The Inverse Domination Number of a Graph

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December 19, 2003

Abstract

Let G be a graph with n vertices and let D be a minimum dominating set of G. If $V - D$ contains a dominating set D' of G, then D' is called an *inverse dominating set of G with respect to D*. The *inverse domination number* $\gamma'(G)$ of G is the cardinality of a smallest inverse dominating set of G. In this paper we characterise graphs for which $\gamma(G) + \gamma'(G) = n$. We give a lower bound for the inverse domination number of a tree and give a constructive characterisation of those trees which achieve this lower bound.

1 Introduction

Let $G = (V, E)$ be a graph with $n \geq 2$ vertices. For any vertex $v \in V$, the *open neighbourhood* of v, denoted $N(v)$, is the set of all vertices adjacent to v. The *closed neighbourhood* of v, denoted $N[v]$, is $N(v) \cup \{v\}$. The *degree* of a vertex v, denoted $\deg(v)$, is $|N(v)|$. The *maximum degree* (*minimum degree*, respectively) of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). Terms not defined here may be found in [5].

A set $D \subseteq V$ is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. The *domination number*, denoted $\gamma(G)$, is the minimum cardinality among all dominating sets of G. Let D be a minimum dominating set in a graph G. If $V - D$ contains a dominating set D' of G, then D' is called an *inverse dominating set with respect to D*. Introduced by Kulli and Sigarkanti [6], the *inverse domination number* $\gamma'(G)$ of G is the cardinality of a smallest inverse dominating set of G. Note that every graph without isolated vertices contains an inverse dominating set, since the complement of any minimal dominating set is also a dominating set. For this reason we restrict ourselves throughout this paper to graphs with no isolated vertices. A dominating set D is called a *γ-set* if $|D| = \gamma(G)$. Also, an inverse dominating set D' is called a *γ'-set* if $|D'| = \gamma'(G)$. For any vertex $v \in D$, we say that $u \in N[v]$ is a *private neighbour of v with respect to D* if $N[u] \cap D = \{v\}$. A characterisation for a dominating set D to be minimal is that every vertex in D must have a private neighbour with respect to D.

A set $I \subseteq V$ is *independent* if no two vertices of I are adjacent. The *independence number*, denoted $\beta_0(G)$, is the cardinality of a maximum independent set in G. In [6], Kulli and Sigarkanti include a proof that $\gamma'(G) \leq \beta_0(G)$ for all graphs G. However, this proof contains an error. We believe the result is true; therefore we make the following conjecture.

Conjecture 1 *For any graph G, $\gamma'(G) \leq \beta_0(G)$.*

One reason for looking at inverse domination is found in the area of computer science. In the event that there is a need for all nodes in a system to have direct access to needed resources (large databases, for example) a dominating set furnishes such a configuration. If a second important resource is needed, then a separate disjoint dominating set provides duplication in case the first is corrupted in some way. Redundancy in system design appears to be a necessary feature to ensure reliability.

In the next section we observe that $\gamma(G) + \gamma'(G) \leq n$ for all graphs G. Then we characterise those graphs which achieve equality. In Section 3 we find a lower bound for the inverse domination number of a tree and provide a constructive characterisation of those trees which achieve this bound. In the final section we consider complexity results.

2 Graphs with $\gamma(G) + \gamma'(G) = n$

A classical theorem in graph theory is due to Gallai [3]. Here, $\alpha_0(G)$ is the *vertex covering number*, the smallest cardinality of a set of vertices with the property that every edge in the graph is incident to at least one vertex in the set.

Theorem 1 *(Gallai [3]) For any graph G, $\alpha_0(G) + \beta_0(G) = n$.*

A *Gallai-type* theorem has the form $x(G) + y(G) = n$ where $x(G), y(G)$ are parameters defined on the graph G. In [2], Cockayne et al. survey Gallai-type theorems. In this vein, we examine the domination and inverse domination numbers and characterise those graphs for which a Gallai-type theorem holds for these parameters.

Recall that the complement of any minimal dominating set in a graph with no isolated vertices is an inverse dominating set. The following observations are immediate.

