Independent sets in Steiner triple systems

A.D. Forbes, M.J. Grannell and T.S. Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM
tonyforbes@ltkz.demon.co.uk
m.j.grannell@open.ac.uk
t.s.griggs@open.ac.uk

Abstract

A set of points in a Steiner triple system (STS(v)) is said to be *independent* if no three of these points occur in the same block. In this paper we derive for each $k \leq 8$ a closed formula for the number of independent sets of cardinality k in an STS(v). We use the formula to prove that every STS(21) has an independent set of cardinality eight and is as a consequence 4-colourable.

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1 Introduction

A *Steiner triple system* of order v, denoted briefly by STS(v), is a pair $(V, \mathcal{B})$ where V is a set of cardinality v, whose elements are called *points*, and $\mathcal{B}$ is a collection of 3-element subsets of V, called *blocks*, such that every 2-element subset of V appears in precisely one block. It is known that Steiner triple systems exist for every $v \equiv 1$ or 3 (mod 6).

Let $S = (V, \mathcal{B})$ be an STS(v). A (weak) χ-*colouring* of S is a function $\phi : V \to C$, where C is a set of cardinality χ, whose elements are called *colours*, such that $|\phi(T)| > 1$ for all $T \in \mathcal{B}$; i.e. every block in the system contains at least two differently coloured points. If a *χ-colouring* exists, S is said to be *χ-colourable*; the least such value of χ is called the *chromatic number* of S.

A subset U of V in an STS(v), $S = (V, \mathcal{B})$, is an *independent set* if no three points of U occur in a single block $T \in \mathcal{B}$. We denote by $I_k(S)$ the number of independent sets of cardinality k that occur in S. If there is no likelihood of confusion, we write I_k or $I_k(v)$ instead of $I_k(S)$.

The main purpose of this article to obtain a formula for $I_k(S)$ in terms of the numbers of occurrences in S of certain *configurations*. This is stated as Theorem 1. From this formula we obtain explicit expressions for $I_k(v)$, $3 \leq k \leq 8$.

Sauer and Schönheim [7] show that an STS(21) cannot have an independent set of cardinality greater than ten. Moreover, with a suitable implementation of Stinson's hill-climbing algorithm [8], it is not too difficult to construct for each $k \in \{8, 9, 10\}$ an STS(21) having an independent set of maximum cardinality k (Colbourn and Rosa [1], Section 17.2). The only remaining possibility is that there might exist an STS(21) whose largest independent set has fewer than eight points. However, once we have obtained the formula for $I_8(v)$ we can prove that every Steiner triple system of order 21 has an independent set of cardinality eight (Theorem 2).

Theorem 2 then allows us to solve the problem of determining those values of χ for which there exists an STS(21) with chromatic number χ. It is well known that STS(21)s with chromatic number 3 exist. Haddad [5] constructs an STS(21) with chromatic number 4, and Forbes, Grannell and Griggs [3] prove that every STS(21) is 5-colourable. Using Theorem 2 it is relatively straightforward to improve this last result and thereby completely determine the spectrum of chromatic numbers for STS(21)s. We shall prove that every Steiner triple system of order 21 is 4-colourable (Theorem 3).

2 Configurations

Definition 1 A *configuration*, X, is a partial Steiner triple system. More formally, $X = (V, \mathcal{B})$, where V is a non-empty set of points and $\mathcal{B}$ is a collection of 3-element subsets of V, called *blocks*, such that $\bigcup_{T \in \mathcal{B}} T = V$ and every 2-element subset of V appears in at most one block. We denote the number of points in X by $p(X)$ and the number of blocks by $b(X)$. If P is a point of X, we call the number of blocks of X containing P the *degree* of P. Two configurations $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ are regarded as identical if there is a bijection $\phi : V \to V'$ which preserves the blocks; i.e. $\phi(T) \in \mathcal{B}'$ if and only if $T \in \mathcal{B}$.

Definition 2 If S is a Steiner triple system and X is a configuration, we denote by $n(X,S)$ the number of occurrences of X in S. If the system S is fixed, we usually abbreviate $n(X,S)$ to $n(X)$. If X is denoted by a subscripted upper-case letter, X_i, say, we usually write the corresponding subscripted lower-case letter, x_i, for $n(X_i)$. A configuration X whose number of occurrences in an STS(v) depends only upon v and not on the actual STS(v) is *constant*, otherwise it is *variable.*

We will need details of the 31 configurations of at most eight points, and for convenience we list them in Table 1. For brevity, set brackets and commas have been omitted. The numbering assigned to the configurations is standard [1, 2].

Of particular relevance are configurations in the table that have no points of degree 1. There are precisely nine of them:

$$C_{16}, D_1, E_1, E_2, E_3, F_1, F_2, F_3, G_1, \tag{1}$$

and they are all variable. The main reason for our interest in these configurations is a theorem established by Horák, Phillips, Wallis and Yucas [6]: *Any constant n-block configuration, together with all m-block configurations for $m \le n$ having all points of degree at least two form a generating set for the n-line configurations.*

This theorem guarantees that for each configuration in Table 1, there is a formula giving its frequency of occurrence in a given STS(v) as a function of v and the frequencies of the configurations (1). We adopt the convention (Definition 2) of using appropriate subscripted lower-case letters, except that we write p for c_{16} (Pasch) and m for d_1 (mitre). For brevity we write n_v for $v(v-1)(v-3)$.

