

# On the Laplacian Spectra of Graphs \*

Xiao-Dong Zhang

Department of Mathematics

Shanghai Jiao Tong University,

1954 Huashan Shanghai, 200030, P.R.China

xiaodong@sjtu.edu.cn

## Abstract

We first establish the relationship between the largest eigenvalue of the Laplacian matrix of a graph and its bipartite density. Then we present lower and upper bounds for the largest Laplacian eigenvalue of a graph in terms of its largest degree and diameter.

Key words: Laplacian matrix, bipartite density, diameter, eigenvalue.

AMS subject classifications: 05C50, 15A18, 15A12

## 1 Introduction.

Let  $G = (V, E)$  be a graph of order  $n$  on vertex set  $V(G)$  and edge set  $E(G)$ . For  $u \in V(G)$ , the degree of  $u$  is denoted by  $d_u$ . We adopt the notations and terminology following from Chung's book [5]. The *Laplacian matrix*  $\mathcal{L}(G)$  of  $G$  (see [5]) is defined to be the matrix

$$\mathcal{L}(u, v) = \begin{cases} 1, & \text{if } u = v \text{ and } d_u \neq 0, \\ -\frac{1}{\sqrt{d_u d_v}}, & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

However, in some literature, the matrix  $D(G) - A(G)$  is called the Laplacian matrix of a graph  $G$ , where  $D(G)$  and  $A(G)$  are degree diagonal and adjacency matrices of a graph  $G$  (see [10] for example). Clearly,  $\mathcal{L}(G)$  is a real symmetric positive semidefinite matrix. The eigenvalues of  $\mathcal{L}(G)$  are denoted by

$$0 = \lambda_0(G) \leq \lambda_1(G) \leq \dots \leq \lambda_{n-1}(G).$$

---

\*Supported by National Natural Science Foundation of China and The Project-sponsored by SRF for ROC'S. SEM.

Moreover,  $\lambda_1(G) > 0$  if and only if  $G$  is connected (see [5] for example). The spectra of  $\mathcal{L}(G)$  can be used to obtain much information about the graphs. For example, Chung and Oden in [6] established the relationship between  $\lambda_1$  and the isoperimetric number of a graph. Araujo and Peña in [1] gave the upper bound for the  $m$ -connectivity index in terms of the spectrum of  $\mathcal{L}(G)$ . For more related results, readers may be referred to [5], [10], [11] and the references therein.

Let  $U$  be a subset of vertices of a graph  $G$  with edge set  $E(G)$ . We denote by  $\partial U$  the set of edges with exactly one vertex in  $U$  and the other in  $V \setminus U$ . The *bipartite density* of a graph  $G$  is defined to be the number

$$b(G) = \max_{U \subseteq V(G)} \frac{|\partial U|}{|E(G)|}.$$

It is a very important parameter in the graph theory. Bondy and Locke in [2], Locke in [8] and Bylka, Idzik and Komar in [3] studied some properties and gave the lower bounds for some classes of graphs. Poljak and Tuza in [11] provided a good survey on maximum cuts and large bipartite subgraphs.

The organization of the paper is as follows. In Section 2, we establish the relationship between the largest eigenvalue of the Laplacian matrix and the bipartite density of a graph. The result, in turn, is used to give lower bounds for the largest eigenvalue of the Laplacian matrix of a graph in terms of its largest degree. In Section 3, we obtain an upper bound for the largest eigenvalue of the Laplacian matrix of a non-bipartite graph in terms of its diameter.

## 2 The lower bound for the largest eigenvalue

We begin with the relationship between the largest eigenvalue of the Laplacian matrix of a graph and its bipartite density.

**Theorem 2.1** *Let  $G$  be a graph on  $n$  vertices with degree  $d_u, u \in V(G)$ . Then*

$$(2.1) \quad b(G) \leq \frac{1}{2} \min\{\lambda_{n-1}(\mathcal{L}(G) + W) \mid \sum_{u \in V(G)} d_u w_u = 0\},$$

where  $W$  is diagonal matrix with diagonal element  $w_u, u \in V(G)$ .

