Strongly Regular Vertices and Partially Strongly Regular Graphs

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Abstract

A strongly regular vertex with parameters (λ, μ) in a graph is a vertex x such that the number of neighbors any other vertex y has in common with x is λ if y is adjacent to x, and is μ if y is not adjacent to x. In this note, we will prove some basic properties of these vertices and the graphs that contain them, as well as provide some simple constructions of regular graphs that are not necessarily strongly regular, but do contain (many) strongly regular vertices. We also make several conjectures and find all regular graphs on at most ten vertices with at least one strongly regular vertex.

1 Introduction

All graphs considered will be finite and simple (no loops or parallel edges). The vertex and edge sets of a graph G will be denoted by $V(G)$ and $E(G)$, respectively (edges will be regarded as 2-subsets of $V(G)$). The complement of G will be denoted by $\overline{G}$. Given two vertices x and y in a graph, we use the notation $x \sim y$ to denote that x and y are adjacent, and $x \nsim y$ to denote that x and y are not adjacent. If we wish to emphasize that x and y are adjacent (non-adjacent) in a particular graph G, we use the notation $x \sim_G y$ ($x \nsim_G y$). If $x \sim y$, then the edge between x and y will be denoted by xy or yx, and y is called a *neighbor* of x (x is also a neighbor of y). The set of all neighbors of x is called the *neighborhood* of x, and is denoted by $N(G; x)$, or just $N(x)$ if there is no danger of confusion. The size of the neighborhood of x is the *degree* of x, and will be denoted by $d_G(x)$ or just $d(x)$. If $x \sim y$, then the number of vertices adjacent to both x and y, or the number of *common neighbors* of x and y, will be denoted by $\lambda_G(x, y)$

or just $\lambda(x,y)$. If $x \not\sim y$, $x \neq y$, then the number of common neighbors of x and y will be denoted by $\mu_G(x,y)$ or $\mu(x,y)$.

A *strongly regular graph* with parameters v, k $(1 \leq k \leq v-2)$, λ, and μ, or a $srg(v,k,\lambda,\mu)$, is a k-regular graph G on v vertices such that for all $x, y \in V(G), x \neq y$, $\lambda(x,y) = \lambda$ if $x \sim y$, and $\mu(x,y) = \mu$ if $x \not\sim y$. For a good introduction to the theory of these graphs, see [2] or [4]. For a good survey of strongly regular graphs, see [1] or [7].

In this note, we treat the concept of strong regularity as a property of vertices rather than as a property of graphs as a whole. Accordingly, we make the following definition.

Definition 1.1 Given a graph G on v vertices, a *strongly regular vertex* in G with parameters λ and μ, or a $srv(\lambda,\mu)$ ($srv_G(\lambda,\mu)$ if we wish to make G explicit) is a vertex $x \in V(G)$ with $1 \leq d(x) \leq v-2$, and such that $\lambda(x,y) = \lambda$ for all $y \sim x$, and $\mu(x,y) = \mu$ for all $y \not\sim x, y \neq x$. Two strongly regular vertices in G will be said to have the same parameters if their parameter tuples are equal regarded as ordered pairs in $\mathbb{N}_0 \times \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of non-negative integers.

Clearly, a graph is strongly regular if and only if it is regular and all of its vertices are strongly regular with the same parameters.

In the forthcoming sections, we will prove some of the basic properties of these vertices and the graphs that contain them, as well as provide some constructions of regular graphs that are not necessarily strongly regular, but do nonetheless possess, in some cases many, strongly regular vertices. We also make some conjectures and find all regular graphs on at most ten vertices with at least one strongly regular vertex.

2 Basic Properties

In this section, we will prove some simple properties of strongly regular vertices and the graphs that contain them. We will see that several results concerning strongly regular graphs have analogues in the context of a regular graph with a strongly regular vertex.

A graph G is a $srg(v,k,\lambda,\mu)$ if and only if $\overline{G}$ is a $srg(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$. The following proposition and its corollary show that a similiar result holds for strongly regular vertices in regular graphs.

