THE DOMINATION PARAMETERS OF THE CORONA AND ITS GENERALIZATION

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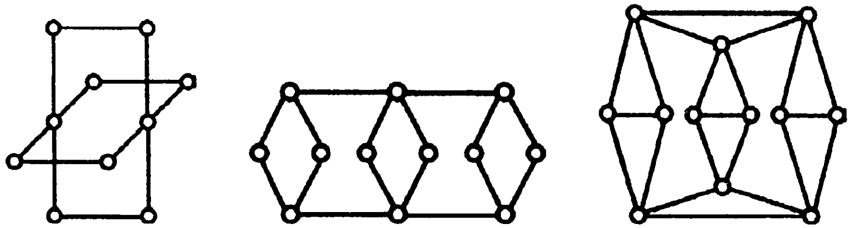
Abstract: Let D be a dominating set of a simple graph $G = (V, E)$. If the subgraph $\langle V - D\rangle_G$ induced by the set $V - D$ is disconnected, then D is called a *split dominating set* of G, and if $\langle D\rangle_G$ has no edges, then D is an *independent dominating set* of G. If every vertex in V is adjacent to some vertex of D in G, then D is a *total dominating set* of G. The *split domination number* $\gamma_s(G)$, *independent domination number* $i(G)$ and *total domination number* $\gamma_t(G)$ equal the minimum cardinalities of a split, independent and total dominating set of G, respectively.. The concept of split domination was first defined by Kulli and Janakiram in 1997 [4], while total domination was introduced by Cockayne, Dawes and Hedetniemi in 1980 [2].
In this paper, we study the split, independent and total domination numbers of corona $G \circ H$ and generalized coronas $kG \circ H$ of graphs.

1. Introduction

Let G be a simple graph with $V(G) = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$. By a *copy of* G we mean the graph G^* with the vertex set $V(G^*) = \{x_1^*, x_2^*, \ldots, x_n^*\}$, where x_i^* corresponds to a vertex $x_i \in V(G)$ and the edge $x_i^* x_j^* \in E(G^*)$ if and only if $x_i x_j \in E(G)$. The *duplication* of a subset $S = \{v_1, v_2, \ldots, v_m\} \subseteq V(G)$ into the copy G^* of a graph G, is a set $S^* \subseteq V(G^*)$ such that $S^* = \{v_1^*, v_2^*, \ldots, v_m^*\}$, where v_i^* corresponds to a vertex v_i. Recall that $G \circ H$ is called the *corona* of graphs if it is obtained from the disjoint union of G and n copies of H (where $n = |V(G)|$) by joining a vertex x_i of G with every vertex from i-th copy of H, for each $i = 1, 2, \ldots, n$ (see [1]).
Let k be a fixed integer, $k \geq 1$. k*−corona* $kG \circ H$ is a graph obtained from k copies of G and $|V(G)|$ copies of H with appropriate edges between each vertex x_i^j of the copy G^j and all of the vertices of the copy H_i. Thus, if G has r vertices and s edges, and H has t vertices and u edges, then the

corona $kG \circ H$ has $kr + rn$ vertices, and $ks + ru + krt$ edges. For example $4K_2 \circ K_1$, $2P_3 \circ \overline{K_2}$ and $2K_3 \circ K_2$ are illustrated in this figure



$4K_2 \circ K_1$ $\quad$ $2P_3 \circ \overline{K_2}$ $\quad$ $2K_3 \circ K_2$

If $k = 1$, then we obtain the corona of graphs G and H.

If $X \subseteq V(G)$, then the notation $\langle X \rangle_G$ means the subgraph of G induced by a subset X. A subset $S \subset V(G)$ is called a *cut set* of G if $\langle V(G) - S \rangle_G$ is disconnected.

