Results and Open Problems on Minimum Saturated Hypergraphs

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Abstract

Let $\mathcal{F}$ be a family of k-graphs. A k-graph G is called *$\mathcal{F}$-saturated* if it a maximal graph not containing any member of $\mathcal{F}$ as a subgraph. We investigate the smallest number of edges that an $\mathcal{F}$-saturated graph on n vertices can have. We present new results and open problems for different instances of $\mathcal{F}$.

1 Introduction

A *k-hypergraph* H is, as usual, a pair $(V(H), E(H))$ (*vertices* and *edges*) where

$$E(H) \subset \binom{V(H)}{k} := \{A \subset V(H) : |A| = k\}.$$

We sometimes call H a *k-graph* or even simply a *graph* when k is understood. The *size* of H is $e(H) := |E(H)|$ and the *order* is $v(H) := |V(H)|$.

Given a family $\mathcal{F}$ of k-graphs (which are typically called *forbidden*), we say that a k-graph H is *$\mathcal{F}$-free* if no $F \in \mathcal{F}$ is a subgraph of H. Next, H is *$\mathcal{F}$-saturated* if it is maximal $\mathcal{F}$-free (that is, H is $\mathcal{F}$-free but the addition of any extra edge to H violates this property). Let

$$\mathrm{SAT}(n, \mathcal{F}) := \{H : H \text{ is } \mathcal{F}\text{-saturated},\ v(H) = n\}$$

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consist of all $\mathcal{F}$-saturated graph of order n. We are interested in the smallest size of a such graph, that is, in

$$\mathrm{sat}(n,\mathcal{F}) := \min\{e(H) : H \in \mathrm{SAT}(n,\mathcal{F})\}. \tag{1}$$

If $\mathcal{F}$ has only a single member F, we write $\mathrm{sat}(n,F)$ instead of $\mathrm{sat}(n,\{F\})$, etc.

The first sat-type results were obtained by Erdős, Hajnal and Moon [11] and by Bollobás [4] (see (2)); the current notation comes from Bollobás' book [6].

Kászonyi and Tuza [16] showed that $\mathrm{sat}(n,\mathcal{F}) = O(n)$ for any (possibly infinite) family $\mathcal{F}$ of 2-graphs. The author [19] proved the estimate $\mathrm{sat}(n,\mathcal{F}) = O(n^{k-1})$ for any *finite* family $\mathcal{F}$ of k-graphs.

Problem 1 *Does* $\mathrm{sat}(n,\mathcal{F}) = O(n^{k-1})$ *for any infinite family* $\mathcal{F}$ *of* k*-graphs?*

The sat-function lacks many natural regularity properties as it is observed by Kászonyi and Tuza [16]. In Section 2 we present a few further results of this type. We demonstrate a pair of connected graphs $F_1 \subset F_2$ on the same vertex set such that $\mathrm{sat}(n,F_1) > \mathrm{sat}(n,F_2)$ for all $n \geq v(F_1)$. Also, for any constant d, we build a 2-graph $F = F(d)$ such that

$$\mathrm{sat}(n,F) < \min\big(\mathrm{sat}(n-1,F), \mathrm{sat}(n+1,F)\big) - d,$$

for a periodic series of values of n.

Tuza [25] made the following conjecture.

Conjecture 2 *For any* 2*-graph* F*, the limit* $\lim_{n\to\infty} \mathrm{sat}(n,F)/n$ *exists.*

The author [19] demonstrated an example of an *infinite* family $\mathcal{F}$ of graphs such that $\mathrm{sat}(n,\mathcal{F})/n$ does not tend to a limit. Here we improve on this by demonstrating a *finite* 'irregular' family $\mathcal{F}$. But Tuza's conjecture remains open as a smallest family that we can construct consists of 4 forbidden graphs.

A number of results have been obtained for special families $\mathcal{F}$ (see e.g. [11, 4, 8, 18, 16, 26, 24, 10, 3, 28, 19, 20, 21]). Here we present a few more.

Bollobás [4] computed the sat-function for the complete k-graph of order m:

$$\mathrm{sat}(n,K_m^k) = \binom{n}{k} - \binom{n-m+k}{k}. \tag{2}$$

This extends the result of Erdős, Hajnal and Moon [11] who had previously proved (2) for $k=2$. Minimum K_m^2-saturated graphs of given minimum degree were studied by Duffus and Hanson [9] and by Alon, Erdős, Holzman and Krivelevich [2]. A result from the latter paper is improved here in Section 3.

In Section 4 we compute $\text{sat}(n, K_l^2 + \bar{K}_m^2)$ for all $n \geq n_0(l, m)$.

In Section 5 we forbid three k-edges such that the symmetric difference of two is contained in the third one and show that the corresponding sat-function equals $n - O(\log n)$. For $k = 3$ we compute the sat-function exactly. (The case $k = 2$ is trivial.)

The paper contains some other results and open problems that are scattered throughout the text.

2 Irregularities

Here we demonstrate some irregularities of the sat-function in the comparison to the *Turán function*

$$\begin{aligned} \text{ex}(n, \mathcal{F}) &= \max\{e(G) : G \in \text{SAT}(n, \mathcal{F})\} \\ &= \max\{e(G) : v(G) = n,\ G \text{ is } \mathcal{F}\text{-free}\}. \end{aligned}$$

Clearly, $\text{ex}(n, F_1) \leq \text{ex}(n, F_2)$ whenever F_1 is a subgraph of F_2. Kászonyi and Tuza [16] demonstrated an example of $F_1 \subset F_2$ with $\text{sat}(n, F_1) > \text{sat}(n, F_2)$ for all large n. Tuza [27, p. 401] asked if there exists a *connected irregular* pair $F_1 \subset F_2$; this is answered in the affirmative by the following simple example.

