Choice Number of Complete Multipartite Graphs with Part Size at Most Three

Kyoji Ohba*

Abstract

We denote by $K(l*r)$ the complete r-partite graph with l vertices in each part, and denote $K(l*r) + K(m*s) + K(n*t) + \cdots$ by $K(l*r, m*s, n*t, \cdots)$. Kierstead showed that the choice number of $K(3*r)$ is exactly $\lceil \frac{4r-1}{3} \rceil$. In this paper, we shall determine the choice number of $K(3*r, 1*t)$, and consider the choice number of $K(3*r, 2*s, 1*t)$.

1 Introduction

A *k-coloring* of a graph G is a mapping $c : V(G) \to \{1, 2, \cdots, k\}$ such that adjacent vertices get different colors. The graph G is called *k-colorable* if there is a k-coloring of G. The *chromatic number*, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable.

Now we define the list-coloring of a graph G. Suppose that each vertex v of G is assigned a set of colors, called a *color list* $L(v)$. An *L-list coloring* of G (or simply an *L-coloring*) is a vertex-coloring c such that:

- $c(u) \neq c(v)$ for any $uv \in E(G)$,
- $c(v) \in L(v)$ for any $v \in V(G)$.

If there is an L-list coloring of G, then G is said to be *L-list colorable* (or simply *L-colorable*). A list-assignment L is called *k-list assignment* if $|L(v)| \geq k$ for every vertex $v \in V(G)$. And G is said to be *k-choosable* if there is an L-list coloring for any k-list assignment L. The *choice number*, denoted by $\mathrm{ch}(G)$, is the smallest integer k such that G is k-choosable. The idea of L-list colorings, the choosability and the choice number was introduced by Vizing [5], as well as Erdős, Rubin and Taylor [2].

*Department of Mathematics, Keio University, Hiyosi 3-14-1,Kohoku-ku,Yokohama 223-8522,Japan E-mail : kyoji@comb.math.keio.ac.jp

We denote by $K(l * r)$ the complete r-partite graph with l vertices in each part, and denote the complete $(r+s+t+\cdots)$-partite graph $K(l*r)+K(m*s)+K(n*t)+\cdots$ by $K(l*r, m*s, n*t, \cdots)$.

There are several results about the problem of determining the choice number of $K(l*r, m*s, n*t, \cdots)$. Erdős, Rubin, and Taylor [2] determined it for $K(2*r)$.

THEOEM A. $\mathrm{ch}(K(2*r)) = r.$

Alon [1] showed the following bounds on $\mathrm{ch}(K(l*r))$.

THEOEM B. *There exist two positive constants c_1 and c_2 such that for every $l \geq 2$ and every $r \geq 2$*

$$c_1 r \log l \leq \mathrm{ch}(K(l*r)) \leq c_2 r \log l.$$

Recently, Kierstead [3] proved the following theorem.

THEOEM C.

$$\mathrm{ch}(K(3*r)) = \left\lceil \frac{4r-1}{3} \right\rceil.$$

Our purpose in this paper is to consider the choice number of $K(3*r, 2*s, 1*t)$, thus extending Theorem C. Here we determine the choice number of $K(3*r, 1*t)$ exactly.

THEOREM 1.

$$\mathrm{ch}(K(3*r, 1*t)) = \max\left(r+t, \left\lceil \frac{4r+2t-1}{3} \right\rceil\right).$$

Another of our intersts is to find a class of graphs whose chromatic numbers are equal to their choice numbers. We call those graphs *chromatic-choosable.*

We have already proposed the following conjecture in [4].

CONJECTURE 1. *If $|V(G)| \leq 2\chi(G)+1$, then G is chromatic-choosable.*

Applying Theorem 1, we obtain a very close result when the independence number is at most 3.

THEOREM 2. *Let G be a graph with $|V(G)| \leq 2\chi(G)$. If the independence number of G is at most 3, then G is chromatic-choosable.*

2 Proofs

Let G be a graph with an list-assignment L. Let $X \subset V(G)$. We denote $L(X) = \bigcup_{x \in X} L(x)$. Let $\langle X \rangle$ denote the subgraph of G induced by X.

We use similar technique as in the proof of Theorem C. The following lemma was also introduced in [3].

LEMMA 3. ([3], Lemma 4) *A graph G is k-choosable if G is L-choosable for every k-list assignment L such that $|L(V(G))| < |V(G)|$.*

Proof of Theorem 1. Let $G = K(3 * r, 1 * t)$. Let $P_1, P_2, \cdots, P_r$ be the partite sets of G of size three, and let $P_i = \{x_i, y_i, z_i\}$ for each $1 \leq i \leq r$. Let $z_{r+1}, z_{r+2}, \cdots, z_{r+t}$ be the vertices in the parts of G of size one. Let $k = \lceil \frac{4r+2t-1}{3} \rceil$.