In this paper we study three of many variations of the domination: the split domination, total domination and independent domination. Recall, a subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex $y \in V(G) - D$ is adjacent to some vertex $x \in D$. We also say that y is dominated by D in G or by x in G. If $D = \{x\}$ is a dominating set of G, then x is called a *dominating vertex*. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The *domatic number* $d(G)$ of G is the maximum number of elements in a partition of $V(G)$ into dominating sets, see [5]. If every vertex $y \in V(G)$ is adjacent to some vertex $x \in D$ in G, then D is called a *total dominating set* of G. The minimum cardinality of a total dominating set of G is the *total domination number* $\gamma_t(G)$. Of course, every total dominating set also is a dominating set of G. A dominating set D of G is a *split dominating set* of G if the induced subgraph $\langle V(G) - D \rangle_G$ is disconnected. We note that the existence of such a subset in a connected graph is assured under the condition that this graph is different from the complete graph. The *split domination number* $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set of G. A dominating set D of G is an *independent dominating* set of G if $\langle D \rangle_G$ has no edges. The *independent domination number* $i(G)$ of G is the minimum cardinality of a independent dominating set of G. For a convenience, a dominating, split dominating, independent dominating and total dominating set of G which realizes the number $\gamma(G)$, $\gamma_s(G)$, $i(G)$ and $\gamma_t(G)$ will be called a $\gamma(G) - set$, a $\gamma_s(G) - set$, a $i(G) - set$ and a $\gamma_t(G) - set$, respectively. Any term not defined in this paper may be found in [1].

2. Domination parameters of the corona

We start with a simple observations concerning mentioned domination parameters with respect to the corona $G \circ H$. The proofs of these observations are self-evident and they will be left to the reader.

Observation 1. *For any graph G and H*

$$\gamma(G \circ H) = |V(G)|.$$

Note that by the definition of the corona it follows that, if H is the complete graph, then $K_1 \circ H$ also is complete. There is no split dominating set of $K_1 \circ H$, in the resulting graph. In conclusion, we assume that H is a noncomplete graph.

Observation 2. *If $|V(G)| \geq 2$ or H is disconnected and $|V(G)| = 1$, then*

$$\gamma_s(G \circ H) = |V(G)|.$$

It is not difficult to observe that, in the above cases, $V(G)$ is the set realizing the domination number and the split domination number of $G \circ H$, respectively.

Observation 3. *If H is a noncomplete connected graph, then*

$$\gamma_s(K_1 \circ H) = \kappa(H) + 1,$$

where $\kappa(H)$ is the minimum cardinality taken over all cut sets of H.

In this case, the sum of $V(K_1)$ and the smallest cut set of H is $\gamma_s(K_1 \circ H)$–set.

Now, we determine the total domination number of the corona. It is easy to see that $\gamma_t(K_1 \circ H) = 2$, for any graph H. This result can be generalized to any graph G, what is the content of the following result.

Observation 4. *For any graph G and H*

$$\gamma_t(G \circ H) = |V(G)| + s,$$

where s is number of isolated vertices of G.

Finally, we determine the independent domination number of the corona.

Observation 5. *For any graph G and H*

$$i(G \circ H) = \beta_0(G) + (|V(G)| - \beta_0(G))\, i(H),$$

where $\beta_0(G)$ is the maximum cardinality of an independent set of G (i.e. $\beta_0(G)$ is the independence number of G).

Proof. Let $V(G) = \{x_1, x_2, ..., x_n\}$ and S be a maximum independent set of G. Let $\mathcal{J}$ be a subset of $\{1, 2, ..., n\}$, such that

$$j \in \mathcal{J} \overset{df}{\Leftrightarrow} S \cap (\{x_i\} \cup V(H_i)) = \emptyset, \text{ for some } i \in \{1, 2, ..., n\}$$

Observe that $|S| = \beta_0(G)$ and $|\mathcal{J}| = |V(G)| - \beta_0(G)$.
Since S also is a dominating set of G, then

$$D = S \cup \bigcup_{j \in \mathcal{J}} D_j,$$

is the smallest independent dominating set of $G \circ H$, where D_j is $i(H_j)$−set. Moreover, $|D| = |S| + \sum_{j \in \mathcal{J}} |D_j| = \beta_0(G) + i(H)\left(|V(G)| - \beta_0(G)\right)$, as required. □

From the above considerations, we obtain

Corollary 2.1. *For any graph H*

$$i(K_1 \circ H) = 1.$$

3. Domination parameters of the generalized corona

Let $V(G) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of a graph G and $G^1, G^2, \ldots, G^k$ and $H^1, H^2, \ldots, H^n$ be the copies of G and H in $kG \circ H$, respectively. Put $V(G^j) = \{x_1^j, x_2^j, \ldots, x_n^j\}$, for $j = 1, 2, \ldots, k$. From the above it follows that for any fixed $i \in \{1, 2, \ldots, n\}$ the vertex $x_i^j \in V(G^j)$ is adjacent to all vertices of H^i in $kG \circ H$, for $j = 1, 2, \ldots, k$. Put $Y_i = \langle V(H^i) \cup \{x_i^1, x_i^2, \ldots, x_i^k\} \rangle_{kG \circ H}$, for $i = 1, 2, \ldots, n$. Then $V(kG \circ H) = \bigcup_{i=1}^{n} V(Y_i)$ and $V(Y_i) \cap V(Y_j) = \emptyset$, for $i \neq j$.
First, we consider the domination number with respect to the k−corona, $kG \circ H$.