Proposition 2.1 *Let G be a graph on v vertices with a $srv_G(\lambda,\mu)$, x, such that every neighbor of x in G has degree k_1 in G, and every non-neighbor of x in G, except possibly x itself, has degree k_2 in G. Then x is a $srv_{\overline{G}}(v-d_G(x)-k_2+\mu-2, v-d_G(x)-k_1+\lambda)$.*

Proof. If $y \sim_{\overline{G}} x$, then $y \not\sim_G x, y \neq x$, and so $\lambda_{\overline{G}}(x,y) = |N(\overline{G};x) \cap N(\overline{G};y)| = |\overline{N(G;x)} \cap \overline{N(G;y)}| - 2 = |\overline{N(G,x) \cup N(G;y)}| - 2 = v - |N(G;x) \cup N(G;y)| - 2 = v - (|N(G;x)| + |N(G;y)| - |N(G;x) \cap N(G;y)|) - 2 = v - d_G(x) - k_2 + \mu - 2$. Similarly, if $y \not\sim_{\overline{G}} x$, $y \neq x$, then $y \sim_G x$, and so $\mu_{\overline{G}}(x,y) = |N(\overline{G};x) \cap N(\overline{G};y)| = v - |N(G;x)| - |N(G;y)| + |N(G;x) \cap N(G;y)| = v - d_G(x) - k_1 + \lambda$. □

Corollary 2.2 *A vertex x in a k-regular graph G with v vertices is a $srv_G(\lambda,\mu)$ if and only if x is a $srv_{\overline{G}}(v - 2k + \mu - 2, v - 2k + \lambda)$.*

Corollary 2.3 *Let G be a k-regular graph on v vertices with a $srv(\lambda,\mu)$. Then $v - 2k + \mu - 2, v - 2k + \lambda \geq 0$.*

Proposition 2.4 *Let G be a graph on v vertices with a $srv(\lambda,\mu)$, x. Then $d(x)[d(x) - \lambda - 1] = [v - d(x) - 1]\mu$.*

Proof. We will count pairs $(y,z) \in V(G) \times V(G)$ such that $x \sim y \sim z \not\sim x, z \neq x$, in two different ways. We can first choose y in $d(x)$ ways, and then choose z in $d(x) - 1 - \lambda$ ways, for a total of $d(x)[d(x) - \lambda - 1]$ such pairs. Or, we can first choose z in $v - 1 - d(x)$ ways, and then choose y in μ ways, for a total of $[v - d(x) - 1]\mu$ pairs. □

Applying Proposition 2.4 to a regular graph with a strongly regular vertex, we obtain the following familiar equation for strongly regular graphs.

Corollary 2.5 *Let G be a k-regular graph on v vertices with a $srv(\lambda,\mu)$. Then $k(k - \lambda - 1) = (v - k - 1)\mu$.*

Proposition 2.6 *Two strongly regular vertices in a graph G have the same degree if and only if they have the same parameters.*

Proof. Let x be a $srv_G(\lambda_x,\mu_x)$, and let $y \neq x$ be a $srv_G(\lambda_y,\mu_y)$. Then, by Proposition 2.4, we have the two equations

$$d(x)[d(x) - \lambda_x - 1] = [v - d(x) - 1]\mu_x \tag{1}$$

and

$$d(y)[d(y) - \lambda_y - 1] = [v - d(y) - 1]\mu_y. \tag{2}$$

Suppose that $d(x) = d(y) = d$. Then the two equations (1) and (2) become

$$d(d - \lambda_x - 1) = (v - d - 1)\mu_x \text{ and } d(d - \lambda_y - 1) = (v - d - 1)\mu_y. \tag{3}$$

If $x \sim y$, then clearly $\lambda_x = \lambda_y = \lambda$, so the equations (3) become

$$d(d-\lambda-1)=(v-d-1)\mu_x \text{ and } d(d-\lambda-1)=(v-d-1)\mu_y. \quad (4)$$

Since $v-d-1>0$, we can solve the equations (4) for μ_x and μ_y to obtain $\mu_x = d(d-\lambda-1)/(v-d-1) = \mu_y$, and so x and y have the same parameters. If $x \not\sim y$, then clearly $\mu_x = \mu_y = \mu$, so the equations (3) become

$$d(d-\lambda_x-1)=(v-d-1)\mu \text{ and } d(d-\lambda_y-1)=(v-d-1)\mu. \quad (5)$$

Since $d>0$, we can solve the equations (5) to obtain $\lambda_x = d-(v-d-1)\mu/d-1=\lambda_y$, and again x and y have the same parameters.