**Proof.** Let  $B = (U, W)$  be a bipartite partition of  $G$  corresponding to the maximum cardinality of  $\partial U$ . Let  $D = \text{diag}(d_u)$  be the degree diagonal matrix of  $G$ . The value of a function  $f : V(G) \rightarrow \mathbf{R}$  at a vertex  $u$  is denoted by  $f_u$ . We can view  $f$  as a column vector corresponding to  $V(G)$ .

For any real-valued function  $f$ , we can consider quadratic form of  $\mathcal{L}(G)+W$  for any diagonal matrix  $W$  with  $\sum_{u \in V(G)} d_u w_u = 0$ .

$$\begin{aligned} \frac{\langle f, (\mathcal{L}(G) + W)f \rangle}{\langle f, f \rangle} &= \frac{\langle D^{1/2}g, \mathcal{L}(G)D^{1/2}g \rangle}{\langle D^{1/2}g, D^{1/2}g \rangle} + \frac{\langle D^{1/2}g, WD^{1/2}g \rangle}{\langle D^{1/2}g, D^{1/2}g \rangle} \\ &= \frac{\langle g, D^{1/2}\mathcal{L}(G)D^{1/2}g \rangle}{\langle g, Dg \rangle} + \frac{\langle g, DWg \rangle}{\langle g, Dg \rangle} \\ &= \frac{\sum_{(u,r) \in E(G)} (g_u - g_r)^2}{\sum_{u \in V(G)} d_u g_u^2} + \frac{\sum_{(u,r) \in E(G)} d_u w_u g_u^2}{\sum_{u \in V(G)} d_u g_u^2}. \end{aligned}$$

where  $f = D^{1/2}g$ . Now let  $g_u = 1$ , if  $u \in U$ ; and  $g_u = -1$ , if  $u \in V(G) \setminus U = W$ . Then  $\sum_{(u,r) \in E(G)} (g_u - g_r)^2 = 4|\partial U|$  and  $\sum_{u \in V(G)} d_u g_u^2 = 2|E(G)|$ . Moreover,  $\sum_{u \in V(G)} d_u w_u g_u^2 = \sum_{u \in V(G)} d_u w_u = 0$ . Hence

$$\lambda_{n-1}(\mathcal{L} + W) = \max_{f \neq 0} \frac{\langle f, (\mathcal{L}(G) + W)f \rangle}{\langle f, f \rangle} \geq \frac{4|\partial U|}{2|E(G)|} = 2b(G).$$

■

**Corollary 2.2** *Let  $G$  be a graph. Then*

$$(2.2) \quad \lambda_{n-1}(G) \geq 2b(G).$$

Moreover, if  $G$  is either bipartite graph or  $G$  is the line graph of an even semi-regular graph (i.e., even semi-regular graph is a bipartite and the degrees of vertices in each bipartite partition of the vertex set are even  $p$  and even  $q$ , respectively), then equality in (2.2) holds.

**Proof.** The (2.2) follows from (2.1) and  $W = 0$ . Moreover, if  $G$  is bipartite, then  $b(G) = 1$ . On the other hand, by [5],  $\lambda_{n-1}(G) = 2 = 2b(G)$ . Now we assume that  $G$  is the line graph of a  $(p, q)$ -semiregular graph with partition  $V(G_1) = (U, W)$ , where  $p$  and  $q$  are even. It is easy to see the Laplacian matrix of  $G$  is  $\mathcal{L}(G) = I - 1/(p+q-2)A(G)$ , where  $A(G)$  is the adjacency matrix of  $G$ . Since  $G$  is the line graph of  $G_1$ , the least eigenvalue of  $A(G)$  is  $-2$ , which yields  $\lambda_{n-1}(G) = (p+q)/(p+q-2)$ . On the other hand, by Theorem 1 in [12], the maximum edge number of bipartite subgraph of  $G$  is  $\frac{1}{4}(p^2|U| + q^2|W|) = (1/4)p|U|(p+q)$ . Hence  $2b(G) = (p+q)/(p+q-2) = \lambda_{n-1}(G)$ . ■