Observation 1 *For any graph G with no isolated vertices, $\gamma(G) + \gamma'(G) \leq n$.*

Observation 2 *For any graph G with no isolated vertices, $\gamma'(G) \geq \gamma(G)$.*

We now determine those graphs for which the inequality of Observation 1 is sharp. We look at two cases, the first concerning graphs with minimum degree at least two.

Theorem 2 *Let G be a connected graph with $\delta(G) \geq 2$. Then $\gamma(G) + \gamma'(G) = n$ if and only if $G = C_4$.*

Proof: Since $\gamma'(C_4) \geq \gamma(C_4) = 2$, then $\gamma(C_4) + \gamma'(C_4) = 4 = n$.

Now suppose that G is a connected graph with $\delta(G) \geq 2$ and $\gamma(G)+\gamma'(G) = n$. Here, for each γ-set D, the set $V - D$ is a minimum inverse dominating set, and thus $V - D$ is a minimal dominating set of G.

Claim: We can find a γ-set D such that $\langle V - D \rangle$ contains an edge.
Let D_1 be a γ-set and suppose $V - D_1$ is an independent set. We will construct a γ-set D for which $\langle V - D \rangle$ contains an edge. Since $V - D_1$ is an independent set, each vertex in $V - D_1$ has at least two neighbours in D_1. Further, the set D_1 must also be an independent set, since if $\langle D_1 \rangle$ contains an edge, the endvertices of this edge have no private neighbours with respect to D_1. Let $u \in D_1$ have neighbours $v, w \in V - D_1$. Then the set $D = (D_1 - \{u\}) \cup \{w\}$ is a γ-set for G and the subgraph $\langle V - D \rangle$ contains the edge uv. This completes the proof of the claim.

So let D be a γ-set for G such that $\langle V - D \rangle$ contains an edge, say xy. Since $V - D$ is a minimal dominating set for G, each of x and y has a private neighbour with respect to $V - D$. Let the private neighbours of x, y be x', y' respectively. Note that $x', y' \in D$ and each of x', y' has no other neighbours in $V - D$.

Claim: $x'y' \in E$.
Suppose $x'y' \notin E$. Since $\delta(G) \geq 2$, x', y' each has a neighbour x'', y'' respectively and $x'', y'' \in D$. (Note that we can have $x'' = y''$.) Then the set $(D - \{x', y'\}) \cup \{x\}$ is a dominating set for G of smaller cardinality than $|D| = \gamma(G)$, a contradiction. Thus we have $x'y' \in E$.

Suppose now that $G \neq C_4$. Then there is a vertex $z \in V$ such that z is adjacent to at least one of x, y, x', y'. If $zx' \in E$, then $z \in D$ (since x' has no other neighbours in $V - D$), and the set $(D - \{x', y'\}) \cup \{y\}$ is a dominating set for G of smaller cardinality than $|D| = \gamma(G)$, a contradiction. Thus $zx' \notin E$ and a similar proof shows $zy' \notin E$.

Suppose $zx \in E$. If $z \in D$, then x' does not have a private neighbour with respect to D, so we must have $z \in V - D$. Since $V - D$ is a minimal dominating set, z must have a private neighbour with respect to $V - D$, say $z' \in D$. The vertex z' has no neighbours in $(V - D) - \{z\}$, and must therefore have a neighbour $z'' \in D$. Note that $z'' \notin \{x', y'\}$. Then the set $(D - \{x', z'\}) \cup \{x\}$ is a dominating set for G of smaller cardinality than $|D| = \gamma(G)$, a contradiction. So $zx \notin E$ and similarly, $zy \notin E$. Thus, there is no vertex in $V - \{x, y, x', y'\}$ and $G = C_4$. □

Next, we characterise connected graphs with minimum degree one for which $\gamma(G) + \gamma'(G) = n$.

