Multidecompositions of the complete graph

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1 Abstract

Let G and H be a pair of non-isomorphic graphs on fewer than m vertices. In this paper, we introduce several new problems about decomposing the complete graph K_m into copies of G and H. We will assume that at least one of G or H is not a cycle. We also begin to examine variations to the problems of subgraph packing, covering, and factorization.

2 Introduction

Among the earliest results concerning graph decomposition is the problem of partitioning the edges of the complete graph into cycles. When the cycles each have length 3, the resulting decomposition of K_m is known as a *Steiner triple system of order m* or $STS(m)$; see Chapter 1 of [17]. Cycle systems with larger cycle length have been studied for some time [18].

The Oberwolfach problem [2, 3, 4, 10, 12, 15, 16] is defined as follows: for $c_1+c_2+\cdots+c_t = m$ and $c_j \geq 3$, partition the edges of K_m into 2-factors consisting of cycles with lengths from the set $\{c_1, c_2, \ldots, c_t\}$. As originally posed, the problem asks if it is possible to seat m guests at t tables with $c_1, c_2, \ldots, c_t$ seats. Solutions have been found for the cases in which the cycles have the same length (so all tables seat the same number of guests).

The Hamilton-Waterloo problem [1, 9] is a variation of the Oberwolfach problem which may be stated as follows: for $c_1 + c_2 + \cdots + c_s = c'_1 + c'_2 + \cdots + c'_t = m$, $c_i, c'_j \geq 3$, and $x + y = (m-1)/2$, partition the edges of K_m into x 2-factors consisting of cycles with lengths from $\{c_1, c_2, \ldots, c_s\}$ and y 2-factors consisting of cycles with lengths from $\{c'_1, c'_2, \ldots, c'_t\}$. This corresponds to the Oberwolfach problem where there are two sets of table sizes. Consider a group of m guests that will be attending two dinner parties. The s tables at Hamilton will seat $c_1, c_2, \ldots, c_s$ guests, and the t tables at Waterloo will seat $c'_1, c'_2, \ldots, c'_t$ guests. For a variation of the

Hamilton-Waterloo problem, see [6].

From here, there are many interesting generalizations of cycle decompositions. Other well-known problems include decomposing the complete graph into hamilton cycles [11, 14], non-isomorphic t-factors with $t \geq 3$ [13], and other factors [5]. The graph decomposition problem has developed an interesting history; for a broad overview, see [8].

We are particularly interested in decomposition problems involving two or more non-isomorphic subgraphs, at least one of which is not a cycle. In the next section, we will introduce several new graph decomposition problems.

The graphs we consider will be simple graphs. Let $V(G) = \mathbb{Z}_m$. Define $[a, b] = \{t \in \mathbb{Z}_m \mid a \leq t \leq b\}$, and let $G[a, b]$ be the subgraph of G induced by the vertices in $[a, b]$. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 is the graph with vertices $V(G_1) \cup V(G_2)$ and edges $E(G_1) \cup E(G_2)$. The *center* of the star $K_{1,n}$ is the vertex of degree n. For other graph-theoretic terminology used but not defined herein, see [7].

3 A new question

Consider the following general problem:

> (G, H)-DECOMPOSITION: Let G and H be non-isomorphic graphs on n_1 and n_2 vertices respectively, where $n_1 \leq n_2 < m$ and at least one of the graphs G or H is not a cycle. For which values of m can the edges of K_m be partitioned into edge-disjoint copies of G and H?

If both G and H are cycles, then the problem posed above is the Hamilton-Waterloo problem. Our interest lies in the problems which arise when either G or H is not a cycle. This problem gives rise to new variations of several classic problems in graph theory. Consider the following notation.

Let $G_1, G_2, \ldots, G_s$ be s edge-disjoint copies of G. We define $G^* = \bigcup_{i=1}^{s} G_i$ with edges $E(G^*) = \bigcup_{i=1}^{s} E(G_i)$. Similarly, let $H_1, H_2, \ldots, H_t$ be t edge-disjoint copies of H, and we define $H^* = \bigcup_{j=1}^{t} H_j$ with edges $E(H^*) = \bigcup_{j=1}^{t} E(H_j)$.