Next, suppose that $\lambda_x = \lambda_y = \lambda$ and $\mu_x = \mu_y = \mu$. Then the two equations (1) and (2) become

$$d(x)[d(x)-\lambda-1]=[v-d(x)-1]\mu \quad (6)$$

and

$$d(y)[d(y)-\lambda-1]=[v-d(y)-1]\mu. \quad (7)$$

Now, if $\mu=0$, then $d(x)-\lambda-1=d(y)-\lambda-1=0$, since $d(x),d(y)>0$. This implies that $d(x)=d(y)$, so we can assume that $\mu>0$. Solving equations (6) and (7) for $d(x)$ and $d(y)$ gives us the following four possible solution sets:

$$d(x), d(y) \in \{\frac{\lambda-\mu+1 \pm \sqrt{(\lambda-\mu+1)^2+4\mu(v-1)}}{2}\}.$$

Since $\mu>0$, we have $(\lambda-\mu+1-\sqrt{(\lambda-\mu+1)^2+4\mu(v-1)})/2<0$. This eliminates all but one of the above solution sets. Thus, $d(x) = (\lambda-\mu+1+\sqrt{(\lambda-\mu+1)^2+4\mu(v-1)})/2 = d(y)$, and so x and y have the same degree. □

Applying Proposition 2.6 to a regular graph gives us the following results.

Corollary 2.7 *All strongly regular vertices in a regular graph have the same parameters.*

Corollary 2.8 *A graph is strongly regular if and only if it is regular and all of its vertices are strongly regular.*

Corollary 2.9 *A graph is strongly regular if and only if all of its vertices are strongly regular with the same parameters.*

The graph obtained by pasting together $n \geq 2$ copies of K_3 at a common vertex x appears at first to be a counter-example to Corollary 2.9, since every pair of distinct vertices has exactly one common neighbor, but the graph is not regular since $d(x) = 2n$ while the other vertices have degree two. However, x is not considered to be a $srv(1,1)$ since it is adjacent to every other vertex.

In view of Corollary 2.7, the following definition makes sense.

Definition 2.10 A *partially strongly regular graph* with parameters v, k $(1 \leq k \leq v-2)$, λ, and μ, or a $psrg(v,k,\lambda,\mu)$, is a k-regular graph G on v vertices with at least one $srv_G(\lambda,\mu)$. We will sometimes call such a graph s-partially strongly regular and use the notation s-$psrg(v,k,\lambda,\mu)$ if G has exactly s $srv(\lambda,\mu)$. If $s < v$, then we say the graph is *strictly* partially strongly regular.

Thus, partially strongly regular graphs are a generalization of strongly regular graphs. Clearly, a $srg(v,k,\lambda,\mu)$ is just a v-$psrg(v,k,\lambda,\mu)$. Other such generalizations of strongly regular graphs have also been studied. For instance, in [3], the authors study what they call Deza graphs. These are regular graphs in which the number of common neighbors of two distinct vertices takes on one of only two values, not necessarily depending on the adjacency of the two vertices.

The following well-known theorem says that the strongly regular graphs can be completely characterized in terms of two matrix equations. In what follows, A denotes a $(0,1)$-adjacency matrix of a graph, and I, J, and $\mathbf{0}$ denote an identity matrix, a square matrix of all 1's, and a square matrix of all 0's, respectively.

Theorem 2.11 *A graph, not complete or edgeless, is a* $srg(v,k,\lambda,\mu)$ *if and only if* $AJ = kJ$ *and* $A^2 - \lambda A - \mu(J - I - A) - kI = \mathbf{0}$.

The following similar results provide a way to identify strongly regular vertices and partially strongly regular graphs.

Proposition 2.12 *A vertex* x *with* $1 \leq d(x) \leq v-2$ *in a graph on* v *vertices is a* $srv(\lambda,\mu)$ *if and only if the row corresponding to* x *consists of all 0's in the matrix* $A^2 - \lambda A - \mu(J - I - A) - d(x)I$.

