GRACEFULNESS OF $P_{a,b}$

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Abstract : In this paper we prove the gracefulness of the class of graphs denoted by $P_{a,b}$
Key words : Graceful labelling - internally disjoint paths.

1. INTRODUCTION

Definition 1.1 A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set {0,1,....q} such that, when each edge xy is assigned the label |f(x) - f(y)|, the resulting edge labels are distinct. A graph which admits such a labelling is called a graceful graph.

Definition 1.2 Let u and v be two fixed vertices. We connect u and v by means of "b" internally disjoint paths of length "a" each. The resulting graph is denoted by $P_{a,b}$.

K.M. Kathiresan [1] established that the graphs $P_{2r,\,2m+1}$ are graceful for all values of r nd m and conjectured that $P_{a,b}$ is graceful except when $a = 2r + 1$ and $b = 4s + 2$.

We prove the conjecture except in one case where $a=4r+1 (r>1)$ with the corresponding $b=4m$ $(m>r)$.

2. MAIN RESULTS

Let $v_0^i, v_1^i, v_2^i, v_3^i, v_4^i \ldots v_a^i$ be the vertices of the i-th copy of the path of length a, where i=1,2...b, $v_0^i = u$ and $v_a^i = v$ for all i. We observe that $P_{a,b}$ has (a-1) b+2 vertices and ab edges.

Definition 2.1 Let x, y, z be functions defined on the set of natural numbers as follows.

i) $$x(t) = \begin{cases} 1 \text{ if } t \le m \\ 0 \text{ if } t > m \end{cases}$$

ii) $$y(t) = \begin{cases} 1 \text{ if } t \equiv 1 \pmod 2 \\ 0 \text{ if } t \equiv 0 \pmod 2 \end{cases}$$

iii) $$z(t) = \begin{cases} 1 \text{ if } t \le 2m \\ 0 \text{ if } t > 2m \end{cases}$$

Theorem 2.1 $P_{2r+1,\,2m+1}$ are graceful for all values of r and m.

Proof: Define f as follows.

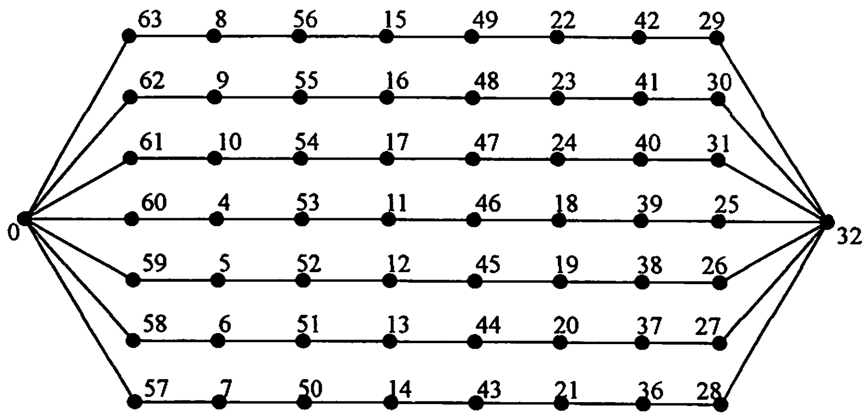
$f(u) = 0$; $f(v) = \dfrac{(2r+1)(2m+1)+1}{2}$

$f(v^i_{2j+1}) = (2r+1)(2m+1) - (j-1)(2m+1) - i+1,$ for $i = 1,2,..,2m+1,\ j = 1,2,...r.$

$f(v^i_{2j}) = x(i)\{(2m+1) + i + (j-1)(2m+1)\} + (1-x(i))\{i + (j-1)(2m+1)\}$

for $i = 1,2,..,2m+1,\ j = 1,2,...r.$

Illustration: The graceful numbering of $P_{9,7}$.



Theorem 2.2 $P_{4r,2m}$ is graceful for all r and m.

Proof: Define f as follows.