First, we show lower bound of the theorem. Since for any graph its choice number is at least its chromatic number, it is clear that G is not $(r+t-1)$-choosable. To prove G is not $(k-1)$-choosable, we construct a $(k-1)$-list assignment L' such that G is not L'-colorable. Let A_1, A_2, A_3 be three pairwise disjoint color sets with $|A_1| = |A_2| = \lceil \frac{k-1}{2} \rceil$, $|A_3| = \lfloor \frac{k-1}{2} \rfloor$, For each i let $L'(x_i) = A_1 \cup A_2$ and $L'(y_i) = A_2 \cup A_3$ for $1 \leq i \leq r$ and $L'(z_i) = A_3 \cup A_1$ for $1 \leq i \leq r+t$. Then any L'-coloring of G must use at least two colors on each P_i, and one color on each z_j for $j \in \{r+1, r+2, \cdots, r+t\}$, thus we need at least $2r+t$ colors on G. However it holds that $|L'(V(G))| = |A_1|+|A_2|+|A_3| = 2\lceil \frac{k-1}{2} \rceil + \lfloor \frac{k-1}{2} \rfloor = \left\lceil \frac{3(k-1)}{2} \right\rceil \leq 2r+t-1$. Thus there does not exist L'-coloring of G and G is not $(k-1)$-choosable.

Next, we will show that G is k-choosable. The theorem states, in other words,

$$\mathrm{ch}(G) = \begin{cases} k & \text{if } t \leq r-1, \\ \chi(G) & \text{if } t > r-1. \end{cases}$$

For any graph G, if G is l-choosable then $G + K_n$ is $(l+n)$-choosable. Thus if it holds that $\mathrm{ch}(G) = \chi(G)$ for $t = r-1$ it holds that $\mathrm{ch}(G) = \chi(G)$ while $t > r-1$. When $t = r-1$, we have $k = r+t = \chi(G)$. Hence it suffices to show the theorem that G is k-choosable with $t \leq r-1$.

Thus in the subsequent argument, we suppose that $t \leq r-1$. We shall prove that G is k-choosable by induction on r. The case where $r = 1$ (and hence $t = 0$) is trivial, so we may assume $r \geq 2$.

Let L be a k-list assignment for G. First, we outline how to color G; we will color G by following four steps.

1. We color two vertices of each P_i for $1 \leq i \leq r$ by one color.
2. We color vertices which are not colored at step 1 of each P_i for $1 \leq i \leq r$ as many as possible.

3. We color all vertices which are colored at neither steps 1 nor 2 by some colors used at step 1. (Here, of course, some vertices lose their colors.)
4. We color the vertices which lost their colors at step 3.

Step1. If there exist $i \in \{1, 2, \cdots, r\}$ such that $\bigcap_{v \in P_i} L(v) \neq \emptyset$ then we assign the same color $c \in \bigcap_{v \in P_i} L(v)$ to each vertex of P_i and finish the proof applying the induction hypothesis. So we assume that

$$\bigcap_{v \in P_i} L(v) = \emptyset, \text{for all } i \in \{1, 2, \cdots, r\}. \tag{1}$$

Let s_i be the number of colors that appear in exactly one of x_i, y_i, z_i and d_i be the number of colors that appear in exactly two of x_i, y_i, z_i. Then it is clear that $s_i + 2d_i = 3k$. We may assume that $|L(V(G))| < |V(G)| = 3r + t$, by Lemma 3. Hence $s_i + d_i \leq |L(V(G))| < 3r + t$ and so

$$r + t \leq d_i, \text{for all } i \in \{1, 2, \cdots, r\}. \tag{2}$$

Thus for each $i \in \{1, 2, \cdots, r\}$, there exist two vertices $u, v \in P_i$ such that

$$|L(u) \cap L(v)| \geq \left\lceil \frac{d_i}{3} \right\rceil \geq \left\lceil \frac{r+t}{3} \right\rceil. \tag{3}$$

We will choose a color α_i for each $i \in \{1, 2, \cdots r\}$ as follows. Suppose that α_h have been chosen for all $h < i$. Choose $\alpha_i \in \bigcup_{u,v \in P_i, u \neq v}(L(u) \cap L(v)) \setminus \{\alpha_h | h < i\}$ and $u, v \in P_i (u \neq v)$ with $\alpha_i \in L(v) \cap L(v)$ so that $|L(u) \cap L(v)|$ is as large as possible. This is possible by (2). We may assume u and v are x_i and y_i. Color both x_i and y_i with α_i.