**Remark 2.3** *Since the problem of determining the bipartite density is NP-problem (see [11] or [14] for example), it is difficult to determine the exact value of the bipartite density of graphs. But the eigenvalue approximation to bipartite density is easy to compute and perhaps useful. On the other hand, we may use the results of the bipartite density to give the lower bounds for the largest eigenvalue of the Laplacian matrix of a graph.*

**Corollary 2.4** Let  $G$  be  $r$ -regular Ramanujan graph (see[9]). Then

$$b(G) \leq \frac{1}{2} + \frac{\sqrt{r-1}}{r}.$$

**Proof.** It follows from the fact that  $\lambda_{n-1}(G) \leq 1 + \frac{2\sqrt{r-1}}{r}$  by [9] and Corollary 2.2. ■

**Corollary 2.5** Let  $G$  be a triangle-free graph with  $d$ -regular graph. Then

$$\lambda_{n-1}(G) \geq \frac{d+4}{d+2}.$$

**Proof.** By [8], it is easy to see that  $b(G) \geq \frac{d+4}{2(d+2)}$ . Therefore the result follows from Corollary 2.2. ■

**Remark 2.6** If  $G$  is not triangle-free, the result may not hold. For example, Let  $K_n$  be complete graph on  $n \geq 4$  vertices. Then  $\lambda_{n-1}(K_n) = n/(n-1) < (n+3)/(n+1)$ .

**Corollary 2.7** Let  $G$  be a triangle-free graph with maximum degree 3. Then  $\lambda_{n-1}(G) \geq 8/5$ .

**Proof.** By [2],  $b(G) \geq 4/5$  which yields the desired result from Corollary 2.2. ■

Denoted by  $m(u)$  the average of the degrees of the vertices adjacent to  $u$ . Then  $d_u m_u$  is the 2-degree of vertex  $u$  (see [4]).

**Lemma 2.8** Let  $G$  be a graph. Then the largest eigenvalue of the adjacency matrix of  $G$ ,  $\lambda_{n-1}(A(G)) \leq m$ , where  $m = \max\{m_u, u \in V(G)\}$ .

**Proof.** Let  $f$  be the real-valued function  $f \mapsto d_u$ . Then by Perron-Frobenius Theorem, we have the largest eigenvalue of the adjacency matrix  $A(G)$  satisfies

$$\lambda_{n-1}(A(G)) \leq \max_{u \in V(G)} \frac{(A(G)f)_u}{f_u} = \max_{u \in V(G)} \{m_u\} = m.$$

■

**Theorem 2.9** Let  $G$  be a graph and  $m = \max\{m_u, u \in V(G)\}$ . Then

$$(2.3) \quad \lambda_{n-1}(G) \geq 1 + \frac{1}{m}$$

with equality if and only if  $G$  is complete graph.

**Proof.** Let  $\chi(G)$  be the chromatic number of  $G$ . By [5], we have

$$(2.4) \quad \chi(G) \geq 1 + \frac{1}{\lambda_{n-1}(G) - 1}.$$

On the other hand, by [13], we have

$$(2.5) \quad \chi(G) \leq 1 + \lambda_{n-1}(A(G)).$$

Combining (2.4) and (2.5), we have  $\lambda_{n-1}(G) \geq 1 + \frac{1}{\lambda_{n-1}(A(G))}$ . Hence (2.3) follows from Lemma 2.8

If  $G$  is complete graph on  $n$  vertices, then  $\lambda_{n-1}(G) = n/(n-1) = 1 + 1/m$ . Conversely, if equality in (2.3) holds, then equality in (2.5) holds. By [13],  $G$  must be either an odd cycle or a complete graph. However, by a simple calculation, the equality in (3.2) does not hold for an odd cycle on  $n > 3$  vertices. Hence  $G$  is a complete graph. ■