Theorem 3 *Let G be a connected graph with $n \geq 3$ and $\delta(G) = 1$. Let $L \subseteq V$ be the set of all degree one vertices (leaves) and let $S = N(L)$ (stems). Then $\gamma(G) + \gamma'(G) = n$ if and only if the following two conditions hold:*

1. *$V - S$ is an independent set and*
2. *for every vertex $x \in V - (S \cup L)$, every stem in $N(x)$ is adjacent to at least two leaves.*

Proof: Suppose $\gamma(G) + \gamma'(G) = n$.

Case 1: $V - (S \cup L) = \emptyset$: In this case $V - S = L$ is an independent set, and both conditions hold.

Case 2: $V - (S \cup L) \neq \emptyset$: We will first show that $V - S$ is an independent set. Assume to the contrary that $V - S$ is not an independent set. Let $N_1 = N(S) - (S \cup L)$ and $N_2 = V - (N_1 \cup S \cup L)$.

Claim 1: $N_2 \neq \emptyset$.
Assume N_2 is empty. There is an edge uv in $\langle N_1 \rangle$. Clearly S is a γ-set and $N_1 \cup L - \{u\}$ is an inverse dominating set. Hence $\gamma(G) + \gamma'(G) \leq n - 1$, a contradiction. Thus $N_2 \neq \emptyset$.

Claim 2: There is a γ-set D for G such that $D \cap (N_1 \cup L) = \emptyset$.
In order dominate any vertex $v \in L$ either v or its neighbour in S must be in D. Thus, we can find a γ-set containing no leaves. Note that for any γ-set D with $D \cap L = \emptyset$, $S \subseteq D$. Of all such γ-sets, let D be one with $D \cap N_1$ having minimum cardinality. Suppose $y \in D \cap N_1$, and let z be a private neighbour of y with respect to D. Note that $z \in N_2$ and all private neighbours of y with respect to D are in N_2. Now y has at least two private neighbours with respect to D, since if z is the only such private neighbour, then $D - \{y\} \cup \{z\}$ is a γ-set of G with fewer vertices in N_1 than D. Since $deg(z) \geq 2$, z must be adjacent to a vertex in $V - D$. Then $V - (D \cup \{z\})$ is an inverse dominating set for G with fewer vertices than $V - D$, a contradiction.

Now we let D be a γ-set for G such that $D \cap (N_1 \cup L) = \emptyset$. We note that any γ-set E for $\langle N_2 \rangle$ yields a γ-set $E \cup S$ for G, since otherwise $D \cap N_2$ is a dominating set for $\langle N_2 \rangle$ with $|D \cap N_2| < |E|$.

Claim 3: $\delta(\langle N_2 \rangle) \geq 1$.
Assume to the contrary, that there is a vertex $z \in N_2$ such that $N(z) \cap N_2 = \emptyset$. Then z is adjacent to at least two vertices in N_1. Let $N(z) = \{y_1, y_2, \ldots, y_k\}, k \geq 2$ with $N(z) \subseteq N_1$. Since $D \cap N_1 = \emptyset$, then $z \in D$. If there is a vertex $y_i \in N(z)$ such that y_i is adjacent to a vertex $x \notin D$, then $D_1 = D - \{z\} \cup \{y_i\}$ is a γ-set for G, and $V - (D_1 \cup \{z\})$ is an inverse dominating set for G. Thus for all $y_i \in N(z)$, $N(y_i) \subseteq D \subseteq N_2 \cup S$. So $D_2 = D \cup \{y_2\} - \{z\}$ is a γ-set for G. Since D_2 is a minimum dominating set, each vertex $u \in N(y_2) \cap D = N(y_2)$ is adjacent to a vertex in $V - (D \cup \{y_2\})$, otherwise $D - \{u\}$ is a dominating set of G. Now $D_3 = D \cup \{y_1\} - \{z\}$ is a γ-set for G and $V - (D_3 \cup \{y_2\})$ is an inverse dominating set for G. Thus $\gamma(G) + \gamma'(G) < n$, a contradiction. Therefore, $\delta(\langle N_2 \rangle) \geq 1$.

