On Infinite Kuratowski Theorems

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Abstract

Širáň and Gvozdjak proved in [9] that the bananas surface, the pseudosurface consisting in the 2-amalgamation of two spheres, does not admit a finite Kuratowski Theorem.

In this paper we prove that pseudosurfaces arising from the n-amalgamation of two closed surfaces, $n \geq 2$, do not admit a finite Kuratowski Theorem, by showing an infinite family of minimal non-embeddable graphs.

1 Introduction

It is well known that classical Kuratowski Theorem [7] gives an obstruction set which characterizes planar graphs.

Robertson and Seymour proved in [8] that for any infinite set of graphs of bounded genus, some member of the set is isomorphic to a minor of another. Since the property of being embeddable in a surface is hereditary (if a graph G can be drawn in a surface S then every minor of G can be drawn in S), every closed surface admits a finite Kuratowski Theorem.

A natural question is to ask whether this result is true for pseudosurfaces in general. Bodendiek and Wagner proved in [3] that the embeddability in a pseudosurface with exactly one singular point is hereditary. In fact, Knor proved in [6] that a pseudosurface is *minor-closed* if and only if it has at most one singular point and some more spheres glued in a tree structure. So the answer to the question is affirmative for pseudosurfaces containing exactly one singular point.

However, if the pseudosurface has two or more singular points, the set of minimal non-embeddable graphs may be non-finite. In [9] Širáň and

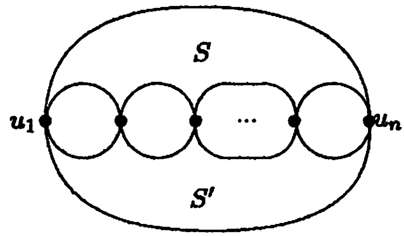

Figure 1: The n-amalgamation of S and S'.

Gvozdjak gave an infinite family of minimal graphs for a pseudosurface with two singular points, the *bananas surface*, arising from two spheres with exactly two points in common.

2 Preliminaries

Orientable surfaces are the surfaces obtainable by adding handles to a sphere. Consequently, we speak of the orientable surface S_q with q handles; the number q is called the genus of the surface. In a like manner we can add crosscaps to a surface, or remove an open disc and identify the pairs of opposite points on the boundary. Every non-orientable surface can be obtained by adding some number q' of crosscaps to the sphere; this is called the non-orientable surface of genus q' (consult [11] for more details).

Let S and S' be two closed surfaces. $S \# S'$ is the surface called connected sum of S and S'. It is obtained when removing a disc from each of S and S', and gluing the boundary circles of these discs together (see [5]).

We denote by $S \overset{n}{\amalg} S'$ the n-amalgamation of S and S', for $n \geq 2$ (the pseudosurface arising when n distinct points in S and another n distinct points in S' are chosen, and you identify each of these points in S with a different point of these in S', as in Figure 1). All of these n points are singular points; S and S' are *bubbles*. In this paper, for any closed surfaces S and S', and $n \geq 2$, we are going to give infinite families of minimal non-embeddable graphs in $S \overset{n}{\amalg} S'$.

¿From now on, all graphs are undirected, without loops or double edges. We follow the standard terminology in Graph Theory (see [4], for instance).

Let G be a graph and m be a positive integer. We denote $mG = \bigcup_{i=1}^{m} G$, and the sphere is S_0. A block of a graph is a maximal non-separable subgraph.

If G and H are two graphs we say that H is a *topological minor* of G

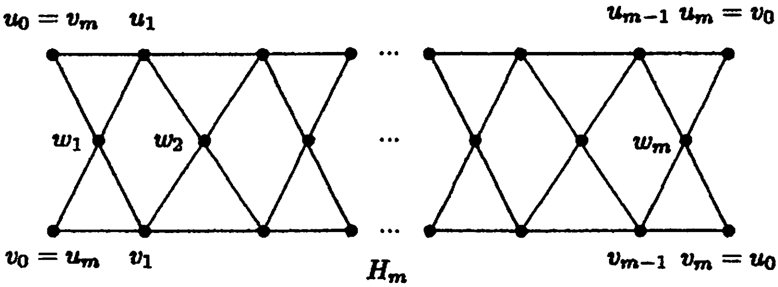

Figure 2: An infinite minimal non-embeddable family in the bananas surface.

when a subdivision of H is a subgraph of G. A graph G will be called minimal non-embeddable for a given (pseudo)surface S if G is not embeddable in S but every H topological minor of G, with $H \neq G$, is embeddable in S.

