Nested BIBDs from affine planes

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Abstract

For any prime power q, there exists an affine plane of order q. The complement of an affine plane is a balanced incomplete block design (BIBD) with block size $q^2 - q$. In this note, a proof is given that the blocks can be split into sub-blocks to form a nested BIBD with parameters $(q^2, q^2+q, q^3+q^2, q^2-1, q^2-q, q-1)$. Alternatively, this is a generalized tournament design with one game each round, involving q teams, each team with $q-1$ players.

1 Introduction

When the blocks of a balanced incomplete block design (BIBD) can be partitioned into sub-blocks that also form a BIBD, then the blocks and sub-blocks are called a nested balanced incomplete block design (NBIBD). Formally, a $(v, b_1, b_2, r, k_1, k_2)$ NBIBD is a (v, b_1, r, k_1) BIBD with blocks partitioned into sub-blocks that form a (v, b_2, r, k_2) BIBD. The recent article by Morgan, Preece, and Rees [2] gives a survey of known results for NBIBDs.

Generalized tournament designs were introduced by Berman, McLaurin, and Smith [1]. Their underlying structure is a NBIBD with the additional condition that games are arranged into rounds. A generalized tournament design is a schedule of games for a tournament involving v players in which teams of fixed size t and varying membership compete in games of c competing teams. A round consists of a fixed number g of games and each player plays in at most one game per round. There are s players sitting out each round, where $s = v - tcg$. The tournament is balanced, so that each pair of players play together as partners (teammates) in $t-1$ games and as opponents in $t(c-1)$ games. A schedule of games satisfying these tournament conditions is denoted by $(t, c, g)\mathrm{GTD}(v)$. The teams and games form the blocks of a NBIBD with parameters $(v, v(v-1)/(tc), v(v-1)/t, v-1, tc, t)$.

Example 1.1 *A* $(2,3,1)GTD(9)$. *Games are enclosed in parentheses and teams are separated by vertical bars.*
(2 3 | 1 5 | 6 7), (3 4 | 0 6 | 5 8), (0 7 | 2 4 | 1 8), (0 4 | 5 7 | 1 6), (4 8 | 2 6 | 3 7), (0 8 | 1 3 | 2 5), (2 7 | 0 5 | 6 8), (1 7 | 3 8 | 4 5), (1 2 | 4 6 | 0 3), (5 6 | 2 8 | 1 4), (3 6 | 0 1 | 7 8), (3 5 | 4 7 | 0 2)

This example is an instance of the general construction given in the next section.

2 Construction

Let q be a prime power. Take an affine plane of order q. We will construct a $(q-1,q,1)GTD(q^2)$ using the blocks of the plane as the sitouts of each round of the tournament. So we must split the complement of each block into q sub-blocks, each of size $q-1$. The complement is one game, and the sub-blocks are the competing teams.

Choose any parallel class on which to base the construction. Call this class the base class and its blocks, base blocks. Choose any bijection σ between the base blocks and the remaining non-base parallel classes. Each base block B is a transversal in its associated class $\sigma(B)$, i.e. each block of $\sigma(B)$ intersects B in a single point. Take each block of $\sigma(B)$ less the intersection point as a team corresponding to the given base block B. This gives q teams, each of size $q-1$ that partition the complement of B.

Next consider a block C not in the base class. Let B denote the base block with $C \in \sigma(B)$, and $x = B \cap C$. For all the other classes, including the base class, take the block D containing x and delete x. The remaining points of D form one team corresponding to the original block C. This gives q teams, each of size $q-1$. Each team is disjoint from C since two blocks of the affine plane intersect in exactly one point and we removed the intersection point x of C and D. Teams are mutually disjoint because an intersection point y would imply two blocks of the affine plane contain both points x and y, a contradiction. Hence, the teams partition the complement of C.