Corollary 2.13 *If* G *is a* s-$psrg(v,k,\lambda,\mu)$, *then* s *is equal to the number of rows of all 0's in the matrix* $A^2 - \lambda A - \mu(J - I - A) - kI$.

3 Constructions

In this section, we provide some constructions of partially strongly regular graphs.

Construction 3.1 Let $m \geq 1$ and $n \geq 3$ be integers, and let $\{R_i\}_{i=1}^{l}$ be any non-empty collection of non-complete $(n-1)$-regular graphs with $v_1, \ldots, v_l$ vertices, respectively. Let $G = mK_n \cup (\cup_{i=1}^{l} R_i)$. Then G is clearly $(n-1)$-regular with $mn + \sum_{i=1}^{l} v_i$ vertices, and the mn vertices that reside in the m copies of K_n are all $srv_G(n-2, 0)$. The remaining vertices are not strongly regular in G. Thus, G is a mn-$psrg(mn + \sum_{i=1}^{l} v_i, n-1, n-2, 0)$.

Construction 3.2 Let H be a $srg(v, k, \lambda, \mu)$ with $\mu > 0$, which is the case as long as H is not a disjoint union of complete graphs, and with $w, x, y, z \in V(H)$ such that $wx, yz \in E(H)$, but $wy, wz, xy, xz \notin E(H)$. Let G be the graph with $V(G) = V(H)$ and $E(G) = (E(H) \setminus \{wx, yz\}) \cup \{wy, xz\}$. Then G is k-regular on v vertices, and the vertices in $\overline{\cup_{u \in \{w,x,y,z\}} N(H; u)} \cup (N(H; w) \cap N(H; z)) \cup (N(H; x) \cap N(H; y))$ will be $srv_G(\lambda, \mu)$. The remaining vertices will not be strongly regular in G.

It is quite possible that $\overline{\cup_{u \in \{w,x,y,z\}} N(H; u)} = \emptyset$ in Construction 3.2. However, since $\mu > 0$, $(N(H; w) \cap N(H; z)) \cup (N(H; x) \cap N(H; y)) \neq \emptyset$. Thus, this construction always provides at least one strongly regular vertex, and so G is a $psrg(v, k, \lambda, \mu)$.

Construction 3.3 Let $k \geq 3$ be an integer and choose k distinguished vertices from the graph $K = kK_k$, one from each copy of K_k. Call this set D. Create a new vertex x and join it to every vertex in D by an edge. Call the resulting graph H. Now, partition the set $V(K) \setminus D$ into 2-subsets such that each 2-subset consists of two vertices residing in different copies of K_k (this can always be done in many ways). Call this partition P. Finally, define the graph G by $V(G) = V(H)$ and $E(G) = E(H) \cup P$. Clearly, G is k-regular and has $k^2 + 1$ vertices. In addition, the vertex x is an $srv_G(0, 1)$. The remaining vertices are not strongly regular in G. Thus, G is a 1-$psrg(k^2 + 1, k, 0, 1)$.

Construction 3.4 Let $k \geq 4$ be an even integer and let $K = K_{k+1,k}$. Let C_1 and C_2 be the color classes of K of sizes $k+1$ and k, respectively. Let x be any vertex of C_1. Now, let $C_1 \setminus \{x\} = \{1, \ldots, k\}$ and $C_2 = \{1', \ldots, k'\}$, and let P be a partition of $C_1 \setminus \{x\}$ into 2-subsets. Finally, let G be the graph defined by $V(G) = V(K)$ and $E(G) = (E(H) \setminus \{ii' : 1 \leq i \leq k\}) \cup P$. Then G is k-regular with $2k+1$ vertices, and x is a $srv_G(0, k-1)$. The remaining vertices are not strongly regular in G. Thus, G is a 1-$psrg(2k+1, k, 0, k-1)$.

Construction 3.5 Let v, k, $\lambda \geq 0$, and $\mu > 0$ be integers such that $k - \lambda - 1 > 0$, k and λ are not both odd, $v - k - 1 > k - \mu \geq 0$, $v - k - 1$ and $k - \mu$ are not both odd, and $k(k - \lambda - 1) = (v - k - 1)\mu$. Let H be a λ-regular graph on k vertices, and let K be a $(k - \mu)$-regular graph on $v - k - 1$ vertices. Create a new vertex x and join it to every vertex of H by an edge. Now, draw a bipartite point-block incidence graph of a 1-$(k, \mu, k - \lambda - 1)$ design between H and K. Call the resulting graph G. Then G is a k-regular graph with v vertices, and x is a $srv_G(\lambda, \mu)$. The other vertices may or may not be strongly regular in G. Therefore, G is a $psrg(v, k, \lambda, \mu)$.