Širáň and Gvozdjak [9] gave the following result:

Theorem 2.1 *If $m \geq 3$ then H_m (see Figure 2) is a minimal non-embeddable graph for $S_0 \overset{2}{\amalg} S_0$.* □

Now we prove some previous results.

Lemma 2.2 *Let G be a connected non-planar graph. Then G is embeddable in $S_0 \overset{3}{\amalg} S_0$ if and only if G has two planar subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{v_1, v_2, v_3\}$ and $E(G_1) \cup E(G_2) = E(G)$, where $v_i \in V(G)$ for $i = 1, 2, 3$.*

Proof. Necessity: An embedding of G in $S_0 \overset{3}{\amalg} S_0$ can be built by drawing G_1 in one bubble and G_2 in the other one so that v_1, v_2 and v_3 lie in the singular points in both cases. This is possible because G_1 and G_2 are planar.

Sufficiency: We need both bubbles of $S_0 \overset{3}{\amalg} S_0$ to draw G in it. We can suppose without losing generality that three vertices of G (v_1, v_2 and v_3) lie in the singular points. Let $G_1 = (V_1 \cup \{v_1, v_2, v_3\}, E_1)$ be a subgraph of G where V_1 and E_1 are, respectively, the sets of vertices and edges lying in one bubble (maybe some of the edges have both ends in $\{v_1, v_2, v_3\}$). In a similar way we define $G_2 = (V_2 \cup \{v_1, v_2, v_3\}, E_2)$. It is easy to see that these subgraphs verify the statement of the Lemma. □

As a consequence of Lemma 2.2 we have the next result.

Lemma 2.3 *H_m admits an embedding in $S_0 \overset{3}{\amalg} S_0$, for any positive integer m.*

Proof. It holds from Lemma 2.2 by taking the subgraph induced by $\{u_0, u_1, w_1\}$ as G_1. □

For a given graph G, let $p(G)$ be the minimum positive integer such that G admits an embedding in $S_0 \coprod^{p(G)} S_0$.

Corollary 2.4 *If $m \geq 3$ then $p(H_m) = 3$.* □

So we have the following result.

Lemma 2.5 *Let $G_1, \ldots, G_k$ be k graphs. Then $p(\bigcup_{i=1}^{k} G_i) = \sum_{i=1}^{k} p(G_i)$.*

Proof. Let us prove this result by induction. The case $k = 1$ is trivial.

- $k = 2$. It is easy to build an embedding of $G_1 \cup G_2$ in $S_0 \coprod^{p(G_1)+p(G_2)} S_0$ from the embedding of G_1 in $S_0 \coprod^{p(G_1)} S_0$ and the embedding of G_2 in $S_0 \coprod^{p(G_2)} S_0$, since singular points not in G_1 can be regarded as lying in one face of the embedding of G_1. So $p(G_1 \cup G_2) \leq p(G_1) + p(G_2)$. Now, let us consider an embedding of $G_1 \cup G_2$ in $S_0 \coprod^{p(G_1 \cup G_2)} S_0$. If the drawing of G_1 contains r singular points of $S_0 \coprod^{p(G_1 \cup G_2)} S_0$, with $r \leq p(G_1 \cup G_2)$, then the drawing of G_2 contains, at most, $p(G_1 \cup G_2) - r$ singular points. So $r \geq p(G_1)$ and $p(G_1 \cup G_2) - r \geq p(G_2)$. From these inequalities we have $p(G_1 \cup G_2) \geq p(G_1) + p(G_2)$.
- Let us suppose that the statement holds for $k \geq 2$. Then we have $p(\bigcup_{i=1}^{k+1} G_i) = p((\bigcup_{i=1}^{k} G_i) \cup G_{k+1}) = (\sum_{i=1}^{k} p(G_i)) + p(G_{k+1}) = \sum_{i=1}^{k+1} p(G_i)$. □