Claim 4: $\gamma(\langle N_2 \rangle) + \gamma'(\langle N_2 \rangle) = |N_2|$.
By Claim 3, $\delta(\langle N_2 \rangle) \geq 1$, so $\langle N_2 \rangle$ has an inverse dominating set, thus $\gamma(\langle N_2 \rangle) + \gamma'(\langle N_2 \rangle) \leq |N_2|$. Let E' be a γ'-set for $\langle N_2 \rangle$ and let E be a γ-set for $\langle N_2 \rangle$ for which $E' \subseteq N_2 - E$. Assuming that the claim does not hold, we know that $|E| + |E'| < |N_2|$. Since $E \cup S$ is a dominating set for G, we know that $E \cup S$ is a γ-set for G (since otherwise $D \cap N_2$ is a dominating set for $\langle N_2 \rangle$ with $|D \cap N_2| < |E|$). Since the set $E' \cup N_1 \cup L$ is an inverse dominating set for G, we have that $\gamma(G) + \gamma'(G) \leq |E \cup S| + |E' \cup N_1 \cup L| < n$, a contradiction. Therefore, $\gamma(\langle N_2 \rangle) + \gamma'(\langle N_2 \rangle) = |N_2|$.

Claim 5: $\delta(\langle N_2 \rangle) \geq 2$.
Assume, to the contrary, that there is a vertex $z \in N_2$ such that $|N(z) \cap N_2| = 1$. Then z is adjacent to $y \in N_1$ and $w \in N_2$. Let F be a γ-set for $\langle N_2 \rangle$ that does not contain z. The set $F \cup S$ is a γ-set for G and $L \cup N_1 - \{y\} \cup (N_2 - F)$ is an inverse dominating set for G. Thus $\gamma(G) + \gamma'(G) < n$, a contradiction. Therefore, $\delta(\langle N_2 \rangle) \geq 2$.

By Claim 4, Claim 5 and Theorem 2 each component of $\langle N_2 \rangle$ is a cycle on four vertices. Let the cycle z_1, z_2, z_3, z_4, z_1 be a component in $\langle N_2 \rangle$. Since G is connected, at least one vertex, say z_1, is adjacent to a vertex $y \in N_1$. The set $D_4 = D \cup \{z_2, z_4\} - \{z_1, z_3\}$ is a γ-set for G. The set $V - (D_4 \cup \{z_1\})$ is an inverse dominating set for G. So $\gamma(G) + \gamma'(G) < n$, a contradiction. Therefore it must be true that $V - S$ is an independent set, and Condition 1 holds in this case.

Now we show that Condition 2 also holds in this case. Since $V - S$ is an independent set, each vertex in $V - (S \cup L)$ must be adjacent to at least two stems.

Now suppose $x \in V - (S \cup L)$, $v \in N(x)$ (hence, v is a stem) and v is adjacent to only one leaf, say u. Since every vertex in $V - S$ is adjacent to a vertex in S, S is a dominating set of G and $\gamma(G) \leq |S|$. In order to dominate the vertices in L, at least $|N(L)| = |S|$ vertices are required, and $\gamma(G) \geq |S|$. Thus, $\gamma(G) = |S|$.

Let $D = (S - \{v\}) \cup \{u\}$. Since every vertex in $V - (S \cup L)$ is adjacent to at least two vertices in S, D is a dominating set of G of size $|D| = |S| = \gamma(G)$. Now, the set $D' = V - D - \{x\} = (V - S - \{u, x\}) \cup \{v\}$ is an inverse dominating set of G. Again, since $x \notin D \cup D'$, then $\gamma(G) + \gamma'(G) \leq n - 1$, a contradiction, and Condition 2 holds.

Conversely, suppose $V - S$ is an independent set and for every vertex $x \in V - (S \cup L)$, every stem in $N(x)$ is adjacent to at least two leaves. Let S_1 be the set of stems adjacent to exactly one leaf, $S_2 = S - S_1$, $L_1 = N(S_1) \cap L$ and $L_2 = N(S_2) \cap L$. Clearly, S_2 is a subset of every γ-set. In addition, it is straightforward to see that no vertex in $V - (S \cup L)$ is in any γ-set, so that $\gamma(G) = |S|$. Furthermore, for any vertex $u \in L_1$, every γ-set contains either u or its neighbour in S_1. Let D be any γ-set and $D_1 \subseteq D$ be the set of vertices which dominate L_1. (Note that $|D| = |S| = \gamma(G)$ and $|D_1| = |L_1| = |S_1|$.)