Combining Construction 3.5 with Corollary 2.5, we obtain the following result.

Corollary 3.6 *Let v, k, $\lambda \geq 0$, and $\mu > 0$ be integers such that $k-\lambda-1 > 0$, k and λ are not both odd, $v - k - 1 > k - \mu \geq 0$, and $v - k - 1$ and $k - \mu$ are not both odd. Then there exists a $psrg(v, k, \lambda, \mu)$ if and only if $k(k - \lambda - 1) = (v - k - 1)\mu$.*

Of course, by Corollary 2.2, we can complement any of the above constructions to obtain more examples of partially strongly regular graphs.

4 Eigenvalues

In this section, we briefly explore the eigenvalues of partially strongly regular graphs.

The eigenvalues of a graph contain much information about the graph. In particular, graphs with a great deal of regularity tend to have few distinct eigenvalues. Conversely, graphs with few distinct eigenvalues quite often exhibit much regularity and symmetry.

The following well-known theorem says that we can recognize if a regular graph is strongly regular simply by examining its number of distinct eigenvalues.

Theorem 4.1 *A regular graph, not complete or edgeless, is strongly regular if and only if it has at most three distinct eigenvalues.*

The eigenvalues of a strongly regular graph are completely determined by its parameters v, k, λ, and μ. However, the parameters v, k, λ, and μ, together with the number s, do not determine the eigenvalues of a partially strongly regular graph. For instance, there are two 2-$psrg(10, 3, 0, 1)$ with different eigenvalues.

Although a strictly $psrg(v, k, \lambda, \mu)$, G, must have more than three distinct eigenvalues by Theorem 4.1, we will see that the eigenvalues of a $srg(v, k, \lambda, \mu)$ are also eigenvalues of G. So, although the eigenvalues of a partially strongly regular graph are not determined by its parameters, the parameters do determine three of the eigenvalues. First, we need a lemma [5], [6].

Lemma 4.2 *Let A be a real symmetric matrix partitioned as follows:*

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix},$$

where $A_{i,i}$ is square for all $i = 1, \ldots, m$. Let $b_{i,j}$ be the average row sum of $A_{i,j}$, and let $B = (b_{i,j})$. If $A_{i,j}$ has constant row sums for all $i, j = 1, \ldots, m$, then every eigenvalue of B is also an eigenvalue of A.

Lemma 4.3 *Let G be a $psrg(v, k, \lambda, \mu)$. Then*

$$\frac{\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}$$

are both eigenvalues of G.

Proof. Let x be a $srv_G(\lambda, \mu)$. Now, partition $V(G)$ into three subsets, $\{x\}$, $N(x)$, and $\overline{N(x)} \setminus \{x\}$. This induces a partition of the adjacency matrix A of G, the average row sums of which are given by

$$B = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k - \lambda - 1 \\ 0 & \mu & k - \mu \end{pmatrix}.$$

The eigenvalues of B are k and $(\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})/2$. Since x is strongly regular, the block matrices in this partitioning of A have constant row sums. Therefore, by Lemma 4.2, these are also eigenvalues of A. □

Theorem 4.4 *There does not exist a connected partially strongly regular graph with exactly four distinct eigenvalues.*

Proof. Let G be a connected $psrg(v, k, \lambda, \mu)$ with exactly four distinct eigenvalues. Since G is connected and regular, G has an eigenvalue k with multiplicity one. Also, by Lemma 4.3, we know that

$$r, t = \frac{\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}$$

are eigenvalues of G. Clearly, $r \neq t$, so let u be the one remaining eigenvalue of G, and let f, g, and h be the multiplicities of r, t, and u, respectively.