Example 3 *There is a pair of connected graphs $F_1 \subset F_2$ on the same vertex set such that* $\text{sat}(n, F_1) > \text{sat}(n, F_2)$ *for all* $n \geq v(F_1)$.

Proof. Let $m \geq 4$. Let F_1 be the star $K_{1,m}$, that is, $V(F_1) = [m+1]$ and $E(F_1) = \{\{1, i\} : i \in [2, m+1]\}$, and let F_2 be obtained from F_1 by adding the edge $\{2, 3\}$. Clearly, $\text{sat}(n, F_2) \leq n - 1$, $n \geq m + 1$, as $K_{1,n-1}$ is an example of an F_2-saturated graph.

On the other hand, in any F_1-saturated graph G, any two vertices of degree at most $m - 2$ must be connected. (Otherwise the addition of this edge cannot create a forbidden subgraph.) If we have $v \in [0, m-1]$ such vertices, then $e(G) \geq \binom{v}{2} + \frac{m-1}{2}(n - v)$, which is easily seen to exceed $n - 1$ for all $n \geq m + 1$. ∎

Clearly, for every $n \geq v(F)$, we have $\text{ex}(n, F) \leq \text{ex}(n + 1, F)$. On the other hand, Kászonyi and Tuza [16] observed that, for any odd $n = 2k - 1$, we have $\text{sat}(n, P_3) = k + 1 > \text{sat}(n + 1, P_3) = k$, where P_3 is the path with three edges. Our next example amplifies this irregularity.

Example 4 *For every constant d, there is a 2-graph $F = F(d)$ such that*

$$\mathrm{sat}(n, F) < \min\left(\mathrm{sat}(n-1, F), \mathrm{sat}(n+1, F)\right) - d,$$

for a periodic series of values of n.

Proof. Let $m = 2d + 3$ and let $F = B_{m,m}$ be the *dumb-bell*

$$E(B_{m,m}) = \binom{[m]}{2} \cup \binom{[m+1, 2m]}{2} \cup \{\{1, m+1\}\},$$

that is, $B_{m,m}$ is the disjoint union of two copies of K_m^2 plus one edge connecting them.

Let us show that the claim is true for any $n = lm$ if $l \in \mathbb{N}$ is large. Clearly, $\mathrm{sat}(lm, F) \leq lm(m-1)/2$ (in fact, this is sharp) as $lK_m^2 \in \mathrm{SAT}(lm, F)$, where lF denotes the union of l disjoint copies of F. On the other hand, let $n = lm - 1$ and suppose that $G \in \mathrm{sat}(n, F)$ has at most $g = lm(m-1)/2 + d$ edges.

Clearly, $\delta(G)$, the minimal degree of G, is at least $\delta(B_{m,m}) - 1 = m - 2$. Suppose that for some $x \in V(G)$ we have $d(x) = m - 2$. Then for every y non-incident to x the edge $\{x, y\} \in E(\bar{G})$ cannot be the bridge in a created $B_{m,m}$-subgraph as the degree of x is too small; that is, x and y fall into the same K_m^2-half. Therefore, y must be connected to all $m - 2$ neighbours of x and $e(G) \geq (m-2)n + O(1)$ which is a contradiction.

Hence $\delta(G) \geq m - 1$. The inequality $\Delta(G) + (m-1)(n-1) \leq 2e(G) \leq 2g$ implies that $\Delta(G) \leq 2(d + m - 1)$. If some $x \in V(G)$ does not belong to an m-clique then any missing edge $\{x, y\}$ must create a K_m^2-subgraph and we arrive at a contradiction again, as $d(x) \leq \Delta(G)$ is bounded. Thus the whole of $V(G)$ is covered by m-cliques.

We want to find a set $X \subset V(G)$ with the *surplus* $s(X) = e(G[X]) - \frac{m-1}{2}|X|$ at least $m - 1$ as then we would obtain a contradiction:

$$e(G) \geq e(G[X]) + \frac{m-1}{2}(n - |X|) \geq \frac{m-1}{2}n + m - 1 > g.$$

As m does not divide n, there are two distinct cliques $A, B \in \binom{V(G)}{m}$ with $i = |A \cap B| > 0$. It is straightforward to verify that

$$s(A \cup B) \geq 2\binom{m}{2} - \binom{i}{2} - \frac{m-1}{2}(2m - i) \geq \frac{m-1}{2}.$$

No m-clique $C \neq A, B$ can intersect some other clique or $A \cup B$. (Otherwise we gain another surplus of at least $(m-1)/2$.) By the divisibility argument, $i = 1$. As a $(2m-1)$-clique has surplus at least $m - 1$, there exists some $E \in E(\bar{G})$ lying within $A \cup B$. It is easy to see that $G + E$ must contain a K_m^2-subgraph on some m-set $C \neq A, B$ intersecting $A \cup B$ in at least two vertices, which implies $s(A \cup B \cup C) \geq m - 1$ as required.

Let $n = ml + 1$ and $G \in \mathrm{SAT}(n, B_{m,m})$. If $\delta(G) = m - 2$, then we argue as above that $e(G) \geq (m-2)n + O(1)$; otherwise $e(G) \geq \frac{m-1}{2} n > g$, which completes the proof. ∎

Next, we present an example of a finite family $\mathcal{F}$ of 2-graphs such that the ratio $\mathrm{sat}(n, \mathcal{F})/n$ does not tend to a limit. The fewest number of elements in $\mathcal{F}$ that our proof gives is four (take $m = 4$). It may be possible that working harder one can further reduce this number but the ultimate aim, a counterexample to Conjecture 2, seems out of reach to our method.