Formulae for the first three configurations are well-known. Indeed, the first is just the formula for the number of blocks in an STS(v):

$$a_0 = \frac{1}{6}v(v-1), \quad a_1 = \frac{n_v}{72}(v-7), \quad a_2 = \frac{n_v}{8}.$$

The next nine equalities are taken from Grannell, Griggs and Mendelsohn [4]:

$$b_2 = \frac{n_v}{48}(v-7)(v-9), \quad b_3 = \frac{n_v}{48}(v-5), \quad b_4 = \frac{n_v}{8}(v-7), \quad b_5 = \frac{n_v}{6},$$

$$c_{10} = \frac{n_v}{8}(v-8) + 3p, \quad c_{11} = \frac{n_v}{4}(v-7),$$

$$c_{12} = \frac{n_v}{4}(v-9) + 12p, \quad c_{14} = \frac{n_v}{4} - 6p, \quad c_{15} = \frac{n_v}{6}.$$

The formulae for the 5-block configurations are given by Danziger, Mendelsohn, Grannell and Griggs [2]:

$$d_2 = 3p, \quad d_3 = \frac{n_v}{2} - 12p, \quad d_4 = \frac{n_v}{2} - 12p - 6m,$$

$$d_5 = 3(v-7)p, \quad d_6 = \frac{n_v}{2} - 12p, \quad d_7 = \frac{n_v}{4} - 6p - 3m.$$

Formulae are now derived for e_6, e_7, e_8 and f_{20} using the technique described in Section 13.1 of Colbourn and Rosa [1].

By considering the different ways of adding a block to configuration D_2 linking two points of degree 2 we obtain the formula

$$e_7 = d_2 - 3e_1,$$

and by linking the point of degree 1 to a point of degree 2,

$$e_6 = 4d_2 - 12e_1.$$

Similarly, by adding a block to the mitre configuration linking two of its points of degree 2,

$$e_8 = 6m,$$

and, finally, by adding a block to E_1 linking two of its points,

$$f_{20} = 3e_1 - 21f_1.$$

3 Independent sets

We now state and prove the main result.

Theorem 1 *Let $S = (V, \mathcal{B})$ be a given Steiner triple system of order v. Then*

$$I_k(S) = \binom{v}{k} + \sum_X (-1)^{b(X)} n(X,S) \binom{v - p(X)}{k - p(X)},$$

where the sum extends over all configurations X consisting of at most k points.

Proof. If W is a subset of V and X is a configuration, denote by $n(X, W)$ the number of occurrences of X in the restriction of S to W. Consider a k-element set W. Suppose W contains exactly l blocks of S and we compute the sum over all possible configurations, X, $\sum_X (-1)^{b(X)} n(X, W)$.

Then we obtain the value $\sum_{i=1}^{l}(-1)^i\binom{l}{i} = -1$ if $l \geq 1$, and zero if $l = 0$. Hence

$$\begin{aligned} I_k(S) &= \binom{v}{k} + \sum_W \sum_X (-1)^{b(X)} n(X,W) \\ &= \binom{v}{k} + \sum_X (-1)^{b(X)} \sum_W n(X,W), \end{aligned}$$

where $\sum_W$ indicates a sum over all possible k-subsets of V.

But $\sum_W n(X,W)$ is the total count obtained by listing the k-element subsets W and scoring 1 for each copy of X in W. The same number is found by taking each copy of X in S and extending it in all possible ways to a set of size k, and this is given by $n(X,S)\binom{v-p(X)}{k-p(X)}$. Therefore

$$I_k(S) = \binom{v}{k} + \sum_X (-1)^{b(X)} n(X,S)\binom{v-p(X)}{k-p(X)}. \qquad \square$$

If k is small, the expression for $I_k(S)$ given by Theorem 1 only has a modest number of terms. Indeed, setting $k = 8$ and recalling from Table 1 the 31 configurations that have at most eight points,

$$\begin{aligned} I_8 = {} & \binom{v}{8} - a_0\binom{v-3}{5} + a_1\binom{v-6}{2} + a_2\binom{v-5}{3} \\ & - b_2 - (b_3+b_4)(v-7) - b_5\binom{v-6}{2} \\ & + c_{10} + c_{11} + c_{12} + (c_{14}+c_{15})(v-7) + p\binom{v-6}{2} \\ & - (m+d_2)(v-7) - d_3 - d_4 - d_5 - d_6 - d_7 \\ & + e_1(v-7) + e_2 + e_3 + e_6 + e_7 + e_8 \\ & - f_1(v-7) - f_2 - f_3 - f_{20} + g_1. \end{aligned}$$