Now let D' be an inverse dominating set for D in G. Then D' must contain the following:

1. all vertices in L_2, in order to dominate $L_2 \cup S_2$;
2. all vertices in $D'_1 = (S_1 \cup L_1) - D_1$, in order to dominate $L_1 \cup S_1$; and
3. all vertices in $V - (S \cup L)$, in order to dominate $V - (S \cup L)$.

Thus $\gamma'(G) \geq |L| + |V - (S \cup L)| = |V - S|$. Since every inverse dominating set lies outside a γ-set, we also know that $\gamma'(G) \leq |V - S|$. So $\gamma'(G) = |V - S|$ and the result follows. □

For any graph G of order $n = 2$ with $\delta(G) = 1$, clearly $G = K_2$, and $\gamma(K_2) + \gamma'(K_2) = 2 = n$. If a graph is disconnected, then the domination number of the graph is the sum of the domination numbers of its components. The next result follows immediately.

Corollary 1 *For any graph G with no isolated vertices, $\gamma(G) + \gamma'(G) = n$ if and only if each component of G is either C_4, K_2 or a graph described in Theorem 3.*

3 A lower bound for trees

In this section we examine the inverse domination number of trees. We begin by finding a lower bound.

Theorem 4 *For any tree T of order $n \geq 2$, $\gamma'(T) \geq \frac{n+1}{3}$.*

Proof: Let T be a tree, let D' be a γ'-set and let D be a γ-set with $D' \subseteq V - D$. Since D is a dominating set of T, we have at least $\gamma'(T)$ edges between D and D'. If $V - (D \cup D') = \emptyset$, then $\gamma(T) + \gamma'(T) = n$. Then $\gamma'(T) \geq \frac{n}{2} \geq \frac{n+1}{3}$, since $n \geq 2$. Otherwise, since D' is also a dominating set of T, every vertex in $V - (D \cup D')$ has at least one edge to D and at least one edge to D'. Hence that the number of edges from $V - (D \cup D')$ is at least $2|V - (D \cup D')|$. Counting edges of T, we get

$$n - 1 \geq 2(n - (\gamma(T) + \gamma'(T))) + \gamma'(T)$$

$$\gamma'(T) + 2\gamma(T) \geq n + 1$$

Since $\gamma'(T) \geq \gamma(T)$, the result follows. □

From the proof of the theorem, if $\gamma'(T) = \frac{n+1}{3}$ then $\gamma(T) = \gamma'(T)$ and every vertex in $V - (D \cup D')$ has degree 2, with one neighbour in D, the other in D'. Furthermore, every vertex in D' has exactly one neighbour in D. This gives us the following lemma.

Lemma 1 *Let T be a tree of order $n \geq 2$ and let D' be a γ'-set with corresponding γ-set D. If $\gamma'(T) = \frac{n+1}{3}$, then*

1. *if $x \in V - (D \cup D')$, then deg $x = 2$, $|N(x) \cap D| = 1$ and $|N(x) \cap D'| = 1$,*
2. *if $y \in D$, then $|N(y) \cap D| = 0$ and $|N(y) \cap D'| = 1$, and*
3. *if $z \in D'$, then $|N(z) \cap D| = 1$ and $|N(z) \cap D'| = 0$.*

We now recursively define all trees which achieve the lower bound in Theorem 4. We will use the expression *attach a P_3, $[x, y, z]$, to vertex w* in a tree T to refer to the operation of adding the path $[x, y, z]$ to T and joining w and x by an edge.