Theorem 3.1. *For two arbitrary graphs G and H and for $k \geq 1$*

$$|V(G)| \leq \gamma(kG \circ H) \leq 2\,|V(G)|\,.$$

Proof. Let D be a $\gamma(kG \circ H)$−set. Let H^i and Y_i be the subgraphs of $kG \circ H$ defined as above. First, we show that $|V(G)| \leq \gamma(kG \circ H)$. Assume that $D \cap V(Y_m) = \emptyset$, for some $m \in \{1, 2, \ldots, n\}$. Then any vertex from $V(H_k)$ is not dominated by D in $kG \circ H$ - a contradiction. Thus, it must be that $D \cap V(Y_i) \neq \emptyset$, for $i = 1, 2, \ldots, n$. In a consequence, $n = |V(G)| \leq \gamma(kG \circ H)$, as desired. To show that $\gamma(kG \circ H) \leq 2\,|V(G)|$, we construct a dominating set S of $kG \circ H$ and $|S| = 2\,|V(G)|$. Let $S = \bigcup_{i=1}^{n} \{h_i\} \cup \bigcup_{i=1}^{n} \{x_i^1\}$, where h_i is some vertex from $V(H^i)$. Since the subset $\{h_i, x_i^1\}$ is a dominating set of Y_i for any $i \in \{1, 2, \ldots, n\}$, thus S is a dominating set of $kG \circ H$ and $|S| = 2\,|V(G)|$, as desired. □

It is interesting to know for which graphs G and H the equality $|V(G)| = \gamma(kG \circ H)$ holds. To answer the question we give some basic results. In all further results we shall take $k \geq 1$.

Theorem 3.2. *Let G be a connected graph. If H has a dominating vertex, then*

$$\gamma(kG \circ H) = |V(G)|.$$

Proof. Duplicating the dominating vertex of H into every copy H^i of $kG \circ H$, we obtain a dominating set of $kG \circ H$ with the cardinality $|V(G)|$. Hence $\gamma(kG \circ H) \leq |V(G)|$. Now, according to Theorem 3.1 we have $\gamma(kG \circ H) = |V(G)|$, as required. □

Proposition 3.3. *Let D be a $\gamma(kG \circ H)-$set with $|D| = n = |V(G)|$. Then $|D \cap V(Y_i)| = 1$, for $i = 1, 2, \ldots, n$.*

Proof. Put D be a set as in the statement of the theorem.
Recall, $Y_i = \left\langle V(H^i) \cup \bigcup_{j=1}^{k} \{x_i^j\} \right\rangle_{kG \circ H}$. Assume that there exists an integer $m \in \{1, 2, \ldots, n\}$, such that $D \cap V(Y_m) = \emptyset$. But it is not possible, since any vertex from $V(H^m)$ would not be dominated by D in $kG \circ H$. Thus, it must be that $D \cap V(Y_i) \neq \emptyset$, for $i = 1, 2, \ldots, n$. Further, since the vertex set $V(kG \circ H)$ is a disjoint union of exactly n subsets $V(Y_i)$ and $|D| = n$, hence $|D \cap V(Y_i)| = 1$, for $i = 1, 2, \ldots, n$. □

Recall that $G^1, G^2, \ldots, G^k$ are the copies of G in $kG \circ H$ and $V(G^j) = \{x_1^j, x_2^j, \ldots, x_n^j\}$, for $j = 1, 2, \ldots, k$.