Example 5 *There exists a finite family $\mathcal{F}$ of 2-graphs such that, for some $c > 0$ and for infinitely many n,*

$$\mathrm{sat}(n, \mathcal{F}) < \min\left(\mathrm{sat}(n-1, \mathcal{F}), \mathrm{sat}(n+1, \mathcal{F})\right) - cn.$$

In particular, the ratio $\mathrm{sat}(n, \mathcal{F})/n$ *does not tend to a limit.*

Proof. Fix $m \geq 4$ and consider the family $\mathcal{F}$ consisting of the dumb-bell $B_{m,m}$ and $F_{m,1}, \ldots, F_{m,m-1}$, where

$$E(F_{m,i}) = \binom{[m]}{2} \cup \binom{[m-i+1, 2m-i]}{2}, \quad i \in [m-1],$$

that is, $F_{m,i}$ is the union of two K_m^2-graphs sharing i common vertices.

Clearly, the disjoint union of K_m^2-graphs is $\mathcal{F}$-saturated as any missing edge connects two different copies and thus creates a $B_{m,m}$-subgraph. Hence, if m divides n then $\mathrm{sat}(n, \mathcal{F}) \leq \frac{n}{m}\binom{m}{2}$.

On the other hand, suppose that m does not divide n and let G be any $\mathcal{F}$-saturated graph on $[n]$. By the definition of $\mathcal{F}$, no vertex can belong to two different K_m^2-subgraphs of G; suppose that the sets $A_i = [m(i-1)+1, mi]$, $i \in [s]$, are all m-sets spanning complete subgraphs in G. Denote $A_I = \cup_{i \in I} A_i$, $I \subset [s]$.

Note the following two properties of G. *Property A*: $G[A_{[s]}] \cong sK_m^2$. (Because $B_{m,m}$ is forbidden.) *Property B*: any missing edge E intersecting $B = [n] \setminus A_{[s]}$ creates a K_m^2-subgraph. (Because it is impossible that $B_{m,m} \subset G + E$ with E being the bridge.)

We claim that these two properties and the fact that $B \neq \emptyset$ (as m is not a divisor of n) imply that

$$e(G) \geq \frac{n}{m}\left(\binom{m}{2} + m - 2\right) - m^2. \tag{3}$$

We use induction on n. If some $E \in \binom{B}{2}$ is not a G-edge then it is easy to check that the graph G' obtained from G by contracting the edge E has the properties in question. The endvertices of E have at least $m-2$ common neighbours in G (because E creates a K_m^2-subgraph) so $e(G) \geq e(G')+m-2$ and (3) follows by induction. (Here we need the inequality $m \geq 4$.)

Suppose that B spans the complete graph in G. If some $E \in E(\bar{G})$ intersects both A_i and B then a K_m^2-subgraph created by E lies within $A_i \cup B$ and so at least $m-2$ G-edges intersect both A_i and B. Therefore,

$$e(G) \geq f(b) = (n-b)\frac{m-1}{2} + \binom{b}{2} + \frac{n-b}{m}(m-2),$$

where $b = |B|$. (We correspondingly count the edges within $A_{[s]}$, within B and in between.) The minimum of f is achieved for $b = \frac{m}{2} + \frac{m-2}{m}$ and our estimate (3) follows rather crudely.

Hence, if we increase/decrease $n = ml$ by one, then $\mathrm{sat}(n, \mathcal{F})$ increases at least by $n\frac{m-2}{m} + O(1)$. ∎

For k-graphs, $k \geq 3$, we are able to prove only the following.

Example 6 *For any $k \geq 3$, there is a finite family $\mathcal{F}_k$ of k-graphs such that* $\mathrm{sat}(n, \mathcal{F}_k) = O(n)$ *but* $\mathrm{sat}(n, \mathcal{F}_k)/n$ *does not tend to any limit as $n \to \infty$.*

Proof. Let $\mathcal{I}_{k,i}$ be the finite family consisting of all (up to isomorphism) k-graphs with at most $k-i+2$ edges whose common intersection has fewer than i vertices.

Note that any $\mathcal{I}_{k,i}$-free k-graph H is *i-intersecting*, that is, $|\cap_{E \in E(H)} E| \geq i$. Indeed, let I be any edge of H and then, as long as possible, if there is $E \in E(H)$ with $I \not\subset E$, replace I by $I \cap E$; if eventually $|I| < i$ then there must be at most $k-i+2$ edges whose intersection has size at most $i-1$, which is forbidden.

Given a 2-graph G, we fix a $(k-2)$-set X disjoint from $V(G)$ and define $C_{k-2}(G)$ by $E(C_{k-2}(G)) = \{E \cup X : E \in E(G)\}$.

Now, let $\mathcal{F}_k = \{C_{k-2}(F) : F \in \mathcal{F}\} \cup \mathcal{I}_{k,k-2}$, where $\mathcal{F}$ is the family constructed in Example 5. It is easy to see that $\mathrm{sat}(n, \mathcal{F}_k) = \mathrm{sat}(n-k+2, \mathcal{F})$ and the claim follows. ∎

Problem 7 *Is there a finite family $\mathcal{F}$ of k-graphs, $k \geq 3$, for which the ratio* $\mathrm{sat}(n, \mathcal{F})/n^{k-1}$ *does not tend to any limit?*

3 Complete Graphs

Duffus and Hanson [9] consider $\mathrm{sat}(n, K_m^2, l)$ which is the minimum size of a graph in

$$\mathrm{SAT}(n, K_m^2, l) := \{G \in \mathrm{SAT}(n, K_m^2) : \delta(G) \geq l\}.$$

Of course, any K_m^2-saturated graph G has minimum degree at least $m-2$, so we assume $l \geq m-1$.