**Corollary 2.10** *Let  $G$  be a graph with the largest degree  $\Delta$ . Then*

$$(2.6) \quad \lambda_{n-1}(G) \geq 1 + \frac{1}{\Delta}$$

*with equality if and only if  $G$  is complete graph.*

**Proof.** It follows from Theorem 2.9 and  $\Delta \geq m$ . ■

**Remark 2.11** *It is obvious that Corollary 2.10 improves the known results  $\lambda_{n-1}(G) \geq n/(n-1)$  in [5].*

### 3 The Upper Bound for Largest Eigenvalue

If  $G$  is bipartite graph, the known result is  $\lambda_{n-1}(G) = 2$  (see [5] for example). Hence we only need to consider the non-bipartite graph for the largest eigenvalue of  $\mathcal{L}(G)$ . In this section, we will give an upper bound for the largest eigenvalue of  $\mathcal{L}(G)$  in terms of its diameter, when  $G$  is a non-bipartite graph.

**Theorem 3.1** *Let  $G = (V, E)$  be a non-bipartite graph on  $n$  vertices with diameter  $D$  and the number  $|E(G)|$  of edges of  $G$ . Then*

$$(3.1) \quad \lambda_{n-1}(G) \leq 2 - \frac{1}{2(D+1)|E(G)|}.$$

**Proof.** Let  $h$  be a real-valued function on vertex set  $V(G)$ .  $h : u \mapsto d_u$ ,  $u \in V(G)$ . We can also view  $h$  as a column vector on vertex set. Then  $h$  is eigenvector of  $\mathcal{L}(G)$  corresponding to eigenvalue 0. Let  $f$  be a real-valued function on vertex set  $V(G)$  such that  $f$  is eigenvector of  $\mathcal{L}(G)$  corresponding to  $\lambda_{n-1}(G)$ . Hence

$$\begin{aligned} \lambda_{n-1}(G) &= \frac{\langle f, \mathcal{L}(G)f \rangle}{\langle f, f \rangle} \\ &= \frac{\langle g, D^{1/2} \mathcal{L}(G) D^{1/2} g \rangle}{\langle g, Dg \rangle} \\ &= \frac{\sum_{(u,v) \in E(G)} (g_u - g_v)^2}{\sum_{u \in V(G)} d_u g_u^2} \\ &= 2 - \frac{\sum_{(u,v) \in E(G)} (g_u + g_v)^2}{\sum_{u \in V(G)} d_u g_u^2} \end{aligned}$$

where  $f = D^{1/2}g$ . Since  $\mathcal{L}(G)$  is symmetric,  $\langle f, h \rangle = \sum f_u h_u = 0$ . Hence neither  $U = \{u \in V(G); f_u \geq 0\}$  nor  $W = \{u \in V(G); f_u < 0\}$  is empty. Without loss of generality, we may assume that there exists  $v_0 \in U$  such that  $f_{v_0} > 0$  and  $f_{v_0} \geq |f_u|$  for any  $u \in V(G)$ . Because  $G$  is a non-bipartite graph, there exists an edge  $e = (u_0, w_0) \in E(G)$  such that either  $u_0, w_0 \in U$  or  $u_0, w_0 \in W$ . Now we consider the following two cases.

**Case 1**  $u_0, w_0 \in U$ . Let  $P$  be a shortest path from  $v_0$  to the set  $\{u_0, w_0\}$ . Then either  $P$  or  $P + (u_0, w_0)$  has odd length which is bounded by  $D + 1$ , since the diameter of  $G$  is  $D$ . Let  $Q$  be such a path from  $v_0$  to  $u_0$  or  $w_0$  (say  $u_0$ ) with odd length  $k \leq D + 1$ . By the Cauchy-Schwartz inequality and  $\sum_{i=1}^{2m+1} |a_i| \geq |\sum_{i=1}^{i-2m+1} (-1)^{i-1} a_i|$ ,

$$\begin{aligned} \sum_{(u,v) \in E(G)} (g_u + g_v)^2 &\geq \sum_{(u,v) \in Q} (g_u + g_v)^2 \\ &\geq \frac{1}{k} \left( \sum_{(u,v) \in Q} |g_u + g_v| \right)^2 \\ &\geq \frac{1}{k} (g_{v_0} + g_{u_0})^2 \\ &\geq \frac{g_{v_0}^2}{D+1}. \end{aligned}$$