Proposition 3.4. *Let D be the $\gamma(kG \circ H)-$set with $|D| = n = |V(G)|$. If $\gamma(H) \geq 2$, then for any $j = 1, 2, \ldots, k$ a subset $D \cap V(G^j)$ is a dominating set of an induced subgraph G^j of $kG \circ H$.*

Proof. Let D be a $\gamma(kG \circ H)-$set and $|D| = n$. By Proposition 3.3 we have $|D \cap V(Y_i)| = 1$, for $i = 1, 2, \ldots, n$. Moreover, since H^i has no dominating vertex, then $D \cap V(H^i) = \emptyset$, for $i = 1, 2, \ldots, n$ (otherwise it would be $|D \cap V(Y_m)| \geq 2$, for some $m \in \{1, 2, \ldots, n,\}$). This means that $D \cap \bigcup_{i=1}^{n} V(H^i) = \emptyset$ and $D \subset \bigcup_{j=1}^{k} V(G^j)$. Put $D_j = D \cap V(G^j)$,for $j = 1, 2, \ldots, k$.
Now, we show that D_j is a dominating set of G^j, for $j = 1, 2, \ldots, k$. Let $x_i^j \in V(G^j) - D_j$, for a fixed $i \in \{1, 2, \ldots, n\}$ and for a fixed $j \in \{1, 2, \ldots, k\}$. Observe that by the definition of $kG \circ H$, it follows that $N_{kG \circ H}(x_i^j) = V(H^i) \cup N_{G^j}(x_i^j)$ (in other words vertex x_i^j has a neighbour in H^i or in G^j). As we noticed $D \cap V(H^i) = \emptyset$, hence also $D_j \cap V(H^i) = \emptyset$. So, it must be that $D_j \cap N_{G^j}(x_i^j) \neq \emptyset$, what means that x_i^j is dominated by D_j in G^j (G^j is meant as an induced subgraph of $kG \circ H$). As a consequence we obtain that D_j is a dominating set of G^j. □

Moreover, duplicating D_j into G, for $j = 1, 2, \ldots, k$ we obtain a partition of $V(G)$ into k dominating sets of G. Hence the following results hold.

Corollary 3.5. *Let D be the $\gamma(kG \circ H)-$set with $|D| = n = |V(G)|$ and $\gamma(H) \geq 2$. Then $k \leq d(G)$, where $d(G)$ is the domatic number of G.*

Corollary 3.6. *If G is a connected graph and $2 \leq k \leq d(G)$, then for any graph H*

$$\gamma(kG \circ H) = |V(G)|.$$

Proof. Putting $|V(G)| = n$ we can observe that any dominating set of $kG \circ H$ contains at least one vertex of Y_i, for $i = 1, 2, \ldots, n$. Hence $n \leq \gamma(kG \circ H)$. To complete the proof we construct a dominating set D of $kG \circ H$ with the cardinality $|D| = n$. Since $k \leq d(G)$, then the vertex set $V(G)$ of G can be partitioned into k dominating sets of G, say $D_1, D_2, \ldots, D_k$. Further, let A^j be a duplication of D_j into G^j and let $D = \bigcup_{j=1}^{k} A^j$. It is obvious that $|D| = n$ and by the definition of the graph $kG \circ H$, we see that D is a dominating set of $kG \circ H$. Hence the proof is complete. □

Theorem 3.7. *Let G be a connected graph. Then $\gamma(kG \circ H) = |V(G)|$ if and only if H has a dominating vertex or $k \leq d(G)$.*

Proof. The necessary condition is a straightforward consequence of Theorem 3.2 and Corollary 3.6. Now we show the sufficient condition. Let D be a $\gamma(kG \circ H)-$set with $|D| = |V(G)| = n$. If $k = 1$, then by Observation 1 the result follows. So, assume that $k \geq 2$. If H has a dominating vertex, then the theorem is true. Hence, suppose that H has no dominating vertex. Since $D \cap V(Y_i) \neq \emptyset$, for $i = 1, 2, \ldots, n$, then $|D \cap V(Y_i)| = 1$, because of there are exactly n disjoint subgraphs Y_i of $kG \circ H$. Moreover, since H^i has no dominating vertex, then $D \cap V(H^i) = \emptyset$, for $i = 1, 2, \ldots, n$ (otherwise it would be $|D \cap V(Y_m)| \geq 2$, for some $m \in \{1, 2, \ldots, n, \}$). This means that $D \cap \bigcup_{i=1}^{n} V(H^i) = \emptyset$ and $D \subset \bigcup_{j=1}^{k} V(G^j)$. Moreover, for any j, holds $\left|D \cap V(G^j)\right| \leq 1$. Denote by $N_{kG \circ H}(V(G^j))$ the sum of the open neighbourhoods of all vertices belonging to $V(G^j)$. It can observe that