Duffus and Hanson [9] proved that, for $n \geq 5$, $\mathrm{sat}(n, K_3^2, 2) = 2n-5$ and, for $n \geq 10$, $\mathrm{sat}(n, K_3^2, 3) = 3n-15$. However, their general lower bound [9, Theorem 2], which states that $\mathrm{sat}(n, K_m^2, l) \geq \frac{l+m-2}{2}n + O(1)$, is far from the actual value. Trying to improve this bound, I showed that $\mathrm{sat}(n, K_m^2, l) = ln + O(\frac{n \log\log n}{\log n})$ for any fixed $l \geq m-1$. Later, I learned that Alon, Erdős, Holzman and Krivelevich [2, Theorem 2] had showed that any $G \in \mathrm{SAT}(n, K_m^2)$ with $O(n)$ edges has an independent set of size $n - O(\frac{n}{\log\log n})$, which implies that $\mathrm{sat}(n, K_m^2, l) = ln + O(\frac{n}{\log\log n})$. However, I decided to present my proof because it improves all these bounds and I think that the general Theorem 8 is of independent interest.

However, the question of Bollobás [7, p. 1271] whether $\mathrm{sat}(n, K_3^2, l) = ln + O(1)$ for any fixed $l \geq 4$, remains open.

Let us give a construction of $G \in \mathrm{SAT}(n, K_m^2, l)$ with $ln + O(1)$ edges: take $G = K_{m-3}^2 + K_{l-m+3,n-l}$ which has minimum degree l for $n \geq 2l-m+3$. The complete bipartite graph $K_{l-m+3,n-l}$ does not contain a triangle but the addition of any new edge violates this; hence, G is K_m^2-saturated.

To prove our lower bound we need some preliminaries. Given any d, define $a_{d-m+2} = 2$ and, consecutively for $j = d-m+1, d-m, \ldots, 1, 0$,

$$\begin{aligned}
c_{j+1} &= (m-2)(a_{j+1}-1)+1\\
b_{j+1} &= (m-2)(c_{j+1}-1)+1\\
b'_{j+1} &= \binom{d-j-1}{m-2}(b_{j+1}-1)+1,\\
a_j &= \binom{d-j-1}{m-2}(b'_{j+1}-1)+2.
\end{aligned}$$

Finally, let $a = (1+2(d-1)+2(d-1)^2)a_0$.

Given a K_m^2-saturated graph G, let A denote the set of G-edges connecting two vertices of degree at most d in G:

$$A = \{\{x,y\} \in E(G) : d(x) \leq d,\ d(y) \leq d\}.$$

The following theorem states that the size of A is bounded by $a = a(d,m)$ which does not depend on n. Note that we do not impose any restriction on the minimal degree of G.

Theorem 8 *For any $G \in \mathrm{SAT}(n, K_m^2)$, $m \geq 3$, we have $|A| < a$.*

Proof. Suppose, on the contrary, that $|A| \geq a$.

We prove, by induction on $j = 0, 1, \ldots, d - m + 2$, that we can find the following configuration in G: a_j-sets X_j and Y_j and j-sets U_j and V_j (all disjoint) such that (i) $X_j \cup Y_j$ induces in G exactly a_j edges which form a perfect matching between X and Y and belong to A; (ii) $\Gamma_{U_j \cup V_j}(x) = U_j$ for any $x \in X_j$ and $\Gamma_{U_j \cup V_j}(y) = V_j$ for any $y \in Y_j$.

For $j = 0$ (when U_0 and V_0 are empty), we take, one by one, edges from A. Once we have selected an edge $E \in A$, cross out all incident to E edges (at most $2(d-1)$ edges) and their neighbouring edges (of which at most $2(d-1)^2$ can belong to A). Hence, we can build an induced matching of size at least $|A|/(1 + 2(d-1) + 2(d-1)^2) \geq a_0$ as required.

Suppose that $j \in [0, d - m + 1]$ and we have X_j, etc., constructed. Choose $x \in X_j$; it has already got $j + 1$ neighbours in G: the neighbour $y \in Y_j$ plus all j vertices of U_j. Let N_x denote the remaining neighbours of x; thus $|N_x| \leq d - j - 1$. For any $z \in Y_j$ distinct from y, the addition of the edge $\{x, z\}$ must create a copy of K_m^2, say on a set $D_z \cup \{x, z\}$. Now, $D_z \subset \Gamma(x) \cap \Gamma(z) \subset N_x$.

Thus some set D_z, $z \in Y_j \setminus \{y\}$, appears at least $b'_{j+1} = \lceil (a_j - 1)/\binom{d-j-1}{m-2} \rceil$ times; suppose it is $D \in \binom{N_x}{m-2}$ which equals D_z for $z \in B' \subset Y_j \setminus \{y\}$, $|B'| = b'_{j+1}$. In a similar manner, we try to connect y to the X_j-matches of B'-vertices and find a set $E \in \binom{N_y}{m-2}$ spanning the complete graph and connected to every z from a set $B \subset X_j$ matched into B' of cardinality $b_{j+1} = \lceil b'_{j+1}/\binom{d-j-1}{m-2} \rceil$.

Clearly, no $z \in B$ can be connected to every vertex of D; otherwise D, z and the match of z in B' span K_m^2. Therefore, some $v \in D$ is not connected to at least $c_{j+1} = \lceil \frac{b_{j+1}}{m-2} \rceil$ vertices of B; let $C \subset B$ consist of all such vertices. Similarly, we can find $u \in E$, not connected to an a_{j+1}-set Y_{j+1} matched into C. Of course, $u \neq v$. Now, let $U_{j+1} = U_j \cup \{u\}$, $V_{j+1} = V_j \cup \{v\}$, and let $X_{j+1} \subset X_j$ consist of the matches of Y_{j+1}, which completes our induction.

