$ is a $v_8 - w_1$ detour in G. Thus S is a detour set of and so $dn(G) \leq |S| = |W| + |X| + 1 = a$. Therefore, $dn(G) = a$. ∎

Using the structure of the graph G of Figure 9, namely, deleting all end-vertices of G, we can construct for each integer $n \geq 17$, a 2-connected graph G_n of order n such that $dn(G_n) = n - 10$. This observation yields the following.

Proposition 4.2 *There is an infinite sequence $\{G_n\}$ of 2-connected graphs G_n of order n such that*

$$\lim_{n \to \infty} \frac{dn(G_n)}{n} = 1.$$

5 Acknowledgments

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