Now we obtain the following three equations by counting eigenvalue multiplicities and by taking the traces of A and A^2.

$$1 + f + g + h = v, \tag{8}$$

$$k + fr + gt + hu = 0, \tag{9}$$

and

$$k^2 + fr^2 + gt^2 + hu^2 = vk. \tag{10}$$

Since $u^2 + (\mu - \lambda)u + \mu - k = 0$ if and only if $u = r$ or t, and r, t, and u are all distinct, we can solve equations (8), (9), and (10) for f, g, and h to obtain

$$h = \frac{(v-k-1)\mu - k(k-\lambda-1)}{u^2 + (\mu-\lambda)u + \mu - k}$$

(we do not give the ugly expressions for f and g). However, $(v-k-1)\mu - k(k-\lambda-1) = 0$ by Corollary 2.5. Therefore, $h = 0$, a contradiction. □

Corollary 4.5 *A connected graph is strongly regular if and only if it is partially strongly regular and has at most four distinct eigenvalues.*

Corollary 4.6 *A connected strictly partially strongly regular graph has at least five disinct eigenvalues.*

Applying Construction 3.1 with $m = 1$, $n = 4$, and $R = \{K_{3,3}\}$, one obtains a 4-$psrg(10, 3, 2, 0)$ with exactly four distinct eigenvalues. Thus, the assumption of connectedness is necessary in Theorem 4.4. Applying Construction 3.1 with $m = 2$, $n = 3$, and $R = \{C_4\}$, where C_4 is the cycle on four vertices, and taking the complement, we obtain a connected 6-$psrg(10, 7, 4, 7)$ with exactly five distinct eigenvalues, all of which happen to be integers. It would be interesting to try to obtain some sort of combinatorial characterization of connected partially strongly regular graphs with five eigenvalues.

5 Another Result

In this section, we will prove a slight strengthening of Corollary 2.8.

Theorem 5.1 *A graph on v vertices is strongly regular if and only if it is s-partially strongly regular with $s \geq v - 3$.*

Proof. Let G be a s-$psrg(v, k, \lambda, \mu)$ with $s \geq v - 3$. Let S be a set of $v - 3$ $srv_G(\lambda, \mu)$, and let $V(G) \setminus S = \{x, y, z\}$ be the remaining three vertices. We must show that x, y, and z are also $srv_G(\lambda, \mu)$. By Proposition 2.2, it suffices to consider the following two cases.

Case 1. Suppose that $x \sim y \sim z \sim x$. We must show that $\lambda(x,y) = \lambda(x,z) = \lambda(y,z) = \lambda$.

We will count the number of pairs $(u,w) \in V(G) \times V(G)$ such that $x \sim u \sim w \nsim x, w \neq x$, in two different ways. There are $(k-2)(k-1-\lambda)$ such pairs with $u \neq y, z$. There are also $k-1-\lambda(x,y)$ pairs with $u = y$, and $k-1-\lambda(x,z)$ pairs with $u = z$. This gives us a total of $(k-2)(k-\lambda-1)+2(k-1)-\lambda(x,y)-\lambda(x,z)$ such pairs. Or, we can just choose w in $v-k-1$ ways, and then choose u in μ ways, for a total of $(v-k-1)\mu$ pairs. This gives us the equation

$$(k-2)(k-\lambda-1)+2(k-1)-\lambda(x,y)-\lambda(x,z) = (v-k-1)\mu. \quad (11)$$

Similarly, counting pairs (u,w) such that $y \sim u \sim w \nsim y, w \neq y$, in two ways gives us the equation

$$(k-2)(k-\lambda-1)+2(k-1)-\lambda(x,y)-\lambda(y,z) = (v-k-1)\mu. \quad (12)$$

Finally, counting pairs (u,w) such that $z \sim u \sim w \nsim z, w \neq z$, gives us the equation

$$(k-2)(k-\lambda-1)+2(k-1)-\lambda(x,z)-\lambda(y,z) = (v-k-1)\mu. \quad (13)$$

Solving equations (11), (12), and (13) for $\lambda(x,y), \lambda(x,z)$, and $\lambda(y,z)$ gives us

$$\lambda(x,y), \lambda(x,z), \lambda(y,z) = \frac{(v-k-1)\mu - k(k-1) + \lambda(k-2)}{2}.$$

Using Corollary 2.5, we can replace $(v-k-1)\mu$ with $k(k-\lambda-1)$ in the above expression. This gives us $\lambda(x,y), \lambda(x,z), \lambda(y,z) = \lambda$, and so G is strongly regular.