At the end, we try to apply our argument again, for $j = d - m + 2$. We obtain that $x \in X_j$ has at least $1 + j + (m-2) > d$ neighbours, which contradicts the fact that $\{x, y\} \in A$, where y is the Y_j-match of x. ∎

Now we are ready to improve the result of Alon et al [2, Theorem 2] mentioned above. Let $\alpha(G)$ denote the maximum size of independent $Y \subset V(G)$.

Lemma 9 *For any $G \in \mathrm{SAT}(n, K_m^2)$ with $O(n)$ edges, we have*

$$\alpha(G) = n - O\left(\frac{n \log\log n}{\log n}\right).$$

Proof. Suppose $e(G) \le Cn$. Let $d = \frac{\varepsilon \log n}{\log\log n}$ for some fixed $\varepsilon > 0$ and let $X = \{x \in V(G) : d(x) > d\}$. Now, $d|X|/2 \le e(G) \le Cn$ implies that

$$|X| \le \frac{2Cn \log\log n}{\varepsilon \log n}.$$

By Theorem 8, $Y = V(G) \setminus X$ spans at most $a \le n^{2\varepsilon(m-2)+o(1)}$ edges. Removing at most a vertices we can make Y independent; it has the required size if $\varepsilon < \frac{1}{2(m-2)}$. ∎

Clearly, $e(G) \ge \alpha(G)\delta(G)$. Therefore, Lemma 9 implies the following result.

Theorem 10 *For any fixed $l \ge m-1$, $\mathrm{sat}(n, K_m^2, l) = ln + O(\frac{n \log\log n}{\log n})$.* ∎

4 Generalised Stars

The graph $S_{l,m} = K_l^2 + \bar{K}_m^2$ can be viewed as a generalisation of a star $K_{1,m}$, so we call it a *generalised star.*

The sat-function for $S_{1,m} = K_{1,m}$ was computed by Kászonyi and Tuza [16]:

$$\mathrm{sat}(n, K_{1,m}) = \begin{cases} \binom{m}{2} + \binom{n-m}{2}, & \text{if } m+1 \le n \le (3m-1)/2, \\ \lceil (m-1)n/2 - m^2/8 \rceil, & n \ge 3m/2 \end{cases} \tag{4}$$

Clearly, $G + K_{l-1}^2$ is $S_{l,m}$-saturated for any $K_{1,m}$-saturated graph G. This shows that

$$\mathrm{sat}(n, S_{l,m}) \le \mathrm{sat}(n-l+1, K_{1,m}) + \binom{l-1}{2} + (l-1)(n-l+1). \tag{5}$$

We can show that this bound is sharp for all sufficiently large n. (This may be true for *all* $n \ge m + l$ but the author was not able to work this out.)

Theorem 11 *There is $n_0 = n_0(l, m)$ such that we have equality in* (5) *for all $n \ge n_0$.*

Proof. We use induction on l. There is nothing to do in the case $l = 1$. Let $l \geq 2$, n be large and G be a minimum $S_{l,m}$-saturated graph of order n. Observe that

$$e(G) \leq \left(l - 1 + \frac{m-1}{2}\right) n + O(1). \qquad (6)$$

If G has a vertex x of degree $n-1$, then we are done by induction as $G - x \in \mathrm{SAT}(n-1, S_{l-1,m})$. Thus we assume that $\Delta(G) \leq n-2$ and try to derive a contradiction. Let the vertices of G be $x_1, \ldots, x_n$ of degrees $d_1 \geq \cdots \geq d_n$ respectively.

Let p be the number of induced paths of length two in G. Observe that every pair of adjacent points of $S_{l,m}$ can be connected by at least $l-1$ edge-disjoint paths of length two. Hence, each edge from $\bar{G}$ contributes at least $l-1$ to p, that is, $p \geq (l-1)e(\bar{G})$. On the other hand, $p \leq \sum_{i=1}^{n} \binom{d_i}{2}$.

Any two vertices of degree at most $l + m - 3$ must be connected in G (otherwise the addition of this edge to G cannot create $S_{l,m}$). Hence, we have at most $l + m - 2$ such vertices and the degrees of G satisfy the following inequalities.

$$n - 2 \geq d_1 \geq \cdots \geq d_{n-l-m+2} \geq l + m - 2. \qquad (7)$$

For any $x \geq y$ the expression $\binom{x}{2} + \binom{y}{2}$ gets larger if we increase x by 1 while decreasing y by 1. Hence, we can find a sequence $(d'_i)_{i \in [n]}$ such that $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$, $\sum_{i=1}^{n} \binom{d_i}{2} \leq \sum_{i=1}^{n} \binom{d'_i}{2}$ and, for some $j \in [l, n-l-m+2]$, we have

$$d'_i = \begin{cases} d_i, & i \in [l-1] \cup [n-l-m+3, n], \\ d_{l-1}, & i \in [l, j-1], \\ l+m-2, & i \in [j+1, n-l-m+2]. \end{cases}$$

(We do not know anything about d'_j except that $d_{l-1} \geq d'_j \geq l + m - 2$.) Thus we obtain

$$(l-1)\binom{n}{2} - O(n) \leq p \leq \sum_{i=1}^{n} \binom{d'_i}{2} \leq \frac{n}{2} \sum_{i=1}^{j} d'_i + \frac{d'_{l-1}}{2}(d'_{l-1} - n) + O(n). \quad (8)$$

Observe that $d'_{l-1} = d_{l-1} = \Omega(n)$ (otherwise $\sum_{i=1}^{n} \binom{d_i}{2} \leq (l-2)\binom{n}{2} + o(n^2)$). From (6) we conclude that $j = O(1)$.