Remark 1. Using (3),

$$|L(x_i) \cap L(y_i)| \geq \left\lceil \frac{r+t}{3} \right\rceil, \text{for all } i \in \{1, 2, \cdots, \left\lceil \frac{r+t}{3} \right\rceil\}. \tag{4}$$

And if $|L(x_i) \cap L(z_i)| \geq i$ then $|L(x_i) \cap L(y_i)| \geq |L(x_i) \cap L(z_i)|$ and similarly if $|L(y_i) \cap L(z_i)| \geq i$ then $|L(x_i) \cap L(y_i)| \geq |L(y_i) \cap L(z_i)|$. Applying this with $i \leq \lceil \frac{k}{2} \rceil$, and using (1),we obtain

$$|L(x_i) \cap L(z_i)|, |L(y_i) \cap L(z_i)| \leq \left\lfloor \frac{k}{2} \right\rfloor, \text{for all } i \in \{1, 2, \cdots, \left\lceil \frac{k}{2} \right\rceil\}. \tag{5}$$

Remark 2. The following inequalities hold:

$$2r + t - k \leq \left\lceil \frac{k}{2} \right\rceil \leq r \tag{6}$$

In order to check the first inequality, consider

$$\begin{aligned}\left\lceil\frac{k}{2}\right\rceil-(2r-t-k)&=\left\lceil\frac{3k}{2}\right\rceil-2r-t\\&\geq\left\lceil\frac{3}{2}\cdot\frac{4r+2t-1}{3}\right\rceil-2r-t=0.\end{aligned}$$

Also, since $t \leq r-1$, we have $k=\lceil\frac{4r+2t-1}{3}\rceil \leq \left\lceil\frac{4r+2(r-1)-1}{3}\right\rceil=2r-1$. Thus the second inequality $\lceil\frac{k}{2}\rceil \leq r$ follows.

Step 2. Let X_p denote the subset of $V(G)$ such that $X_p=\{x_i,y_i|1\leq i\leq r\}\cup\{z_i|1\leq i\leq p\}$. Let q be the largest number such that there exist an L-coloring of $\langle X_q\rangle$, called f, such that $f(x_i)=f(y_i)$ for any $i\in\{1,2,\cdots,r\}$. We may assume $f(x_i)=f(y_i)=\alpha_i$ for all $i\in\{1,2,\cdots,r\}$ and let $f(z_i)=\beta_i$ for all $i\in\{1,2,\cdots,q\}$. We claim that

$$q\geq 2r+t-k. \tag{7}$$

Let $R=f(X_q)$ and $D=\{j\in\{1,2,\ldots,q\}|\{\alpha_j,\beta_j\}\subset L(z_{q+1})\}$. Then $L(z_{q+1})\subset R$, for otherwise we could color z_{q+1} with a new color. Thus $|D|\geq k-r$. Suppose that $j\in D$ and $\gamma\notin R$. We can conclude $\gamma\notin L(x_j)\cap L(y_j)$, since otherwise we could color z_{q+1} with α_j and both x_j and y_j with γ. By a similar reason, we have $\gamma\notin L(z_j)$. Thus we conclude that $(L(x_j)\cap L(y_j))\cup L(z_j)\subset R$ for all $j\in D$. Now if there exists $j\in\{1,2,\cdots,\lceil\frac{r+t}{3}\rceil\}\cap D$, then, by (1) and (4), it holds that

$$\begin{aligned}\left\lceil\frac{r+t}{3}\right\rceil+k&\leq|L(x_j)\cap L(y_j)|+|L(z_j)|\\&=|(L(x_j)\cap L(y_j))\cup L(z_j)|\\&\leq|R|\\&=r+q,\end{aligned}$$

and so $q\geq\lceil\frac{r+t}{3}\rceil+k-r$. If $\{1,2,\cdots,\lceil\frac{r+t}{3}\rceil\}\cap D=\emptyset$, then $q\geq|\{1,2,\cdots,\lceil\frac{r+t}{3}\rceil\}|+|D|\geq\lceil\frac{r+t}{3}\rceil+k-r$. Thus we have $q\geq\lceil\frac{r+t}{3}\rceil+k-r$ in either case. And hence

$$\begin{aligned}q&\geq\left\lceil\frac{r+t}{3}\right\rceil+k-r\\&=2r+t-k+\left\lceil\frac{6k-8r-2t}{3}\right\rceil\\&\geq 2r+t-k+\left\lceil\frac{2t-2}{3}\right\rceil\end{aligned}$$

$$= \left\lceil \frac{2t-2}{3} \right\rceil$$
$$\geq 2r + t - k.$$

This proves (7)