Case 2. Suppose that $x \sim y \nsim z \nsim x$. We must show that $\lambda(x,y) = \lambda$ and $\mu(x,z) = \mu(y,z) = \mu$. We will count in two different ways the number of pairs $(u,w) \in V(G) \times V(G)$ such that $x \sim u \sim w \nsim x, w \neq x$. The number of pairs with $u \neq y$ is $(k-1)(k-1-\lambda)$, and the number of pairs with $u = y$ is $k-1-\lambda(x,y)$. This gives us a total of $(k-1)(k-\lambda-1)+k-\lambda(x,y)-1$ pairs. Also, the number of pairs with $w \neq z$ is $(v-2-k)\mu$, and the number of pairs with $w = z$ is $\mu(x,z)$. This gives us $(v-k-2)\mu(x,z)$ such pairs. Therefore, we have the equation

$$(k-1)(k-\lambda-1)+k-\lambda(x,y)-1 = (v-k-2)\mu(x,z). \quad (14)$$

Similarly, counting pairs (u, w) such that $y \sim u \sim w \nsim y, w \neq y$, gives us the equation

$$(k-1)(k-\lambda-1)+k-\lambda(x,y)-1=(v-k-2)\mu(y,z). \qquad (15)$$

Finally, we will count the number of pairs (u, w) such that $z \sim u \sim w \nsim z, w \neq z$, in two ways. The number of pairs with $w \neq x, y$ is $(v-3-k)\mu$, the number of pairs with $w = x$ is $\mu(x, z)$, and the number of pairs with $w = y$ is $\mu(y, z)$. This gives us a total of $(v-k-3)\mu+\mu(x,z)+\mu(y,z)$ such pairs. Or, we could just choose u in k ways, and then choose w in $k-1-\lambda$ ways, for a total of $k(k-\lambda-1)$ such pairs. This gives us the equation

$$(v-k-3)\mu+\mu(x,z)+\mu(y,z)=k(k-\lambda-1). \qquad (16)$$

Solving equations (14), (15), and (16) for $\lambda(x, y), \mu(x, z)$, and $\mu(y, z)$ gives us

$$\lambda(x,y)=\frac{(v-k-1)\mu-k(k-1)+\lambda(k-2)}{2}$$

and

$$\mu(x,z),\mu(y,z)=\frac{k(k-\lambda-1)-(v-k-3)\mu}{2}.$$

Using Corollary 2.5, we can substitute $k(k-\lambda-1)$ for $(v-k-1)\mu$ in the above expression for $\lambda(x, y)$, and we can substitute $(v-k-1)\mu$ for $k(k-\lambda-1)$ in the above expression for $\mu(x, z)$ and $\mu(y, z)$. This gives us $\lambda(x, y) = \lambda$ and $\mu(x, z) = \mu(y, z) = \mu$, and again G is strongly regular. □

Applying Construction 3.1 with $n = 3$ and $R = \{C_4\}$, we can obtain $(v-4)$-partially strongly regular graphs on v vertices for arbitrarily large v. Thus, Theorem 5.1, as weak as it is, is best possible. By Corollary 2.2, the assumption of connectedness would not change the situation. One wonders if the assumption that both G and $\overline{G}$ are connected would allow a stronger theorem to be proved.

6 Conjectures

In this section, we make several conjectures concerning partially strongly regular graphs. The first three conjectures are very similar to Theorem 5.1.

First, we conjecture the existence of a subtractive term, larger than the three in Theorem 5.1, for connected partially strongly regular graphs with connected complement.

Conjecture 6.1 There exists an integer $c \geq 4$, independent of v, such that a connected s-partially strongly regular graph on v vertices with connected complement is strongly regular if and only if $s \geq v - c$.

The next conjecture is a stronger multiplicative version of Conjecture 6.1.

Conjecture 6.2 There exists a constant $c < 1$, independent of v, such that a connected s-partially strongly regular graph on v vertices with connected complement is strongly regular if and only if $s \geq cv$.

Specializing Conjecture 6.2, we obtain the next conjecture, which the author's intuition tells him must be true (so it probably is not).