If $n - d_{l-1} = \Omega(n)$, then $d'_1 + \cdots + d'_j \geq (l-1)n + \Omega(n)$ by (8). Also, $\sum_{i=j+1}^{n} d_i \geq (l + m - 2)n + O(1)$. But then we obtain the contradiction

$$e(G) = \frac{1}{2} \sum_{i=1}^{n} d'_i \geq \left(l - 1 + \frac{m-1}{2}\right) n + \Omega(n).$$

Hence, each of $d_1, \ldots, d_{l-1}$ is $n + o(n)$. Let $X = \{x_1, \ldots, x_{l-1}\}$. Choose some $y \in V(G) \setminus \Gamma(x_1)$. Let $H = G - X - y$ and $A = \Gamma_{\bar{X}}(y) \subset V(H)$. For any $z \in V(H) \setminus A$, the addition of the edge $\{y, z\}$ to G creates a copy of $S_{l,m}$ which contains a set B of at least $l-1$ vertices connected to both y and z. Of course, $x_1 \notin B$; hence, $B \not\subset X$. Let $v \in B \setminus X \subset A$; we have $\{v, z\} \in E(H)$. As $z \in V(H) \setminus A$ was arbitrary, we conclude that $A \subset V(H)$ is a dominating set in H.

We know that we have at most $l + m - 2$ vertices of G-degree less than $l + m - 2$. For any other vertex $x \in V(H)$ we clearly have: $d_H(x) \geq m - 1$ if $x \in V(H) \setminus A$ and $d_H(x) \geq m - 2$ if $x \in A$. Let $a = |A|$. Note that

$$\sum_{x \in A} d_H(x) \geq \max\big(a(m-2), n-a\big) + O(1)$$

because A is a dominating set. Hence,

$$\begin{aligned} a + e(H) &\geq a + \frac{1}{2}((n-a)(m-1) + \max(a(m-2), n-a)) + O(1) \\ &\geq \left(\frac{m-1}{2} + \frac{1}{2(m-1)}\right) n + O(1), \end{aligned}$$

where the latter inequality is obtained by the straightforward minimisation with respect to a (the minimum occurs when $a = \frac{n}{m-1} + O(1)$). We have

$$e(G - X) \geq |A| + e(H) \geq \left(\frac{m-1}{2} + \frac{1}{2(m-1)}\right) n + O(1).$$

This gives at least $(l - 1 + \frac{m-1}{2} + \frac{1}{2(m-1)})n + o(n)$ edges in G, which is the desired contradiction. ∎

The above construction generalises to the following settings. Let $S^k_{l,m}$ have $l + m$ vertices and consist of all edges intersecting some fixed l-set of vertices called the *centre*. Thus, $S_{l,m} = S^2_{l,m}$; also, for example, $e(S^k_{l,m}) = \binom{l+m}{k} - \binom{m}{k}$. The value of $\mathrm{sat}(n, S^k_{1,k})$ was asymptotically computed by Erdős, Füredi and Tuza [10] and $\mathrm{sat}(n, S^k_{1,m})$ by the author [20]. What is $\mathrm{sat}(n, S^k_{l,m})$ in general?

We have the following construction. Given $n \geq l + m$, let $A = [l-1]$, $u = m - k + 2$ and $n' = \lceil (n - l + 1)/u \rceil$. Partition $[n] \setminus A$ into *blocks* $B_1, \ldots, B_{n'}$ of size u each except possibly the last one. Our k-graph G consists of the edges intersecting A plus those edges intersecting the first block they meet in at least 2 vertices. It is easy to see that $S^k_{l,m}$ is not a subgraph of G but the addition of any new edge E creates an $S^k_{l,m}$-subgraph on $E \cup A \cup B_j$ centred at $A \cup \{v\}$, where B_j is the first block meeting E and $\{v\} = B_j \cap E$. Thus, G is $S^k_{l,m}$-saturated and we have an upper bound which we conjecture to be asymptotically sharp.

Conjecture 12 *For any fixed positive integers k, l, m with $m \geq k$, we have*

$$\mathrm{sat}(n, S^k_{l,m}) = \frac{m + 2l - k - 1}{2\,(k-1)!}\, n^{k-1} + o(n^{k-1}).$$

5 Triangular Families

The notion of a triangle-free 2-graph can be extended to hypergraphs in the following way suggested by Katona [17]: a k-graph is *triangle-free* if the symmetric difference of any two distinct edges is not contained in a third edge. Clearly, this is the same as forbidding the *triangular family* $\mathcal{T}_k$ which consists of all k-graphs with three edges E_1, E_2, E_3 such that $E_1 \triangle E_2 \subset E_3$.

We have the following obvious example of a $\mathcal{T}_k$-saturated graph: the *pyramid* P^k_n which consists of all k-subsets of $[n]$ containing the set $[k-1]$. Indeed, any missing edge E intersects $[k, n]$ in at least 2 points and creates a forbidden subgraph on the set $E \cup [k-1]$. Thus

$$\mathrm{sat}(n, \mathcal{T}_k) \leq n - k + 1, \quad n \geq k + 1.$$

and this might be sharp.

In the general case we are able to prove only the following.

Theorem 13 *Let $k \geq 3$ be fixed. Then*

$$n - O(\log n) \leq \mathrm{sat}(n, \mathcal{T}_k) \leq n - k + 1.$$

Proof. We have to prove the lower bound. Let G be a minimum $\mathcal{T}_k$-saturated graph on $[n]$; $e(G) \leq n-k+1$. Consecutively choose $G_1, G_2, \ldots \subset G$ as follows: let e_{j+1} be the largest integer such that the k-graph H_j,

$$E(H_j) = E(G) \setminus \big(E(G_1) \cup \ldots \cup E(G_j)\big),$$

contains a $P^k_{e_{j+1}+k-1}$-subgraph and let G_{j+1} be any such subgraph. We terminate the procedure when $b_j = n - e_{[j]} - j(k-1)$ is less than $\max(j, k)$. (We denote $e_{[j]} = \sum_{i \in [j]} e_i$, etc.)