$f(u) = r-1$; $f(v) = 4rm - (r+1)$

$f(v^1_j) = y(j)\{8rm - \frac{(2r-1-j)}{2}\} + y(j+1)\{\frac{(2r-2-j)}{2}\}$ $\quad j = 1, 2. ... 2r-1$

$= y(j)\{4rm + \frac{(j-2r-1)}{2}\} + y(j+1)\{4rm-1- \frac{(j-2r)}{2}\}$ $\quad j=2r, 2r+1, ..., 4r-1$

For $i = 2,3,...2m$

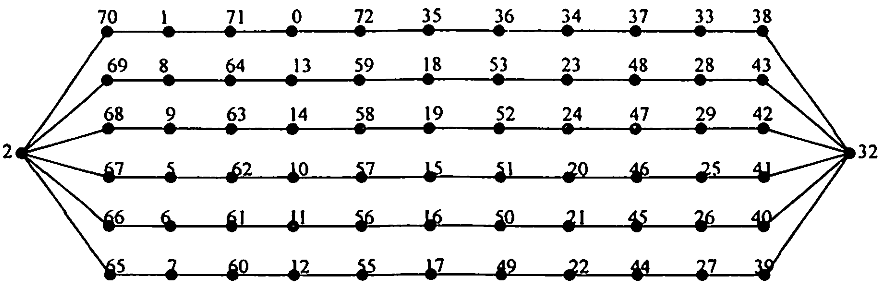
$f(v^i_j) = 8rm - r - (i-2) - (\frac{j-1}{2})(2m-1)$ $\quad j = 1, 3, . . . , 2r-1$

$= 8rm - r - 1 - (i-2) - (\frac{j-1}{2})(2m-1)$ $\quad j=2r+1, 2r+3, ...4r-1$

$= x(i)\{2m + r - 1 + (i-2) + (\frac{j-2}{2})(2m-1)\} +$
$(1-x(i))\{m + r - 1 + (i-m-1) + (\frac{j-2}{2})(2m-1)\}$ $\quad j=2, 4, . . . , 4r-2$

f gives a graceful labeling.

Illustration: The graceful numbering of $P_{12,6}$.



Theorem 2.3 $P_{4r+2,2m}$ are graceful for all r and m

Proof: Define f as follows.

$f(u) = r$; $f(v) = (4r+2)m - r$

$$f(v_j^1) = y(j)\{4r+2)2m - (\frac{2r+1-j}{2})\} + y(j+1)\{(\frac{2r-j}{2})\} \qquad j=1,2,\ldots 2r+1$$

$$= y(j)\{(2r+1)2m+1+(\frac{j-2r-3}{2})\} + y(j+1)\{(2r+1)2m - (\frac{j-2r-2}{2})\}$$

$j = 2r+2, 2r+3, \ldots 4r+1$

For $i = 2,3,\ldots 2m$

$$f(v_j^i) = x(i)\{r+2m+(i-2) + (\frac{j-2}{2})(2m-1)\} + (1-x(i))\{r+m+(i-m-1) + (\frac{j-2}{2})(2m-1)\}$$

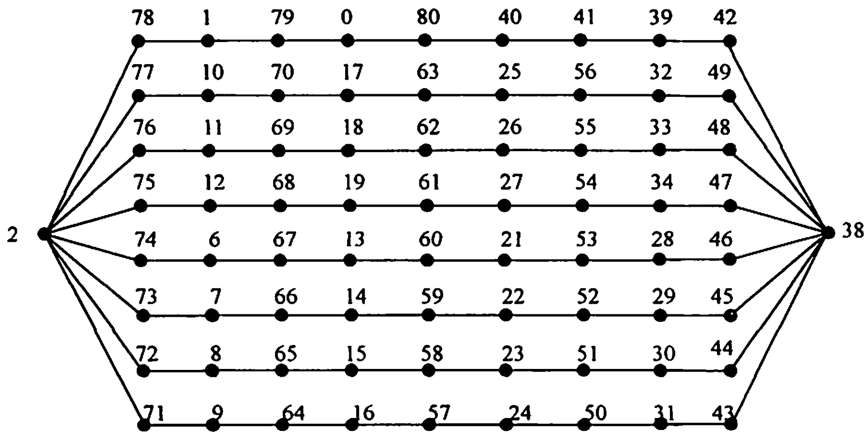
$j=2,4,..2r$

$$= x(i)\{(r+2m+1) + (i-2) + (\frac{j-2}{2})(2m-1)\} +$$
$$(1-x(i))\{(r+m+1) + (i-m-1) + (\frac{j-2}{2})(2m-1) \qquad j = 2r+2, 2r+4, \ldots, 4r$$

$$= (4r+2)2m - (r+1)-(i-2) - (\frac{j-1}{2})(2m-1) \qquad j = 1,3,5,\ldots 4r+1.$$

f gives a graceful labeling.