Moreover,

$$\sum_{u \in V(G)} d_u g_u^2 \leq g_{v_0}^2 \sum_{u \in V(G)} d_u = g_{v_0}^2 2|E(G)|.$$

Therefore the inequality in (3.1) holds.

**Case 2**  $u_0, w_0 \in W$ . Using an similar argument of Case 1, there exists a path  $R$  from  $v_0$  to  $u_0$ (say) with even length  $l \leq D + 1$ . Furthermore, we can use an argument similar to that in the previous case show that

$$\sum_{(u,v) \in E(G)} (g_u + g_v)^2 \geq \frac{(g_{v_0} - g_{u_0})^2}{D + 1} \geq \frac{g_{v_0}^2}{D + 1}.$$

Hence the inequality in (3.1) holds. This completes the proof. ■

**Remark 3.2** *The upper bound in (3.1) is best possible up to a constant factor in second term. We consider the odd cycle  $C_n$  on  $n = 2D + 1$  vertices. It follows from the fact that*

$$\lambda_{n-1}(G) = 1 + \cos \frac{\pi}{n} \geq 2 - \frac{\pi^2}{2(D+1)|E(G)|}.$$

*On the other hand, by Theorem 3.1, we have*

$$\lambda_{n-1}(G) \leq 2 - \frac{1}{2(D+1)|E(G)|}.$$

*In some sense, our result is best possible.*

## References

- [1] O. Araujo and J.A. de la Peña, The connectivity index of a weighted graph. *Linear Algebra and its Applications*, **283** (1998), 171-177.
- [2] J. A. Bondy and S. C. Locke, Largest bipartite subgraphs in triangle-free graphs with maximum degree three. *J. of Graph Theory*, **10**(1986), 477-504.
- [3] S. Bylka, A. Idzik, and J. Komar, Bipartite subgraphs of graphs with maximum degree three. *Graphs and Combin.*, **15** (1999), 129-136.
- [4] D. Cao, Bounds on eigenvalues and chromatic numbers. *Linear Algebra and its Applications*, **270** (1998), 1-13.
- [5] F.R.K. Chung, *Spectral Graph Theory*, CBMS Lecture Notes 92, AMS Publication, 1997.
- [6] F.R.K. Chung and Oden, Weighted graph Laplacians and isoperimetric inequalities. *Pacific J. Mathematics*, **192**(2) (2000), 257-273.
- [7] P. Erdős, On some extremal problems in graph theory. *Israel J. Mathematics*, **3**(1965), 113-116.

- [8] S. C. Locke. A note on bipartite subgraphs of triangle-free regular graphs. *J. of Graph Theory* , **14**(1990), 181-185.
- [9] A. Lubotzky, R. Phillips and P. Sarnak. Ramanujan graphs. *Combinatorica*. **8(3)** (1988), 261-277.
- [10] R. Merris. Laplacian matrices of graphs: a survey. *Linear Algebra and its Applications*. **197-198** (1994), 143-176.
- [11] S. Poljak and Z. Tuza. Maximum cuts and large bipartite subgraphs. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*. **20** (1995), 181-244.
- [12] S. Shiraishi. On the maximum cut of line graphs. *Combinatorics, Probability and Computing*. **7** (1998), 335-337.
- [13] H.S. Wilf. The eigenvalues of a graph and its chromatic number. *J. London Math. Soc.* , **42** (1967), 330-332.
- [14] M. Yannakakis. Node and edge deletion NP-complete problems. *Proc. Tenth Ann.A.C.M. Symp. on theory of computing, A.C.M.*, New York, 1978, 253-264.