3 Minimal non-embeddable families for $S_0 \overset{n}{\amalg} S_0$

In this section we are going to give an infinite family of minimal non-embeddable graphs in $S_0 \overset{n}{\amalg} S_0$, for every $n \geq 3$. As a consequence, we obtain that $S_0 \overset{n}{\amalg} S_0$ does not admit a finite Kuratowski Theorem. Let us consider the class of graphs H'_m depicted in Figure 3. We have the next result:

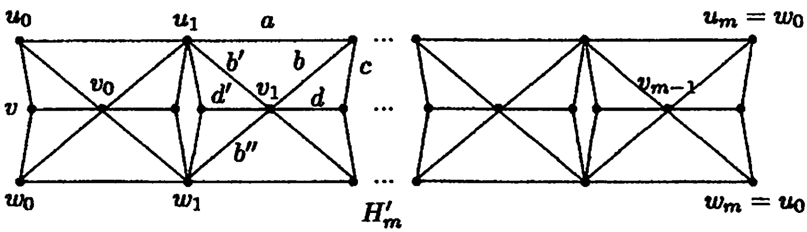

Figure 3: An infinite minimal non-embeddable family in $S_0 \overset{3}{\amalg} S_0$.

Theorem 3.1 *If $m \geq 3$ then H'_m is a minimal non-embeddable graph for $S_0 \overset{3}{\amalg} S_0$.*

Proof. First we will prove that H'_m is non-embeddable in $S_0 \overset{3}{\amalg} S_0$. Let us suppose that the assertion is false.

Let us consider a triple of vertices $\{x, y, z\}$ of H'_m. In a first case, let us suppose that $\{x, y, z\}$ is a cut-set. The only two types of three vertices cut-sets are $\{u_i, v_i, w_i\}$, $i = 0, ..., m-1$ and $\{u_j, v_{j-1}, w_j\}$, $j = 1, ..., m$. Without losing generality we can consider that $x = u_0$, $y = v_0$ and $z = w_0$. It is easy to check that the subgraph $H'_m - v - \{u_0, v_0\} - \{v_0, w_0\}$ is not planar and this is contradictory with the statement of Lemma 2.2.

In a second case, let us suppose that $\{x, y, z\}$ is not a cut-set. The subgraph induced by $\{x, y, z\}$ is, at most, a triangle, or one $K_{1,2}$ that does not belong to any triangle. The subgraph obtained removing a triangle from H'_m is, by symmetry, $H'_m - \{u_0, v_0\} - \{v_0, u_1\} - \{u_0, u_1\}$ (for $\{u_0, v_0, u_1\}$), or greater than $H'_m - v - \{u_0, v_0\} - \{v_0, w_0\}$ (for $\{u_0, v_0, v\}$). It is not difficult to check that this subgraph is not planar. In a similar way, it is easy to check that the subgraph obtained removing one $K_{1,2}$ from H'_m is not planar (there are 8 types of vertices here). These two possible situations mean a contradiction with the statement of Lemma 2.2.

To finish the proof, we show that $H'_m - e$ is embeddable in $S_0 \overset{3}{\amalg} S_0$, for every $e \in E(H'_m)$. Taking into account the symmetry, there are only four types of edges in H'_m, denoted as a, b, c and d in Figure 3.

$H'_m - a$, $H'_m - b$, $H'_m - c$ and $H'_m - d$ satisfy the statement of Lemma 2.2 taking the subgraphs induced respectively by $\{b, c, d\}$, $\{a, c\}$, $\{a, b\}$ and $\{a, b, b'\}$ as G_1, and $G_2 = (H'_m - e) - G_1$. □

In the general case we have the next result.