Illustration: The graceful numbering of $P_{10,8}$



Theorem 2.4 P_{4r+1}, 4m are graceful for $r \geq m$.

Proof: Define f as follows.

$f(u) = r-m$; $\quad f(v) = (4r+1)2m + (r+m)$

$f(v_j^1) = y(j)\{(4r+1)4m - (r-m) + (\frac{j-1}{2})\} + y(j+1)\{(r-m)-1-(\frac{j-2}{2})\}$ $\quad j=1,2,..2(r-m)+1$

$= y(j)\{2m(4r+1)+1+(\frac{j-2(r-m)-3}{2})\} + y(j+1)\{2m(4r+1)-(\frac{j-2(r-m)-2}{2})\}$ $\quad j=2(r-m)+2, 2(r-m)+3,..4r$

For j=1,3,..4r-1

$f(v_j^i) = (4r+1)4m - (r-m)-(i-1)-(4m-1)(\frac{j-1}{2})$ $\quad i=2,3,..,4m$

For j=2,4,...2r

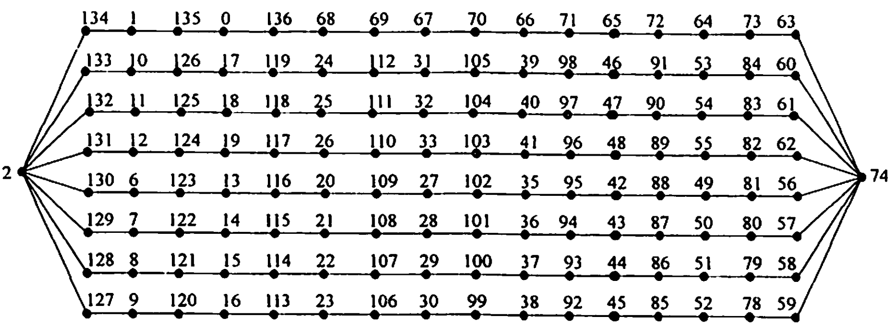
$f(v_j^i) = z(i)\{r+3m+(i-2)+(4m-1)(\frac{j-2}{2})\} + (1-z(i))\{r+m+(i-2m-1)+(4m-1)(\frac{j-2}{2})\}$ $\quad i=2,3,....,4m$

For j = 2r+2, 2r+4...4r

$f(v_j^i) = z(i)\{r+3m+(i-1)+(4m-1)(\frac{j-2}{2})\} + (1-z(i))\{r+m+(i-2m)+(4m-1)(\frac{j-2}{2})\}$ $\quad i=2,3,...,4m$

f gives a graceful labeling.

Illustration: The graceful numbering of $P_{17,8}$ (r=4, m=2)



Theorem 2.5 $P_{4r-1,\,4m}$ are graceful for $r \geq m$.

Proof: Define f as follows.

$f(u) = r-m$; $f(v) = r(8m+1)-m-1$

Case 1: r = m

$f(v_j^1) = y(j)\{2m(4r-1)+(\frac{j-1}{2})\} + y(j+1)\{2m(4r-1)-1-(\frac{j-2}{2})\}$ j=1,2,...4r-2

For j = 1,3,...,2r-1

$f(v_j^i) = 4m(4r-1) - (i-2) - (4m-1)(\frac{j-1}{2})$ i = 2,3,...,4m

For j = 2r+1, 2r+3, ...,4r-3

$f(v_j^i) = 4m(4r-1) - 1 - (i-2)-(4m-1)(\frac{j-1}{2})$ i = 2,3,...4m

For j = 2,4,...4r-2

$f(v_j^i) = z(i)\{r+3m+(i-2)+(4m-1)(\frac{j-2}{2})\}+(1-z(i))\{r+m+(i-2m-1)+(4m-1)(\frac{j-2}{2})\}$

i = 2,3,...,4m

Case 2: r > m

$f(v_j^1) = y(j)\{r(16m-1)-3m+1+(\frac{j-1}{2})\}+y(j+1)\{r-m-(\frac{j-2}{2})\}$ j=1,2,...,2(r-m)

$= y(j)\{2m(4r-1)+(\frac{j-2(r-m)-1}{2})\}+y(j+1)\{2m(4r-1)-1-(\frac{j-2(r-m)-2}{2})\}$

j = 2(r-m)+1, 2(r-m)+2,....4r-2

For j=1,3,...2r-1

$f(v_j^i) = r(16m-1) - 3m - (i-2)-(4m-1)\left(\frac{j-1}{2}\right)$ $i = 2,3,...4m$