Let $\mathcal{T}$ be the family of trees T such that $\gamma'(T) = \frac{n+1}{3}$. Clearly, any tree in $\mathcal{T}$ must have $3k+2$ vertices for some nonnegative integer k. Let $X(P_2) = \emptyset$. Also, let $\mathcal{C}$ be the family of labeled trees T_j that can be obtained from a sequence $T_1, \dots, T_j$ ($j \geq 1$) of trees such that T_1 is a path P_2, and if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by attaching a $P_3 = [x, y, z]$ to a vertex w in T_i, where $xw \in E(T_{i+1})$, $w \notin X(T_i)$, and let $X(T_{i+1}) = X(T_i) \cup \{x\}$.

Theorem 5 *The families $\mathcal{T}$ and $\mathcal{C}$ are equal.*

Proof: Suppose $T \in \mathcal{C}$. Then T has $3k + 2$ vertices for some nonnegative integer k and $\gamma'(T) \geq k+1$ by Theorem 4. Clearly if $n = 2$, then $T = P_2 \in \mathcal{T}$. Let the vertices of the original P_2 be u and v, and let the k copies of P_3 have vertices x_i, y_i, z_i for $1 \leq i \leq k$ where $d(u, x_i) < d(u, y_i) < d(u, z_i)$. We proceed by induction on n to prove that any tree $T \in \mathcal{C}$ has a γ-set D and a γ'-set D' with respect to D such that $D \cup D' = \{u, v\} \cup \{y_i | 1 \leq i \leq k\} \cup \{z_i | 1 \leq i \leq k\}$ and $|D| = |D'| = \frac{n+1}{3}$.

If $n = 2$ then $T = [u, v]$ so $D \cup D' = \{u, v\}$ and $|D| = |D'| = 1$.

Now suppose $n \geq 5$ and any tree $T_1 \in \mathcal{C}$ with $n_1 < n$ vertices ($n_1 = 3k_1+2$) has a γ-set D_1 and a γ'-set D_1' with respect to D_1 such that $D_1 \cup D_1' = \{u, v\} \cup \{y_i | 1 \leq i \leq k_1\} \cup \{z_i | 1 \leq i \leq k_1\}$ and $|D_1| = |D_1'| = \frac{n_1+1}{3}$.

Let $[x_k, y_k, z_k]$ be the last P_3 added in the construction of T. Let $T_1 = T - \{x_k, y_k, z_k\}$, so that $T_1 \in \mathcal{C}$. Note that T_1 has $n_1 = n - 3$ vertices and $n_1 = 3k_1 + 2$ where $k_1 = k - 1$. By the inductive hypothesis, T_1 has a γ-set D_1 and a γ'-set D_1' with respect to D_1 such that $D_1 \cup D_1' = \{u, v\} \cup \{y_i | 1 \leq i \leq k-1\} \cup \{z_i | 1 \leq i \leq k-1\}$ and $|D_1| = |D_1'| = \frac{(n-3)+1}{3} = k$. By the construction of T, x_k is adjacent to a vertex $w \in D_1 \cup D_1'$.

If $w \in D_1$, let $D = D_1 \cup \{z_k\}$ and $D' = D_1' \cup \{y_k\}$. If $w \in D_1'$, let $D = D_1 \cup \{y_k\}$ and $D' = D_1' \cup \{z_k\}$.

Clearly D and D' are minimum dominating sets of T, since if D is not a γ-set for T, then there is a smaller dominating set for T which in turn gives us $\gamma(T_1) < k$, a contradiction. So $\gamma'(T) = |D'| = k + 1 = \frac{n+1}{3}$ and $T \in \mathcal{T}$. Thus $\mathcal{C} \subseteq \mathcal{T}$.

Conversely, suppose $T \in \mathcal{T}$. Then $\gamma'(T) = \frac{n+1}{3}$ and $n = 3k+2$ for some nonnegative integer k.