$$N_{kG \circ H}(V(G^j)) = \bigcup_{i=1}^{n} V(H^i) \cup V(G^j).$$

If there exists an integer m, such that $D \cap V(G^m) = \emptyset$, then $N_{kG \circ H}(V(G^m)) \cap D = \emptyset$. Because of (as we noticed) $D \cap \bigcup_{i=1}^{n} V(H^i) = \emptyset$. It contradicts the fact that D is a dominating set of $kG \circ H$. Hence it must be that $k \leq n$. Further, put

$$D_j = D \cap V(G^j),$$

for $j = 1, 2, \ldots, k$. It is clear that D_j is a dominating set of G^j (G^j is meant as a subgraph of $kG \circ H$) and $D_i \cap D_j = \emptyset$. Finally, duplicating D_j into G, for $j = 1, 2, \ldots, k$ we obtain a partition of $V(G)$ into k dominating sets of G. It means that $k \leq d(G)$. □

Next we discuss the split domination number of the generalized corona, $kG \circ H$, for $2 \leq k \leq d(G)$. First note that for a disconnected graph G the generalized corona $kG \circ H$ also is disconnected, for any k. Thus $\gamma(kG \circ H) = \gamma_s(kG \circ H)$, if G is disconnected. Consequently, in the future investigations we shall assume that G is connected.
We shall give some propositions concerning the split domination number of special generalized coronas. Simple observation shows that the next proposition follows.

Proposition 3.8. *For any graph H and for $k \geq 2$*

$$\gamma_s(kK_1 \circ H) = |V(H)|.$$

Proposition 3.9. *Let G be a connected graph and let $k \geq 2$, then*

$$\gamma_s(kG \circ K_1) = |V(G)|.$$

Proof. If $G \cong K_1$ and $k \geq 2$, then $kG \circ K_1$ is isomorphic to the star with at least three vertices and then $\gamma_s(kK_1 \circ K_1) = 1 = |V(G)|$, as desired.
Suppose that G is different from K_1. Let $|V(G)| = n$. We show that $\gamma_s(kG \circ H) = n$, for $H \cong K_1$. Since $n \geq 2$, then $kG \circ H$ has n copies of H, say $H^1, H^2, \ldots, H^n$. Of course $H^i \cong K_1$, for $i = 1, 2, \ldots, n$. Putting $V(H^i) = \{h^i\}$, for $i = 1, 2, \ldots, n$ we state that the subset $\{h^1, h^2, \ldots, h^n\} \subset V(kG \circ H)$ is a split dominating set of $kG \circ H$ with the cardinality n, as required. □

Proposition 3.10. *Let G be a connected graph different from K_1 and let $k \geq 2$, then for any graph H*

$$\gamma_s(2G \circ H) = |V(G)|.$$

Proof. Let D be a $\gamma(G)$−set and let G^1 and G^2 be two copies of G in $2G \circ H$. Observe that $V(G) - D \neq \emptyset$ is a dominating set of G (otherwise D would not be a minimal dominating set of G). Further, duplicating D into G^1 and $V(G) - D$ into G^2 we obtain a split dominating set of $2G \circ H$, with the cardinality $|V(G)|$. According to Theorem 3.1 the result follows. □

Finally, we discuss the $\gamma_s(kG \circ H)$, for $3 \leq k \leq d(G)$ and $H \not\cong K_1$.

Theorem 3.11. *Let G be a connected graph different from K_1. If H is different from K_1 and $3 \le k \le d(G)$, then*