For j = 2r+1,3r+3,...4r-3

$f(v_j^i) = r(16m-1) - 3m - 1 - (i-2)-(4m-1)\left(\frac{j-1}{2}\right)$ $i = 2,3,...4m$

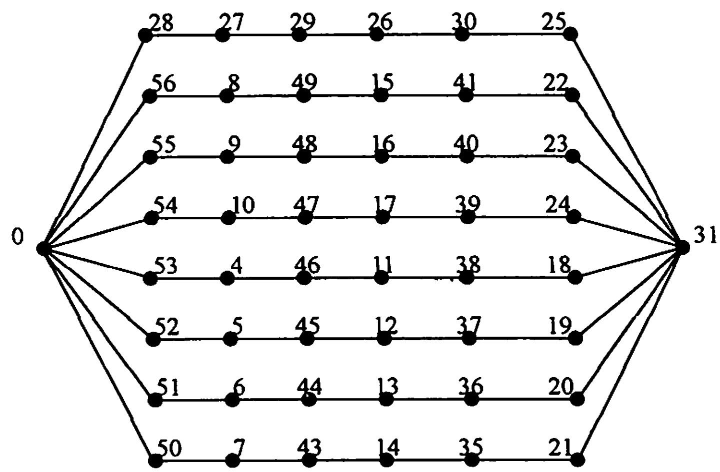
For j = 2, 4, ... , 4r-2

$f(v_j^i) = z(i)\{r+3m+(i-2)+(4m-1)\left(\frac{j-2}{2}\right)\}+(1-z(i))\{r+m+(i-2m-1) + (4m-1)\left(\frac{j-2}{2}\right)\}$

$i=2,3,...4m$

f gives a graceful labeling.

Illustration: The graceful numbering of $P_{7,8}$ (r = 2, m=2).



Theorem 2.6 $P_{3,4m}$ are graceful for all m $\geq$ 2.

Proof : Define f as follows.

$f(u) = 0$: $f(v) = m-1$

$f(v_1^i) = 12m - i + 1$ $i = 1,2,...,2m-1$

$\quad\quad = 8m-i+1$ $i = 2m,2m+1,....,4m$

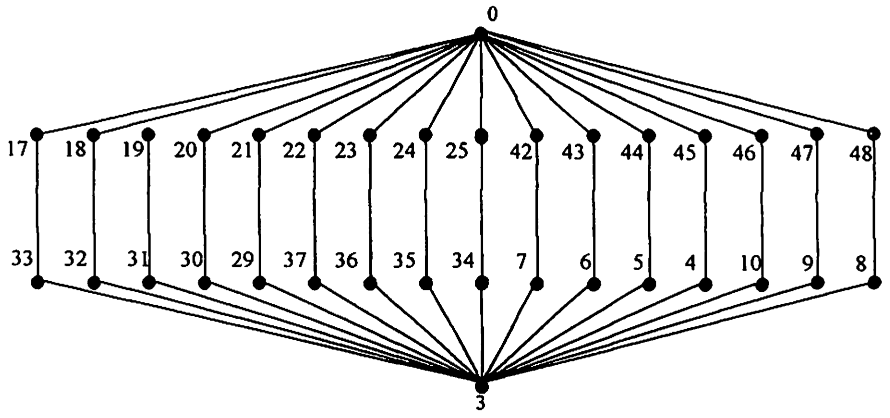
$f(v_2^i)$ = 2m+i-1   i=1,2,...,m-1

= i   i=m,m+1,...,2m-1

= 6m + i + 2   i=2m,2m+1,...,3m-1

= 4m + i + 1   i=3m,3m+1,3m+2,...,4m

Illustration: The graceful numbering of $P_{3,16}$.



Theorem 2.7 $P_{5,4m}$ are graceful for m ≥ 2.

Proof: Define f as follows.