Let $j \geq 0$ and suppose we have chosen $G_1, \ldots, G_j$. Let B_j consist of some b_j vertices not covered by an edge of G_i, $i \in [j]$; B_j exists as $v(G_i) = e_i + k - 1$. (We let $b_0 = n$.) Label all $(k-1)$-subsets of $[n]$ by $A_1, \ldots, A_l$, $l = \binom{n}{k-1}$. Let d_i be the number of edges of H_j containing A_i, $i \in [l]$. Clearly,

$$d_{[l]} = ke(H_j) \leq k(n - k + 1 - e_{[j]}) = k(b_j + (j-1)(k-1)) < k^2 b_j. \quad (9)$$

The number of ways to add an element of $\binom{B_j}{k}$ creating a forbidden subgraph with any given $E_1, E_2 \in \binom{[n]}{k}$ is at most $\binom{b_j - 2}{k-2} + O(1)$ if $|E_1 \cap E_2| =$

$k-1$ and it is $O(b_j^{k-4})$ otherwise. As the addition of any $E \in \binom{B_j}{k} \setminus E(H_j)$ to H_j creates a forbidden subgraph (because E is disjoint from any edge of G_i, $i \in [j]$), we conclude that

$$O(b_j^{k-4})\binom{e(H_j)}{2} + \binom{b_j-2}{k-2}\sum_{i\in[l]}\binom{d_i}{2} \geq \binom{b_j}{k} - e(H_j), \qquad (10)$$

which implies by (9) that

$$\sum_{i\in[l]}\binom{d_i}{2} \geq \frac{b_j^2}{k(k-1)} - O(b_j). \qquad (11)$$

We have $e_{j+1} = \max_{i\in[l]} d_i$. The convexity of the $\binom{x}{2}$-function implies that the left-hand side of (11) does not exceed $\frac{d_{[l]}}{e_{j+1}}\binom{e_{j+1}}{2} < \frac{1}{2}k^2 b_j e_{j+1}$. Therefore, we obtain that

$$e_{j+1} \geq \frac{2b_j}{k^3(k-1)} - O(1).$$

From this inequality (and from the fact that $e_{j+1} \geq 1$ if $b_j \geq k$) we deduce the following inequality

$$b_{j+1} \leq \min\left(\left(1 - \tfrac{2}{k^3(k-1)}\right)b_j + O(1),\ b_j - k\right). \qquad (12)$$

It is clear that, starting with $b_0 = n$, we stop after $j = O(\log n)$ steps. Now,

$$e(G) \geq e_{[j]} = n - b_j - j(k-1) = n - O(\log n).$$

The theorem is proved. ∎

Let us consider the case $k = 3$; note that $\mathcal{T}_3$ contains only 2 non-isomorphic graphs, $S_{1,3}^3$ and T_3:

$$\begin{aligned} E(S_{1,3}^3) &= \{\ \{1,2,3\},\ \{1,2,4\},\ \{1,3,4\}\ \}, \\ E(T_3) &= \{\ \{1,2,3\},\ \{1,2,4\},\ \{3,4,5\}\ \}. \end{aligned}$$

Theorem 14 *For any $n \geq 4$, $\mathrm{sat}(n, \mathcal{T}_3) = n - 2$.*

Proof. Let G be any $\mathcal{T}_3$-saturated graph on $[n]$. Make a list of all edges of G and, consecutively and as long as possible, *merge* together any two sets in the list sharing at least 2 vertices (that is, replace then by their union.) Call the resulting sets $C_1, \ldots, C_l \subset [n]$ *components*. Let $v_i = |C_i|$. Define the 2-graph H on $[n]$ by

$$E(H) = \left\{\{x,y\} \in \binom{[n]}{2} : \{x,y\} = E_1 \triangle E_2 \text{ for some } E_1, E_2 \in E(G)\right\}.$$

Consider any component C. It is easy to see by induction on $|C|$ that C is composed of at least $|C| - 2$ edges of G.

Note that if $E \in E(H[C])$ then any $E_1, E_2 \in E(G)$ with $E_1 \Delta E_2 = E$ share two vertices and so belong to the same component C'; but $E \subset C' \cap C$ so necessarily $C' = C$.

Claim 1 For every component C, $\Delta(H[C]) \leq e(G[C]) - 1$.

Proof of Claim. Let $x \in C$ be arbitrary. For each $\{x, y\} \in E(H[C])$, choose $D_y, E_y \in E(G)$ with $D_y \Delta E_y = \{x, y\}$ and $E_y \ni y$. If $\{x, z\}$ is another edge of $H[C]$ then $E_y \neq E_z$: indeed, otherwise $D_z \Delta E_z = \{x, z\} \subset D_y$ and G contains a forbidden subgraph. Hence, $d(x) \leq e(G[C]) - 1$ (we must have at least one G-edge incident to x) and the claim is proved. ▮

Claim 2 If $e(G[C]) \leq |C| - 1$ then for any $x \in [n] \setminus C$ there is a component $C' \ni x$ intersecting C.

Proof of Claim. By Claim 1, there exists $\{a, b\} \in E(\bar{H}[C])$. As $x \notin C$, $E = \{a, b, x\} \notin E(G)$. Consider a forbidden subgraph F created by E. We are home if $\{a, x\}$ or $\{b, x\}$ is covered by E_1 or E_2, where $E(F) = \{E, E_1, E_2\}$. If $\{a, b, y\} \in E(F)$ then $y \in C$ and the remaining edge of F contains both x and y. Finally, if $E_1 \Delta E_2 \subset E$ then, as $\{a, b\} \notin E(H)$, x belongs to the component containing E_1 and E_2 which is the required one. The claim is proved. ▮

If every component C spans at least $|C|$ edges, then we are done as the components cover all but at most one vertex; so assume otherwise. Now, Claim 2 implies that $C_{[l]} = [n]$.