After substituting from the formulae in Section 2 and simplifying, this gives

$$\begin{aligned} I_8 = {} & n_v(v^5 - 80v^4 + 2575v^3 - 41820v^2 + 344724v - 1167600)/8! \\ & + p(v^2 - 37v + 354)/2 \\ & - (v-22)m + (v-25)e_1 + e_2 + e_3 \\ & - (v-28)f_1 - f_2 - f_3 + g_1. \end{aligned} \qquad (2)$$

In a similar manner and with somewhat less effort we obtain formulae for I_k, $3 \le k \le 7$:

$$I_7 = \frac{n_v}{7!}(v^4 - 52v^3 + 1014v^2 - 8808v + 28905) + (v-15)p - m + e_1 - f_1,$$

$$I_6 = \frac{n_v}{6!}(v-9)(v-10)(v-12) + p, \quad I_5 = \frac{n_v}{120}(v-7)(v-9),$$

$$I_4 = \frac{n_v}{24}(v-6) \quad \text{and} \quad I_3 = \frac{n_v}{6}.$$

4 STS(21)s

Theorem 2 *Every Steiner triple system of order 21 has an independent set of cardinality eight.*

Proof. Let S be a given STS(21). By (2), the number of occurrences in S of eight independent points is

$$I_8(21) = 315 + 9p + m - 4e_1 + e_2 + e_3 + 7f_1 - f_2 - f_3 + g_1. \tag{3}$$

To deal with the terms in (3) with negative coefficients, we define three 9-point, 7-block configurations,

$$F_{37}: \quad \{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,6\},\{1,4,7\},\{2,4,8\},\{5,6,7\}\},$$
$$F_{39}: \quad \{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,6\},\{1,4,7\},\{2,5,7\},\{4,5,8\}\},$$
$$F_{44}: \quad \{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,6\},\{1,4,7\},\{4,6,8\},\{2,7,8\}\},$$

and use the corresponding subscripted lower-case letters f_{37}, f_{39}, f_{44} to denote their frequencies of occurrence in S.

By considering the addition of a block to configuration E_3 linking one of these pairs of points of degree 2: {0, 5}, {0, 6}, {5, 6}, {1, 4}, {1, 7}, {4, 7}, thus converting the E_3 to either an F_{44} or an F_3, we obtain the formula

$$6e_3 = 3f_3 + f_{44} \ge 3f_3.$$

Similarly, by adding a block to configuration E_8 linking points 0 and 7, or points 4 and 5, we obtain $e_8 = 3f_3 + f_{39}$, which, combined with the formula for e_8 from Section 2, gives another upper bound for f_3, namely

$$6m = 3f_3 + f_{39} \ge 3f_3.$$

Finally, by adding a block to configuration E_2 linking points 2 and 4,

$$e_2 = 2f_2 + f_{37} \ge 2f_2.$$

From the formulae for d_2 and e_6 from Section 2 we obtain $e_1 \le p$, and from those above, $f_3 \le e_3 + m$ and $f_2 \le \frac{1}{2}e_2$. Hence

$$I_8(21) \ge 315 + 5p + \frac{1}{2}e_2 + 7f_1 + g_1.$$

The proof of the theorem shows that, in fact, every STS(21) has at least 315 independent sets of cardinality eight. □

Theorem 3 *Every Steiner triple system of order 21 is 4-colourable.*

Proof. By Theorem 2, every STS(21) has at least eight independent points. Given an STS(21), choose eight independent points and colour them red. Let U be the configuration consisting of the 13 points that are not coloured red and the 18 blocks that do not contain a red point. (There are 70 blocks altogether of which 28 contain two red points and 24 contain exactly one red point.) Denote by $p * q$ the third point in the block of the STS(21) that contains points p and q.

Suppose there exists a point x that occurs in exactly five blocks of U,

$$\{x, a, b\}, \{x, c, d\}, \{x, e, f\}, \{x, g, h\}, \{x, i, j\}.$$

Let k and l be the remaining points of U. We can assume that the points are labelled in such a way that $b * k$ is not equal to l, that $a * c$ is not equal to e or f, and that $a * d$ is not equal to e or f. A valid 4-colouring of the STS(21) is achieved by assigning colours as follows, yellow: $\{a, c, d, e, f\}$, blue: $\{g, h, i, j\}$ and green: $\{x, b, k, l\}$.

Alternatively, suppose y is a point that occurs in only four blocks of U,

$$\{y, a, b\}, \{y, c, d\}, \{y, e, f\}, \{y, g, h\},$$

with points $\{i, j, k, l\}$ of U remaining. We assume that the points are labelled such that $j * k$ is not equal to l, $a * i$ is not equal to c or d, and $b * i$ is not equal to c or d. Now colour $\{a, b, c, d, i\}$ yellow, $\{e, f, g, h\}$ blue and $\{y, j, k, l\}$ green.

Finally, a simple counting argument shows that there must exist a point of degree 4 or 5 in U. Let n be the number of points of degree 6 in U. Then $n \le 4$ since five such points would require at least $6 + 5 + 4 + 3 + 2 > 18$ blocks. Hence $(3 \cdot 18 - 6n)/(13 - n) > 3$ and therefore it is impossible for all the remaining points to have degree less than 4 in U. □

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