Conjecture 6.3 A connected s-partially strongly regular graph on v vertices with connected complement is strongly regular if and only if $s \geq v/2$.

By Theorem 5.1, a strictly s-partially strongly regular graph on v vertices must have $s \leq v - 4$. Our next conjecture concerns the structure of such graphs that are extremal with respect to this inequality.

Conjecture 6.4 Let G be a $(v-4)$-partially strongly regular graph on v vertices. Then $G \cong mK_3 \cup C_4$ or $G \cong \overline{mK_3 \cup C_4}$ for some integer $m \geq 1$.

Corollary 4.6 states that a connected strictly partially strongly regular graph has at least five distinct eigenvalues. Our final conjecture is an upper bound on the number of disinct eigenvalues of a partially strongly regular graph.

Conjecture 6.5 An s-partially strongly regular graph on v vertices has at most $2\lceil v/s \rceil + 1$ distinct eigenvalues.

Of course, Conjecture 6.5 says nothing when $s = 1$ or 2. However, it is probably not possible to say much about graphs with such a small s-value anyway. For instance, there is a 1-$psrg(9, 4, 1, 2)$ with all nine eigenvalues distinct.

7 Small Graphs

In this final section, we list all strictly partially strongly regular graphs on at most ten vertices. The following table lists, for each of the 31 such graphs, the parameters (v, k, λ, μ), the number s, and the number of distinct eigenvalues ($L(G)$ denotes the line graph of a graph G, and C is the unique cubic graph with six vertices and girth three).

no.	v	k	(λ,μ)	s	distinct eigenvalues	comments
1.	7	2	(1,0)	3	4 integer	$K_3 \cup C_4$
2.	7	4	(1,4)	3	5 integer	$\overline{K_3 \cup C_4}$
3.	8	2	(1,0)	3	4 (2 integer)	$K_3 \cup C_5$
4.	8	5	(2,5)	3	5 (3 integer)	$\overline{K_3 \cup C_5}$
5.	9	2	(1,0)	3	4 integer	$K_3 \cup C_6$
6.	9	4	(0,3)	1	7 (1 integer)	Construction 3.4, 3.5
7.	9	4	(1,2)	1	7 integer	Construction 3.5
8.	9	4	(1,2)	1	7 (5 integer)	Construction 3.5
9.	9	4	(1,2)	1	7 (5 integer)	complement of #8
10.	9	4	(1,2)	1	7 (5 integer)	Construction 3.5
11.	9	4	(1,2)	1	9 (5 integer)	Construction 3.5
12.	9	4	(1,2)	3	5 integer	Const. 3.2 ($H = L(K_{3,3})$)
13.	9	4	(1,2)	3	5 integer	complement of #12
14.	9	4	(2,1)	1	7 (1 integer)	complement of #6
15.	9	6	(3,6)	3	5 integer	$\overline{K_3 \cup C_6}$
16.	10	2	(1,0)	3	5 (2 integer)	$K_3 \cup C_7$
17.	10	2	(1,0)	6	4 integer	$2K_3 \cup C_4$
18.	10	3	(0,1)	1	6 (3 integer)	Construction 3.3, 3.5
19.	10	3	(0,1)	2	7 (5 integer)	Construction 3.5
20.	10	3	(0,1)	2	7 (5 integer)	Construction 3.5
21.	10	3	(0,1)	4	5 integer	Const. 3.2 ($H = \overline{L(K_5)}$)
22.	10	3	(2,0)	4	4 integer	$K_4 \cup K_{3,3}$
23.	10	3	(2,0)	4	5 integer	$K_4 \cup C$
24.	10	6	(2,6)	4	5 integer	$\overline{K_4 \cup K_{3,3}}$
25.	10	6	(2,6)	4	6 integer	$\overline{K_4 \cup C}$
26.	10	6	(3,4)	1	6 (3 integer)	complement of #18
27.	10	6	(3,4)	2	7 (5 integer)	complement of #19
28.	10	6	(3,4)	2	7 (5 integer)	complement of #20
29.	10	6	(3,4)	4	5 integer	complement of #21
30.	10	7	(4,7)	3	6 (3 integer)	$\overline{K_3 \cup C_7}$
31.	10	7	(4,7)	6	5 integer	$\overline{2K_3 \cup C_4}$

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