Step 3. Let X be a maximal subset of $V(G)$ such that both $X \supset X_q$ and there exists an L-coloring c of $\langle X \rangle$ such that $c(x_i) = c(y_i)$ for all $i \in \{1, 2, \cdots, r\}$. Let $Q = c(X)$ and $U = V(G) \setminus X$. Let $S \subset \{q+1, q+2, \cdots, r+t\}$ such that $U = \{z_i \mid i \in S\}$. Again we may assume that $c(x_i) = c(y_i) = \alpha_i$ for all $i \in \{1, 2, \cdots, r\}$ and $c(z_i) = \beta_i$ for all $i \in \{1, 2, \cdots, r+t\} \setminus S$. For each $i \in S$, $L(z_i) \subset Q$, for otherwise we could color z_i with a new color. For each $i \in S$, let $D_i = \{h \in \{1, 2, \cdots, 2r+t-k\} | \{\alpha_h, \beta_h\} \subset L(z_i)\}$. Then it holds

$$k = |L(z_i)| \leq r + |D_i| + \{(r+t) - (2r+t-k) - |S|\},$$

and hence we have

$$|S| \leq |D_i|.$$

Thus there exists an injection $\phi : S \to \{1, 2, \cdots, 2r+t-k\}$ such that $\phi(i) \in D_i$ for all $i \in S$. Let $U' = \{x_h, y_h | h \in \phi(S)\}$.

Color z_i with $\alpha_{\phi(i)}$ for all $i \in S$.

Step 4. Suppose $i \in S$ and $\phi(i) = h$. Then $L(z_h) \subset Q$, for otherwise we could color z_h with a new color and color z_i with β_h. Since $h \leq 2r+t-k \leq \lceil \frac{k}{2} \rceil$ by (6), it holds by (5) that

$$|L(x_h) \setminus Q| = |L(x_h) \setminus ((L(x_h) \cap L(z_h)) \cup (Q \setminus L(z_h)))|$$
$$\geq k - \left\lfloor \frac{k}{2} \right\rfloor - (2r + t - |S| - k)$$
$$\geq |S|.$$

Similarly, it holds

$$|L(y_h) \setminus Q| \geq |S|.$$

Since $\langle U' \rangle$ is isomorphic to $K(2 * |S|)$ and $|L(v) \setminus Q| \geq |S|$ for every $v \in U'$, by Theorem A there exist an L-coloring c' of $\langle U' \rangle$ that does not use any color of Q. Thus we can define an L-coloring C of G by

$$C(v) = \begin{cases} c(v) & \text{if } v \in V(G) \setminus (U \bigcup U'), \\ \alpha_{h(j)} & \text{if } v = z_i \in U, \text{ and} \\ c'(v) & \text{if } v \in U'. \end{cases}$$

■

In order to prove Theorem 2, we use the following lemma in [4].

Lemma 4. ([4], Lemma 6) *Let G be a graph and n be a positive integer. If G is k-choosable, and it holds $(n-1)(|V(G)|+n) \leq n(k+1)$, then $G+\overline{K_n}$ is $(k+1)$-choosable.*

Proof of Theorem 2.

Let G be a graph satisfying the assumption of Theorem 2. By the assumption, there exists a complete multi-partite graph $K = K(3*r, 2*s, 1*t)$ with $r \leq t$ such that G is a subgraph of K, and $\chi(G) = \chi(K)$. Since $\text{ch}(G) \leq \text{ch}(K)$ and $\chi(G) \leq \text{ch}(G)$, it is sufficient to prove the theorem that K is $\chi(G)$-choosable.

We will show that if $r \leq t$ then $K(3*r, 2*s, 1*t)$ is chromatic-choosable, by induction on s. The base step $s = 0$ is clear by Theorem 1, then we assume $s > 0$. By the hypothesis of induction, we assume $K(3*r, 2*(s-1), 1*t)$ is $(r+s-1+t)$-choosable. Since $K(3*r, 2*s, 1*t) = K(3*r, 2*(s-1), 1*t) + \overline{K_2}$ and the order of $K(3*r, 2*s, 1*t)$ is less than or equal to twice the number of its chromatic number, we conclude that $K(3*r, 2*s, 1*t)$ is $(r+s+t)$-choosable by applying Lemma 4 with $n = 2$, and this completes the proof. ■

References

[1] N. Alon, Restricted colorings of graphs, pp.1-33 in *Surveys in Combinatorics*, (Proc. 14th British Combinatorial Conf.), K. Walker, Ed.,Cambridge Univ. Press, Cambredge, 1993

[2] P. Erdős, A.L. Rubin, and H. Taylor, Choosability in graphs, *Congressus Numer.* **26** (1980), 122–157.

[3] H. A. Kierstead, On the choosability of complete multepartite graphs with part size three, *Discrete Math.* **211** (2000), 255–259.

[4] K. Ohba, On chromatic-choosable graphs, *J. Graph Theory* **40** (2002), 130–135.

[5] V.G. Vizing, Colouring the vertices of a graph in prescribed colours, *Diskret. Anal.* **29** (1976), 3–10.