Theorem 2.1 *The construction just outlined produces a NBIBD*$(q^2, q^2+q, q^3+q^2, q^2-1, q^2-q, q-1)$, *or equivalently, a* $(q-1,q,1)GTD(q^2)$.

Proof: We claim that for every block B of the affine plane, and for every point $x \in B$, the team $B - x$ occurs in the construction. If B is a base block, and $C \in \sigma(B)$ satisfies $x = B \cap C$, then $B - x$ is a team associated to C. Next suppose C is a non-base block and B is the base block satisfying $C \in \sigma(B)$. If $x = B \cap C$, then $C - x$ is a team associated to B. If

$x \neq B \cap C$, let D be the base block containing x and let E be the block in $\sigma(D)$ satisfying $x = D \cap E$. Then $C - x$ is a team associated to E.

We next prove the tournament balance conditions. Every pair of points is in one block of the affine plane, so occur as partners in $q-2$ teams. The complement of an affine plane is a BIBD with every pair of points in $q^2 - q - 1$ blocks. Each occurrence is either as a partner pair or an opponent pair. Hence, the number of times a pair occurs as opponents is $q^2 - q - 1 - (q-2) = (q-1)^2$ times. ∎

Example 2.2 *Begin with the affine plane of order 3 with parallel classes:*
base class: 048, 127, 356
σ(048): 238, 015, 467
σ(127): 134, 026, 578
σ(356): 037, 245, 168
The construction produces the GTD given in Example 1.1

Example 2.3 *For* $v = 25$ *here are two different designs produced by the construction. The first design has an automorphism group of order 100, the second has one of order 20. They were obtained by using different bijections.*

The first design has initial blocks:
((0,0)(2,0)(0,1)(4,3) | (3,4)(1,1)(3,2)(4,2) | (2,1)(1,2)(1,4)(2,2) | (3,1)(1,0)(3,3)(4,0) | (3,0)(2,4)(1,3)(0,2)),((0,3)(0,4)(2,3)(4,4) | (1,4)(1,0)(3,4)(0,0) | (2,0)(2,1)(4,0)(1,1) | (3,1)(3,2)(0,1)(2,2) | (4,2)(4,3)(1,2)(3,3)), cycled mod(5,5). The second block is fixed under (x,y)→(x+1,y+1), so has an orbit of length 5 only.

The second design has initial blocks:
((I,0)(0,1)(1,2)(2,4) | (I,1)(0,0)(1,4)(2,2) | (I,2)(0,4)(1,1)(2,0) | (I,3)(0,3)(1,3)(2,3) | (3,0)(3,1)(3,2)(3,4)),((I,0)(0,2)(2,3)(3,1) | (I,1)(0,3)(2,4)(3,2) | (I,2)(0,4)(2,0)(3,3) | (I,3)(0,0)(2,1)(3,4) | (I,4)(0,1)(2,2)(3,0)),((I,0),(I,1)(I,2)(I,3) | (0,0)(1,1)(2,3)(3,2) | (0,1)(1,3)(2,2)(3,0) | (0,2)(1,0)(2,1)(3,3) | (0,3)(1,2)(2,0)(3,1)), ((0,0)(1,0)(2,0)(3,0) | (0,1)(1,1)(2,1)(3,1) | (0,2)(1,2)(2,2)(3,2) | (0,3)(1,3)(2,3)(3,3) | (0,4)(1,4)(2,4)(3,4)), cycled mod(4,5), where I is invariant. The second block has an orbit of length 4, the third one of length 5, and the fourth is fixed.

3 Acknowledgements

I thank the referee for helpful suggestions including the observation that the construction can produce different designs for the same affine plane (Example 2.3).

References

[1] D.R. Berman, S.C. McLaurin, and D.D. Smith, Generalized tournament designs, *Congressus Numerantium* **142** (2000), 33-40.

[2] J.P. Morgan, D.A. Preece, and D.H. Rees, Nested balanced incomplete block designs, *Discrete Mathematics* **231** (2001), 351-389.