$$|V(G)| + 1 \le \gamma_s(kG \circ H) \le |V(G)| + k - 1.$$

Proof. Let D be a $\gamma(kG \circ H)-$set. Hence $|D| = |V(G)| = n$. We show that D is not a split dominating set of $kG \circ H$. More precisely, we prove that $\langle V(kG \circ H) - D \rangle_{kG \circ H}$ is connected.
Recall $Y_i = \left\langle V(H^i) \cup \bigcup_{j=1}^{k} \{x_i^j\} \right\rangle_{kG \circ H}$, for $i = 1, 2, ..., n$. Note that $Y_i \cong kK_1 \circ H$ and additionally observe that the last graph is connected, for all $k \ge 1$.
Consider two adjacent vertices of G. Without loss of generality we may assume that $x_1 x_2 \in E(G)$. Thus by the definition of $kG \circ H$, it follows that $x_1^j x_2^j \in E(kG \circ H)$, for $j = 1, 2, \ldots, k$. Further, consider the subgraphs $Y_1 = \left\langle V(H^1) \cup \bigcup_{j=1}^{k} \{x_1^j\} \right\rangle_{kG \circ H}$ and $Y_2 = \left\langle V(H^2) \cup \bigcup_{j=1}^{k} \{x_2^j\} \right\rangle_{kG \circ H}$ of $kG \circ H$. According to Proposition 3.3 and Proposition 3.4, exactly one vertex from D belongs to $\bigcup_{j=1}^{k} \{x_1^j\}$ and $\bigcup_{j=1}^{k} \{x_1^j\}$, respectively. Thus $\langle V(Y_1) - D \rangle_{kG \circ H}$ and $\langle V(Y_1) - D \rangle_{kG \circ H}$ is connected, because of they are isomorphic to $(k-1)K_1 \circ H$ and $k - 1 \ge 2$. Since $k \ge 3$, then we can find $m \in \{1, 2, \ldots, k\}$ such that $x_1^m, x_2^m \in V(kG \circ H) - D$ and $x_{i_1}^m x_{i_2}^m \in E(kG \circ H)$. This means that $\langle (V(Y_{i_1}) \cup V(Y_{i_2})) - D \rangle_{kG \circ H}$ is connected.
Now, it is obvious that if $x_{i_1}, x_{i_t} \in V(G)$ are joined by a path in G containing successive vertices $x_{i_1} x_{i_2} \ldots x_{i_t}$, then the induced subgraph $\langle (V(Y_{i_1}) \cup \ldots \cup V(Y_{i_t})) - D \rangle_{kG \circ H}$ is connected. Further, since G is connected, hence every two vertices are joined by a path in G. As a consequence, we obtain that $\langle V(kG \circ H) - D \rangle_{kG \circ H}$ is connected, as required.
Finally, $|V(G)| = \gamma(kG \circ H) < \gamma_s(kG \circ H)$ and the lower bound of $\gamma_s(kG \circ H)$ holds.

To show the upper bound we construct a split dominating set S of $kG \circ H$ with the cardinality $|V(G)| + k - 1$. Let $\{A_1, A_2, \ldots, A_k\}$ be the partition of $V(G)$ into k dominating sets of G (such a partition exists, since k is not greater than the domatic number of G). Put S_j be a duplication of A_j into G^j, for $j = 1, 2, \ldots, k$. It is easy to observe that $\bigcup_{m=1}^{k} S_m$ is a dominating set of $kG \circ H$ with $\left|\bigcup_{m=1}^{k} S_m\right| = |V(G)|$. Further, let $S = \bigcup_{m=1}^{k} S_m \cup \bigcup_{j=1}^{k} \{x_1^j\}$. Note that $\langle V(kG \circ H) - S \rangle_{kG \circ H}$ is disconnected, since H^1 is one of the connected components of it. All this together gives that S is a split dominating set with required cardinality not greater than $|V(G)| + k - 1$. □

Theorem 3.12. $\gamma_t(kG \circ H) = |V(G)| \Longleftrightarrow$ *G has no isolated vertices and $V(G)$ can be partitioned into k total dominating sets.*

Proof. Let D be a $\gamma(kG \circ H)-$set with $|D| = n = |V(G)|$.
Recall $Y_i = \left\langle V(H^i) \cup \bigcup_{j=1}^{k} \{x_i^j\} \right\rangle_{kG \circ H}$, for $i = 1, 2, ..., n$. Observe that $D \cap V(Y_i) = \{x_i^j\}$, for some $j \in \{1, 2, ..., k\}$. Otherwise D would not be a dominating set of the generalized corona or $|D|$ would be greater than n, what contradicts the assumption of D. Since D is a total dominating set, then $\{x_i^j\}$ is adjacent to some vertex of D in $kG \circ H$, say x_m^j, $i \neq m \in \{1, 2, ..., n\}$. Thus x_i is not an isolated vertex of G, because of it is adjacent to x_m in G. Next, note that $D \cap V(G^j)$ is a total dominating set of G^j and duplicating this set into $V(G)$, for $j = 1, 2, ..., k$, we obtain the partition of $V(G)$ into k total dominating sets, as desired.