If $K_m - (E(G^*) \cup E(H^*)) = \overline{K_m}$, the empty graph on m vertices, then we say K_m is (G, H)-*multidecomposable* and refer to the decomposition as a $(K_m; G, H)$-*multidecomposition* or $(K_m; G, H)$-*design.*

We believe a variety of nice results may be obtained for this new question. For instance,

Theorem 3.1 *There is a $(K_m; K_n, K_{1,n})$-design for all $n \geq 3$ and $m \equiv 0, 1 \pmod{n}$.*

Proof. Let $n \geq 3$. Suppose $m = pn$ for some positive integer p. If $p = 1$, then there is no multidesign, since there is just one copy of K_n and no copies of $K_{1,n}$. Assume $p \geq 2$. If $|a - b| = n - 1$, clearly $K_m[a, b] \cong K_n$. Let $S_0 = \{s \in \mathbb{Z}_m \mid s \equiv 0 \mod n\}$, and define $K_n^* = \bigcup_{i \in S_0} K_m[i, i + n - 1]$. Since $|S_0| = p$, it is clear that $K_m - E(K_n^*) \cong K_{n(p)}$, the complete multipartite graph with p parts of size n.

Let $i, j \in S_0$, where $i \neq j$. Since $[i, i + n - 1] \cap [j, j + n - 1] = \phi$, the vertices $[i, i + n - 1] \cup [j, j + n - 1]$ induce a subgraph of $K_{n(p)}$ isomorphic to $K_{n,n}$. Because $K_{n,n}$ is the union of n copies of $K_{1,n}$, we may complete the multidecomposition with $\binom{p}{2}n$ copies of $K_{1,n}$.

Suppose $m = pn + 1$ for some integer $p \geq 2$. Let $V(K_m) = \mathbb{Z}_{m-1} \cup \{\infty\}$. As described above, the edges of $K_m[0, m - 2]$ can be partitioned into p copies of K_n and $\binom{p}{2}n$ copies of $K_{1,n}$.

Let $S_1 = \{s \in \mathbb{Z}_{m-1} \mid s \equiv 0 \mod n\}$. For each $s \in S_1$, the edges $E_s = \big\{\{\infty, v\} \mid v \in [s, s + n - 1]\big\}$ form a copy of $K_{1,n}$. Since $|S_1| = p$, we may complete the multidecomposition with these p copies of $K_{1,n}$. □

If the conditions do not allow a (G, H)-multidecomposition, we will try to use the edges of K_m as efficiently as possible. This means we may use only some maximal subset of edges, or we may reuse a small subset of edges. If $K_m - \big(E(G^*) \cup E(H^*)\big) \cong L$, where L is a non-empty graph, then we say the (G, H)-*multipacking* of K_m has *leave* L. If $(G^* \cup H^*) - E(K_m) \cong P$, where P is a non-empty graph, then we say the (G, H)-*multicovering* of K_m has *padding* P. We may seek either a maximum (G, H)-multipacking or a minimum (G, H)-multicovering; in both cases, L or P have as few edges as possible.

When $m \equiv m' \pmod n$ and $m' \geq 2$, it is natural to ask how "efficiently" we may multidecompose K_m into copies of K_n and $K_{1,n}$. Consider first the problem of a maximum packing of K_m with copies of K_n and $K_{1,n}$. Using the construction suggested in the proof of Theorem 3.1, we obtain the following.

Theorem 3.2 *If $n \geq 3$ and $m \equiv m' \pmod n$, then there is a maximum $(K_m; K_n, K_{1,n})$-multipacking with leave $L \cong K_{m'}$.*

Proof. By Theorem 3.1, the proof is clear when $m \equiv 0, 1 \pmod n$. We may assume that $m = pn + m'$, where $p \geq 2$ and $m' \geq 2$.

We can multidecompose $K_m[0, pn - 1]$ into p copies of K_n and $\binom{p}{2}n$ copies of $K_{1,n}$. For each $v \in [pn, pn + m' - 1]$ and $1 \leq k \leq p$, the edges $E_{kv} = \big\{\{s, v\} \mid s \in [(k-1)n, kn - 1]\big\}$ form a copy of $K_{1,n}$. From the edges of K_m, we have formed p copies of K_n and $\binom{p}{2}n + pm'$ copies of $K_{1,n}$.

The remaining edges of K_m form a subgraph isomorphic to $K_{m'}$. Since $m' < n$, this subgraph contains no further copies of K_n or $K_{1,n}$. □

Next, consider the problem of a minimum covering of K_m with copies of K_n and $K_{1,n}$.