If every component C spans at least $|C| - 1$ edges then we are home: by Claim 2 relabel components $C_1, \ldots, C_l$ so that $C_i \cap C_{[i-1]} \neq \emptyset$, $i \in [2, l]$, and it is easy to show by induction on i that $C_{[i]}$ is made of at least $|C_{[i]}| - 1$ edges, which gives $e(G) \geq n - 1$.

So, suppose that, for example, $e(G[C_1]) = |C_1| - 2$. If for every $x \in [n] \setminus C_1$, there are two distinct components containing x and intersecting C_1 then are home:

$$\begin{aligned} e(G) &\geq \sum_{i \in [l]} (v_i - 2) = v_1 - l - 1 + \sum_{i \in [2,l]} (v_i - 1) \\ &\geq v_1 - l - 1 + \max(2l - 2, 2(n - v_1)) \geq n - 2. \end{aligned} \tag{13}$$

So let C_2 be the only component containing some vertex $x \notin C_1$ and intersecting C_1. Let $\{y\} = C_1 \cap C_2$.

Let $z \in [n] \setminus C_{[2]}$ be arbitrary. If $\{x, z\} \subset C_i$, for some $i \in [3, l]$, then, by the choice of x, $C_i \cap C_1 = \emptyset$ and, by Claim 2, there exists another component through z intersecting C_1.

If no component contains both x and z then, for every $y' \in C_1 \setminus \{y\}$, $E = \{x, y', z\} \notin E(G)$ and considering a forbidden subgraph created by E we conclude that, for some $i \in [3, l]$, $\{y', z\} \subset C_i$ (as $\{x, y'\}$ cannot lie within a component by the definition of x). As $|C_1| \geq 3$, we have at least 2 distinct components containing z and intersecting C_1.

Now the argument similar to (13) shows that $C_{[3,l]}$ is made of at least $n - |C_1 \cup C_2|$ edges, which gives $e(G) \geq n - 3$. (The conclusion is true in the case $l = 2$ as well: then $C_1 \cup C_2 = [n]$.)

Can we have $e(G) = n - 3$? If we have the equality then every C_i, $i \in [3, l]$, must intersect $C_1 \cup C_2$ in exactly one vertex and $e(G[C_j]) = |C_j| - 2$, $j \in [l]$. By Claim 1, there exists $y_i \in C_i$ such that $\{y, y_i\} \notin E(H)$, $i = 1, 2$. But then $\{y, y_1, y_2\} \notin E(G)$ (e.g. because it intersects C_1 in two vertices) and the consideration of a created forbidden graph yields a component containing both y_1 and y_2. Hence, $e(G) > n - 3$ as required. ∎

Remark. Our further analysis has not yet yielded any characterisation of all extremal graphs: we have got stuck considering different cases and, even if we had succeeded, the proof would have been rather long. The difficulty is that an extremal graph may be not unique. For example, there is another minimum $\mathcal{T}_3$-saturated graph of order 7: let $V(G) = [7]$ and

$$E(G) = \{\ \{1,2,5\},\ \{1,3,6\},\ \{1,4,7\},\ \{2,3,4\},\ \{5,6,7\}\ \}.$$

6 Some Other Open Problems

The following definition comes from the Ramsey theory. We say that a graph F *arrows* a t-tuple $(F_1, \ldots, F_t)$ of graphs, which is denoted as $F \to (F_1, \ldots, F_t)$, if any t-colouring of $E(F)$ contains a monochromatic F_i-subgraph of colour i for some $i \in [t]$.

Hanson and Toft [13] made the following conjecture (which we restate here in the sat-type notation).

Conjecture 15 *Given $t \geq 2$ and numbers $m_i \geq 3$, $i \in [t]$, let*

$$\mathcal{F} = \{F : F \to (K^2_{m_1}, \ldots, K^2_{m_t})\}.$$

Let $r = r(K^2_{m_1}, \ldots, K^2_{m_t})$ be the classical Ramsey number, that is, the minimum order of a complete graph arrowing $(K^2_{m_1}, \ldots, K^2_{m_t})$. Then

$$\mathrm{sat}(n, \mathcal{F}) = \binom{r-2}{2} + (r-2)(n-r+2).$$

Obviously, they had the graph $S = K^2_{r-2} + \bar{K}^2_{n-r+1}$ in mind. Observe that this graph is $\mathcal{F}$-saturated, that is, a maximal graph not arrowing $(K^2_{m_1}, \ldots, K^2_{m_t})$. Indeed, $K^2_{r-1} \subset S$ can be properly coloured and this colouring extends on the whole of S by 'cloning' some coloured vertex. On the other hand, the addition of any edge to S creates $K^2_r \in \mathcal{F}$.

Tuza [27] defines the *local density* $d(F)$ of a k-graph F by

$$d(F) = \min_{E \in E(F)} \max_{\substack{E' \in E(F) \\ E' \neq E}} |E \cap E'|$$

and conjectures that

$$\mathrm{sat}(n, F) = O(n^{d(F)}). \tag{14}$$

(Or, even more strongly, that $\mathrm{sat}(n, F) = cn^{d(F)} + O(n^{d(F)-1})$.) This conjecture is motivated by his results on so-called *monotonically saturated* graphs, see [27].

Also, there are many interesting results and open problems on the related notion of *weak saturation* which is studied in e.g. [5, 12, 14, 15, 1, 10, 27, 29, 22, 23].

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