Suppose that G has no isolated vertices and $V(G)$ can be partitioned into k total dominating sets. Then duplicating each element of the partition into different copy of G in the generalized corona we obtain the required total dominating set of $kG \circ H$. Thus the theorem is true. □

Corollary 3.13. *Let $k \geq 1$, then for any graph H*

$$\begin{aligned} \gamma_t(kP_n \circ H) &= n \Longleftrightarrow k = 1. \\ \gamma_t(kC_n \circ H) &= n \Longleftrightarrow (k = 2 \text{ and } n \equiv 0(\text{mod } 4)) \text{ or } k = 1. \\ \gamma_t(kK_n \circ H) &= n \Longleftrightarrow k \leq \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Arguing similarly as in the proof of Theorem 3.7, but with respect to an independent dominating set, we obtain the result due to Theorem 3.7

Theorem 3.14. *Let G be a connected graph. Then $i(kG \circ H) = |V(G)|$ if and only if H has a dominating vertex or $V(G)$ can be partitioned into k independent dominating sets.*

Proposition 3.15. *Let $n \geq 3$, then*

$$\begin{aligned} d_i(C_n) = 3 &\iff n \equiv 0(\text{mod } 3), \\ d_i(C_n) = 2 &\iff n \text{ is even and } n \equiv 1, 2(\text{mod } 3), \\ d_i(C_n) = 0 &\iff n \text{ is odd and } n \equiv 1, 2(\text{mod } 3), \end{aligned}$$

where $d_i(C_n)$ is the maximum number of elements in a partition of $V(C_n)$ into independent dominating sets. Moreover, if n is even and $n \equiv 0(\text{mod } 3)$, then $V(C_n)$ can be partitioned into 2 or 3 independent dominating sets.

Proof. Let D be an independent dominating set of C_n, $n \geq 3$. Since C_n is $2-$regular, thus $V(C_n)$ can not be partitioned into more than 3 dominating sets and of course 3 independent dominating sets. Now, we consider all cases with respect to the complement of D in C_n.

Assume that $\langle V(C_n) - D \rangle_{C_n} \cong \overline{K_s}$, then n is even. Moreover, $\{D, V(C_n) - D\}$ is the unique partition of $V(C_n)$ into independent dominating sets.

Assume that $\langle V(C_n) - D \rangle_{C_n} \cong sK_2$ and $D = \{x_i : i \equiv 1(\text{mod } 3)\}$. Then it is not difficult to observe that, $\{D, \{x_i : i \equiv 2(\text{mod } 3)\}, \{x_i : i \equiv$

$0(\bmod 3)\}\}$ is the unique partition of $V(C_n)$ into independent dominating sets and $n \equiv 0(\bmod 3)$.

Assume that there exist $x_i, x_j \in V(C_n) - D$, such that $N_{C_n}(x_i) \subset D$ and $N_{C_n}(x_j) \not\subseteq D$. From the fact that $N_{C_n}(x_i) \subset D$ it follows that $V(C_n)$ can not be partitioned into more than 2 independent dominating sets. But $V(C_n) - D$ is not an independent dominating set of C_n, since x_j is adjacent to some vertex of $V(C_n) - D$ in C_n, by the assumption of x_j. Thus, in this case, there is no partition of $V(C_n)$ into independent dominating sets.

Reassuming, if $n \equiv 0(\bmod 3)$, then $V(C_n)$ can be partitioned into 2 or 3 independent dominating sets. If n is even and $n \equiv 1, 2(\bmod 3)$, then $V(C_n)$ can be partitioned into 2 independent dominating sets, and if n is odd and $n \equiv 1, 2(\bmod 3)$, then there does not exist any partition of $V(C_n)$ into independent dominating sets.

Corollary 3.16. *Let H be a graph with $\gamma(H) \geq 2$, then*

$$i(kP_n \circ H) = n \iff k = 2,$$

$$i(kC_n \circ H) = n \iff (k = 2 \text{ and } n \text{ is even}) \text{ or } (k = 3 \text{ and } n \equiv 0(\bmod 3)),$$

$$i(kK_n \circ H) = n \iff k = n,$$

$$i(kK_{m,n} \circ H) = m + n \iff k = 2.$$

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