Theorem 3.2 *The following two statements are true:*

1. *$\{H'_m \cup \frac{n-3}{2} K_5\}$ is an infinite family of minimal non-embeddable graphs in $S_0 \overset{n}{\amalg} S_0$, with $m \geq 3$ and n odd, $n \geq 3$.*

2. $\{H_m \cup \frac{n-2}{2}K_5\}$ *is an infinite family of minimal non-embeddable graphs in* $S_0 \overset{n}{\amalg} S_0$, *with* $m \geq 3$ *and* n *even,* $n \geq 2$.

Proof. Since H'_m is embeddable in $S_0 \overset{4}{\amalg} S_0$ and $p(K_5) = 2$, the statement is a consequence of Corollary 2.4 and Lemma 2.5. □

Remark 3.3 It can also be proved that $\{2H_m \cup \frac{n-5}{2}K_5\}$ is another infinite family of minimal non-embeddable subgraphs for $S_0 \overset{n}{\amalg} S_0$ with $m \geq 3$ and n odd, $n \geq 3$.

As a consequence of Theorems 2.1, 3.1 and 3.2 we state the main result of this section.

Theorem 3.4 $S_0 \overset{n}{\amalg} S_0$ *does not admit a finite Kuratowski Theorem for* $n \geq 2$. □

4 Minimal non-embeddable families for $S_p \overset{n}{\amalg} S_q$

Let S_p be a surface of genus p, orientable or not. Now we are going to give an infinite family of minimal non-embeddable graphs in $S_p \overset{n}{\amalg} S_q$, for every $n \geq 2$ and $p, q \geq 1$, in both orientable and non-orientable cases. As a consequence, $S_p \overset{n}{\amalg} S_q$ does not admit a finite Kuratowski Theorem.

We need some previous results. Battle, Harary, Kodama and Youngs proved in [1] that $\gamma(G) = \sum_{i=1}^{k} \gamma(G_i)$ where G_i is a block of a given graph G for $i = 1, \ldots, k$. From Stahl and Beineke results [10] we obtain the following property for the non-orientable genus:

Lemma 4.1 *Let* G *be a graph such that* $\tilde{\gamma}(G_i) \geq 1$ *for every* G_i *block of* G, $i = 1, \ldots, k$. *Then* $\tilde{\gamma}(G) = \sum_{i=1}^{k} \tilde{\gamma}(G_i)$. □

According to the above Lemma 4.1, we obtain the genus of the families H_m and H'_m.

Lemma 4.2 $\gamma(H_m) = \tilde{\gamma}(H_m) = \gamma(H'_m) = \tilde{\gamma}(H'_m) = 1$, *for every* $m \geq 3$.

Proof. For $m \geq 3$, H_m and H'_m cannot be embedded in the plane since they contain $K_{3,3}$ as a minor. However it is easy to build an embedding of them in the torus and in the projective plane. □

Theorem 4.3 *Let S_p and S_q be two surfaces of genus p and q respectively (orientable or not) with $p, q \geq 1$. The following two statements are true:*

1. $\left\{(p+q+1)H'_m \cup \frac{n-3}{2}K_5\right\}$ *is an infinite family of minimal non-embeddable subgraphs in $S_p \overset{n}{\amalg} S_q$, with $m \geq 3$ and n odd, $n \geq 3$.*

2. $\left\{(p+q+1)H_m \cup \frac{n-2}{2}K_5\right\}$ *is an infinite family of minimal non-embeddable subgraphs in $S_p \overset{n}{\amalg} S_q$, with $m \geq 3$ and n even, $n \geq 2$.*

Proof. To prove the first statement let us suppose that, for some $m \geq 3$, $(p+q+1)H'_m \cup \frac{n-3}{2}K_5$ can be embedded in $S_p \overset{n}{\amalg} S_q$ with n odd, $n \geq 3$. Notice that $S_p \overset{n}{\amalg} S_q = (S_p \# S_0) \overset{n}{\amalg} (S_0 \# S_q)$,where $\#$ denotes the connected sum of two surfaces.