$f(u) = 0$; $f(v) = 5m+3$

$f(v_1^i) = 20m-(i-1)$,   i=1,2,3,...4m

$f(v_j^i) = 10m+3+i+(2m-1)\left(\frac{j-2}{2}\right)$ i = 1,2,...,m-1; j = 2, 4

$= 8m+4+i+(2m-1)\left(\frac{j-2}{2}\right)$ i = m,m+1,....,2m-1; j = 2, 4

$f(v_j^i) = i+2+(2m+1)\left(\frac{j-2}{2}\right)$ i = 2m,2m+1,...3m-1; j = 2, 4

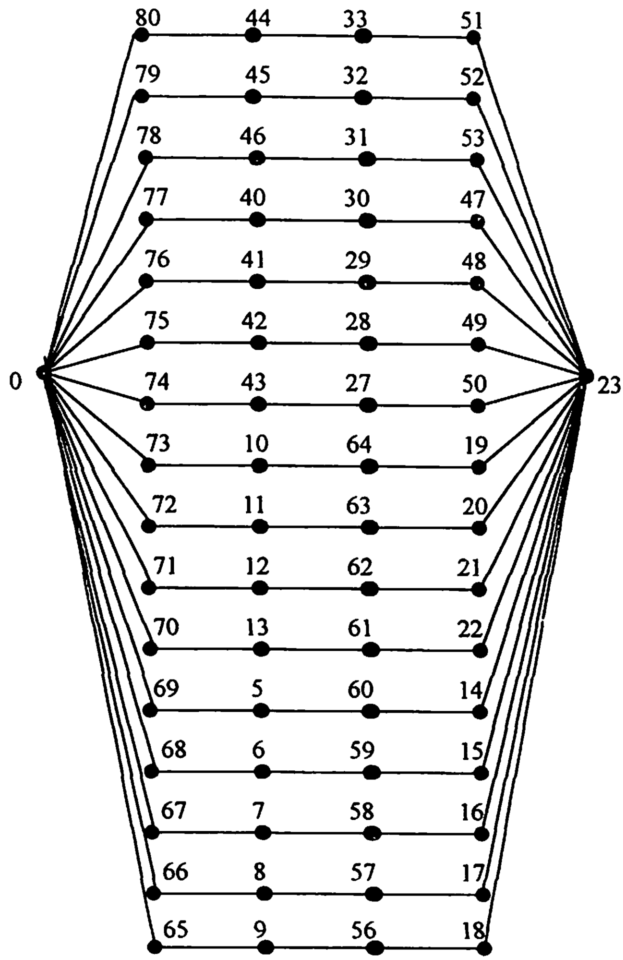
$= i+1-2m+(2m+1)\left(\frac{j-2}{2}\right)$ i = 3m,3m+1,...4m; j = 2, 4

$f(v_3^i) = 8m+2-i$ i = 1,2,...,2m-1

$= 18m - i$ i = 2m,2m+1,....4m

f gives a graceful labeling.

Illustration: The graceful numbering of $P_{5,16}$.



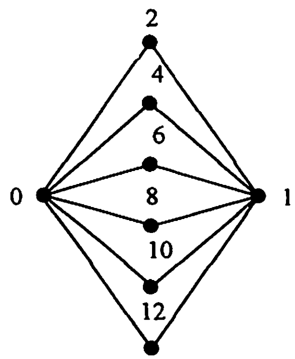
Theorem 2.8 $P_{2,m}$ is graceful for all $m \geq 1$.

Proof: Define f as follows.

$f(u) = 0$; $f(v) = 1$; $f(v_1^i) = 2i$, $i=1,2,...,m$

f gives a graceful labeling.

Illustration: The graceful numbering of $P_{2,6}$.



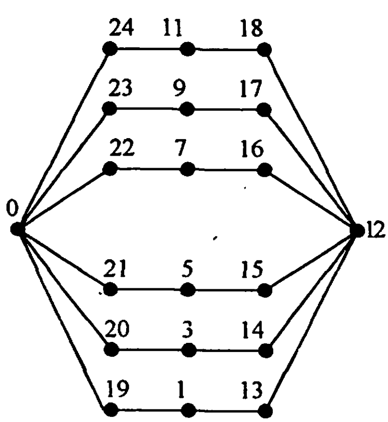
Theorem 2.9 $P_{4,m}$ are graceful for all $m \geq 1$.

Proof: Define f as follows.

$f(u) = 0$; $f(v) = 2m$; $f(v_{2j+1}^i) = 4m-jm-i+1$, $i = 1,2,......m$; $j = 0,1$

$f(v_2^i) = 2m+1-2i$, $i=1,2,...m$.

Illustration: The graceful numbering of $P_{4,6}$.



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References

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