A note on the difference between the upper irredundance and independence numbers of a graph [1]

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Abstract

Let $G = (V, E)$ be a simple graph. Let α and IR be the independence number and upper irredundance number of G respectively. In this paper, we prove that for any graph G of order n with maximum degree $\Delta \geq 1$, $IR(G) - \alpha(G) \leq \frac{\Delta-2}{2\Delta}n$. When $\Delta = 3$, the result was conjectured by Rautenbach.

Keywords: Independence number; Upper irredundance number

1. Introduction

Let $G = (V, E)$ be a simple graph. Denote by $\Delta(G)$ and $\chi(G)$ the maximum degree and chromatic number of G, respectively. For a subset S of V, denote by $G[S]$ the subgraph induced by S. A subset S of V is a dominating set of G, if for any vertex $v \in V - S$, there is some vertex $u \in S$ such that $uv \in E(G)$. A set S is independent if $E(G[S]) = \emptyset$. The independence number of G, denoted by $\alpha(G)$, is the maximum size of a (maximal) independent set of G. For any $S \subseteq V$ and any vertex $x \in S$ we define the private neighborhood $PN(x, S)$ of x with respect to S as $N[x] - N[S - \{x\}]$ where $N[x] = N(x) \cup \{x\}$ denotes the closed neighborhood of x and $N[S] = \cup_{y \in S} N[y]$. A set S is irredundant, if $PN(x, S) \neq \emptyset$ for any $x \in S$. The upper irredundance number of G, denoted by $IR(G)$, is the maximum size of a maximal irredundant set of G. A maximal irredundant set of cardinality $IR(G)$ is called a IR-set of G.

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[1]The project was partially supported by NNSFC 19871036.

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Rautenbach [3] investigated $IR(G) - \alpha(G)$ and obtained the following result.

Theorem 1 *([3]) For any graph G of order n with maximum degree $\Delta \geq 1$,*

$$IR(G) - \alpha(G) \leq \lfloor \frac{(\Delta - 1)^2}{2\Delta^2} n \rfloor.$$

Furthermore, he posed a conjecture as follows.

Conjecture 1 *([3]) If G is a graph with maximum degree $\Delta(G) \leq 3$, then*

$$IR(G) - \alpha(G) \leq \lfloor \frac{n}{6} \rfloor.$$

The conclusion in Theorem 1 is not best possible. In what follows, we will improve it and obtain a sharp upper bound on $IR(G) - \alpha(G)$. This bound will imply Conjecture 1.

Lemma 1 *([1, P. 223]) Let $\Delta \geq 3$ and let G be a graph such that $\Delta(G) \leq \Delta$ and $K_{\Delta+1} \not\subseteq G$, then $\chi(G) \leq \Delta$.*

By Lemma 1, it is easy to prove the following theorem.

Theorem 2 *If G is a connected graph of maximum degree $\Delta \geq 3$ and $G \neq K_{\Delta+1}$, then*

$$\alpha(G) \geq \frac{n}{\Delta}.$$

Now we are in a position to prove our main result.

Theorem 3 *For any graph G of order n with maximum degree $\Delta \geq 2$,*

$$IR(G) - \alpha(G) \leq \frac{\Delta - 2}{2\Delta} n.$$

Furthermore, this bound is sharp.

Proof If $\Delta = 2$, the conclusion is trivial. We may assume that $\Delta \geq 3$. Let I be an IR-set of G. Let $S = \{x_1, x_2, \ldots, x_s\}$ be the set of vertices $x \in I$ for which $PN(x, I) - I \neq \emptyset$ and $R = I - S$. Note that for any $x_i \in S$ where $1 \leq i \leq s$, $PN(x_i, I) - I \neq \emptyset$. Therefore we can choose a vertex $y_i \in PN(x_i, I) - I$ and let $S' = \{y_1, y_2, \ldots, y_s\}$. Since $G[S \cup S'] \neq K_{\Delta+1}$, by Theorem 2, we have

$$\alpha(G[S \cup S']) \geq \frac{2s}{\Delta}.$$

Let I_1 be a maximum independent set of $G[S \cup S']$. It is obvious that $I_1 \cup R$ is an independent set of G. Thus we have

$$IR(G) - \alpha(G) \leq IR(G) - |I_1 \cup R| \leq |R| + s - \frac{2s}{\Delta} - |R| = \frac{(\Delta - 2)s}{\Delta} \leq \frac{\Delta - 2}{2\Delta}n.$$

To see the sharpness of the bound, we construct an infinite class $\mathcal{G}$ of connected graphs such that $IR(G) - \alpha(G) = \frac{\Delta-2}{2\Delta}n$ for each $G \in \mathcal{G}$ as follows.

Let k be a positive integer. For $0 \leq i \leq k-1$, we let $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,\Delta}$, $Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,\Delta}\}$ and $Z_i = (Y_i - \{y_{i,1}\}) \cup \{y_{i+1,1}\}$ where the subscript is modulo k. Let $V(G) = \bigcup_{i=0}^{k-1}(X_i \cup Y_i)$. To complete the construction of graphs in $\mathcal{G}$, join $x_{i,j}$ to $y_{i,j}$ and let $G[X_i] \cong G[Z_i] \cong K_\Delta$. Observe that every independent set of G contains at most one vertex in each X_i and Z_i. Hence $\alpha(G) \leq 2k$. Again since $\alpha(G) \geq \frac{2k\Delta}{\Delta} = 2k$, we have $\alpha(G) = 2k$. Furthermore, using the fact that $\bigcup_{i=0}^{k-1} X_i$ is an irredundant set of G, we have $IR(G) \geq |\bigcup_{i=0}^{k-1} X_i| = k\Delta$. Thus $\frac{\Delta-2}{2\Delta}n \geq IR(G) - \alpha(G) \geq k\Delta - 2k = (\Delta - 2)k = \frac{\Delta-2}{2\Delta}n$. This completes the proof of the theorem. ∎

Corollary 1 *If G is a graph with maximum degree $\Delta(G) \leq 3$, then*

$$IR(G) - \alpha(G) \leq \lfloor \frac{n}{6} \rfloor.$$

Thus, Conjecture 1 is true.

References

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