Let k be the number of K_5 embedded completely either in S_p or in S_q. So there is a $\left(\frac{n-3}{2} - k\right)$ K_5 embedded in $S_0 \overset{n}{\amalg} S_0$. As K_5 is not planar, every embedding of K_5 must contain at least 2 singular points of $S_0 \overset{n}{\amalg} S_0$.

Let h be the number of H'_m embedded in S_p or S_q. So there is a $(p+q+1-h)$ H'_m embedded in $S_0 \overset{n}{\amalg} S_0$. Having into consideration that H'_m is not embeddable in $S_0 \overset{3}{\amalg} S_0$ (by Theorem 3.1), every embedding of H'_m must contain at least 4 singular points of $S_0 \overset{n}{\amalg} S_0$.

Notice that $k, h \geq 0$, and $p+q \geq k+h$ by Lemma 4.2. We can state another relation using the number of singular points: $n - 2(\frac{n-3}{2} - k) - 4(p+q+1-h) \geq 0$, and hence $2k + 4h \geq 1 + 4p + 4q \geq 1 + 4k + 4h$. Thus $1 + 2k \leq 0$ which is a contradiction.

To prove the minimality of $(p+q+1)H'_m \cup \frac{n-3}{2}K_5$ we are going to embed $((p+q+1)H'_m \cup \frac{n-3}{2}K_5) - e$ in $S_p \overset{n}{\amalg} S_q$. First, let us consider that e is removed from one K_5. Then we can embed p H'_m in S_p, q H'_m in S_q, $H'_m \cup \frac{n-5}{2}K_5$ in $S_0 \overset{n}{\amalg} S_0$ (by Remark 3.3) and $K_5 - e$ in any plane face since it is planar. Finally, if e is removed from H'_m, we can embed p H'_m in S_p, q H'_m in S_q and $(H'_m - e) \cup \frac{n-3}{2}K_5$ in $S_0 \overset{n}{\amalg} S_0$, using Remark 3.3.

The second statement can be proved analogously. □

References

[1] J. Battle, F. Harary, Y. Kodama, J.W.T. Youngs, "Additivity of the genus of a graph". *Bull. Amer. Math. Soc.* **68** (1962), 565–568.

[2] R. Bodendiek, P. Gvozdjak, J. Širáň, "On minimal graphs for bananas surface B_2". In: R. Bodendiek ed.: Contemporary Methods in

Graph Theory, in Honour of K. Wagner (Bibliographisches Inst. Wissenschaftsverlag, pp. 8–25), Mannheim Wien Zurich, 1990.

[3] B. Bodendiek, K. Wagner, "Solution to König's graph embedding problem". *Math. Nachr* **140** (1989), 251–272.

[4] F. Harary, "Graph Theory". *Addison Wesley Reading Mass*, 1969.

[5] L.C. Kinsey, "Topology of Surfaces". Undergraduate Texts in Mathematics, *Springer-Verlag*, pp. 55–89, Philadelphia, 1993.

[6] M. Knor, "Characterization of minor-closed pseudosurfaces". *Ars Comb.* **43** (1996), 246–256.

[7] K. Kuratowski, "Sur le problème des courbes gauches en topologie". *Fund. Math.* **15** (1930), 271–283.

[8] N. Robertson, P.D. Seymour, "Graph Minors. VIII. A Kuratoski Theorem for general surfaces". *J. Comb. Theory Ser. B* **48** (1990), 255–288.

[9] J. Širáň, P. Gvozdjak, "Kuratowski-Type Theorems do not extend to pseudosurfaces". *J. Comb. Theory Ser. B* **54** (1992), 209–212.

[10] S. Stahl, L.W. Beineke, "Blocks and the nonorientable genus of graphs". *J. Graph Theory* **1** (1977), 75–78.

[11] A.T. White, L.W. Beineke, "Topological graph theory". In: L.W. Beineke and R.J. Wilson, ed.: Selected Topics in Graph Theory, pp. 15–49, *Academic Press*, 1978.