Theorem 3.3 *If $n \geq 3$ and $m \equiv m'$ (mod n), then there is a minimum $(K_m; K_n, K_{1,n})$-multicovering with padding P, where*

$$P \cong \begin{cases} K_n - E(K_{n-m'+1}) & \text{if } m' < (n+1)/2 \\ K_n - E(K_{m'}) & \text{if } m' \geq (n+1)/2 \end{cases}$$

Proof. As in Theorem 3.2, we may assume that $m = pn + m'$, where $p \geq 2$ and $m' \geq 2$. We first multidecompose $K_m - E(K_m[pn, pn + m' - 1])$ into p copies of K_n and $\binom{p}{2}n + pm'$ copies of $K_{1,n}$.

It remains to cover the edges of $K_m[pn, pn + m' - 1] \cong K_{m'}$. We may use a single copy of K_n to cover this subgraph. However, it is possible to form a padding with fewer edges by using $m' - 1$ copies of $K_{1,n}$. Suppose $m < (n+1)/2$ and consider the following:

$$\begin{aligned} m' &< \frac{n+1}{2} = \frac{n-1}{2} + 1 \\ n(m'-1) &< \frac{n(n-1)}{2} = \binom{n}{2} \end{aligned}$$

So we have

$$n(m'-1) - \binom{m'}{2} < \binom{n}{2} - \binom{m'}{2} = \left|E(K_n - E(K_{m'}))\right| .$$

That is, when $m' < (n+1)/2$, a padding obtained using $m' - 1$ copies of $K_{1,n}$ will have fewer edges than the padding resulting from just one copy of K_n. We first consider the covering by stars.

Suppose $m' < (n+1)/2$. Let $R = [pn, pn + m' - 1]$ and $S = [0, n - m']$. For each $v \in R$, let $K_{1,n}(v)$ be the star with n edges centered at v with pendant vertices $(R - \{pn, v\}) \cup S$. Let C^* be the graph with vertices $R \cup S$ and edges $\bigcup_{v \in R - \{pn\}} (K_{1,n}(v))$, where the edge $\{u, v\}$ has multiplicity 2 whenever $u, v \in R - \{pn\}$. Note that $|E(C^*)| = n(m' - 1)$, since C^* is constructed from $m' - 1$ copies of $K_{1,n}$.

Clearly, the graph C^* covers the edges of $K_m[pn, pn + m' - 1]$. In addition, C^* contributes the edges $\big\{\{s, t\} \mid s \in [0, n - m'], t \in [pn + 1, pn + m' - 1]\big\}$ to the padding P as well as the edges $E(K_m[pn + 1, pn + m' - 1])$. Since $K_m[0, n - m'] \cong K_{n-m'+1}$, we have $P = C^* - K_m \cong K_n - E(K_{n-m'+1})$.

If $m' \geq (n+1)/2$, then we use a single copy of K_n to cover the edges of $K_{m'}$. The padding $P \cong K_n - E(K_{m'})$ has $\binom{n}{2} - \binom{m'}{2}$ edges. For $m' =$

$(n+1)/2$, we note that the padding with $m'-1$ stars has as many edges as the padding with one copy of K_n, but we prefer the simplicity of the latter. □

4 Conclusion

Our constructions provide an effective way to decompose complete graphs which are sufficiently large into subgraphs that are reasonably well-behaved. We believe there are many interesting directions from this starting point. It is natural to ask what results might be obtained for given subgraph pairings G and H, or for 3 or more initial subgraphs. There appear to be many exciting results possible.

We are especially interested in other constructions for the multidecomposition problems we have introduced. For instance, how might we achieve a more "balanced" $(K_m; G, H)$-multidecomposition, in which the number of copies of G is as close as possible to the number of copies of H? More generally, for a given pair of subgraphs G and H, find a multidecomposition of K_n into g copies of G and h copies of H for all possible values of $g \geq 1$ and $h \geq 1$.

The Oberwolfach and Hamilton-Waterloo problems involve prescribed 2-factorizations of K_m, requiring that m is odd. When m is even, the so-called "spouse-avoidance" variation of these problems look at prescribed 2-factorizations of $K_m - F$, where F is a 1-factor. Removing a particular factor is a common technique in graph decomposition. We are certain that many results may be obtained for the problem of $(K_m; G, H)$-*multifactorizations.* That is, our general multidecomposition problem may take an interesting turn if one or both of the subgraphs G and H is a spanning subgraph or factor of K_m.

We welcome any and all comments and suggestions on the problems mentioned here or related problems.

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