We will proceed by induction on n. Clearly if $n = 2$, then $T = P_2 \in \mathcal{C}$. Now suppose $n \geq 5$ and any tree T_1 with $n_1 < n$ vertices and $\gamma'(T_1) = \frac{n_1+1}{3}$ is a tree in $\mathcal{C}$. Let D' be any γ'-set of T and D be the γ-set corresponding to D'. It is immediate to see that the only star in $\mathcal{T}$ is P_2, which is an element of $\mathcal{C}$. So we may assume that there are vertices in T with distance at least 2 from a leaf. Root T at any such vertex r. Let z be a vertex furthest from r. Then z is a leaf. Let $N(z) = y$ and x be the parent of y. Since z is a leaf, either $z \in D$ and $y \in D'$, or $z \in D'$ and $y \in D$. Now, suppose deg $y > 2$. Then y has at least one more child t. Since z is a vertex furthest from r, then t must also be a leaf. By Lemma 1, since $y = N(z)$ is either in D or D', then $t \notin D$ and $t \notin D'$. Hence, t is either not dominated by D or D', a contradiction. Thus, deg $y = 2$. By Lemma 1, since $\{y, z\} \subseteq D \cup D'$, then $x \in V-(D \cup D')$ and deg $x = 2$. Let $w \in V(G)$ be such that $N(x) = \{w, y\}$. Now, let $T_1 = T - \{x, y, z\}$.

Since $x \notin D \cup D'$, then $D_1 = D - \{y, z\}$ and $D_1' = D' - \{y, z\}$ are both dominating sets of T_1. Clearly D_1 and D_1' are minimum dominating sets of T_1, since if D_1 is not a γ-set for T_1, then there is a smaller dominating set for T_1 which in turn gives us $\gamma(T) < \frac{n+1}{3}$, a contradiction. So $\gamma'(T_1) = |D_1'| = \gamma'(T) - 1 = \frac{n+1}{3} - 1 = \frac{n-2}{3} = \frac{n_1+1}{3}$. By the inductive hypothesis, $T_1 \in \mathcal{C}$.

Suppose $w = x_i$ where $x_i \in V(T_1)$. Since z is a vertex furthest from r, z_i is a leaf and either $z_i \in D$ and $y_i \in D'$ or $z_i \in D'$ and $y_i \in D$. So $x_i \in V - (D \cup D')$ and x is not dominated by either D or D'. Thus $w \neq x_i$ and $T \in \mathcal{C}$. Hence $\mathcal{T} \subseteq \mathcal{C}$.

Therefore, the families $\mathcal{T}$ and $\mathcal{C}$ are equal. □

4 Complexity results

Finally, we show that the problem of finding an inverse dominating set of cardinality at most k in a graph is NP-complete even when restricted to chordal graphs. First recall that the problem of finding a dominating set of cardinality at most k in a graph is NP-complete (see [4]).

DOMINATING SET

INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$.

QUESTION: Does G have a dominating set of cardinality at most k?

Theorem 6 *DOMINATING SET is NP-complete.*

We now turn our attention to inverse dominating sets.

INVERSE DOMINATING SET

INSTANCE: Graph $G = (V, E)$, positive integer $k \leq |V|$.

QUESTION: Does G have an inverse dominating set of cardinality at most k?

Theorem 7 *INVERSE DOMINATING SET is NP-complete even when restricted to chordal graphs.*

Proof: We will use a transformation from DOMINATING SET. Let G be any graph. Form the graph $G^* = G + K_1$, by adding a new vertex x and making it adjacent to every vertex in G. Note that this construction can be done in polynomial time.

Claim: G has a dominating set of cardinality at most k if and only if G^* has an inverse dominating set of cardinality at most k.
If G has a dominating set of cardinality at most k, then this set will be an inverse dominating set in G^*, since $\{x\}$ is a minimum dominating set of G^*. Conversely, suppose that G^* has an inverse dominating set of cardinality at most k. Call this set D'. If $x \in D'$, by minimality, G^* contains a dominating set D of cardinality 1 that does not contain x, and this is clearly a dominating set of G. If $x \notin D'$, then the vertices of D' form a dominating set of G of cardinality at most k. This completes the proof of the Claim.

Since the join operation preserves chordality, INVERSE DOMINATING SET is NP-complete even when restricted to chordal graphs (see [1]). □