Partitioning a strong tournament into k cycles

Hao Li [1]

L.R.I., Bât. 490, Université de Paris-Sud, 91405, Orsay Cedex, France

Jinlong Shu [2]

Department of Mathematics, East China Normal University, Shanghai 200062, China

Abstract

A digraph T is called strongly connected if for every pair of vertices u and v there exists a directed path from u to v and a directed path from v to u. Denote the in-degree and out-degree of a vertex v of T by $d^-(v)$ and $d^+(v)$, respectively. We define $\delta^- = \min_{v \in V(T)}\{d^-(v)\}$, and $\delta^+ = \min_{v \in V(T)}\{d^+(v)\}$. Let T_0 be a 7-tournament which contains no transitive 4-subtournament. Let T be a strong tournament, $T \not\cong T_0$ and $k \geq 2$. In this paper, we show that if $\delta^+ + \delta^- \geq \frac{k-2}{k-1}n + 3k - 1$, then T can be partitioned into k cycles. When $n \geq 3k(k-1)$ a regular strong n-tournament can be partitioned into k cycles and a almost regular strong n-tournament can be partitioned into k cycles when $n \geq (3k+1)(k-1)$. Finally, if a strong tournament T can be partitioned into k cycles, q is an arbitrary positive integer not large than k. We prove that T can be partitioned into q cycles.

Key words: Strong Tournament, Regular Tournament, Transitive Tournament, Quasi-transitive Tournament.

AMS classification: 05C20, 05C75.

1 Introduction and Notation

A digraph is called *strongly connected* or *strong* if for every pair of vertices u and v there exists a directed path from u to v and a directed path from v to u. A digraph is *disconnected* if it is not strong. Let k be a positive integer. A digraph T is *k-connected* if the removal of any set of fewer than k vertices results in a strong digraph. A digraph T *can be partitioned into k subgraphs*, say as, T_1, T_2, $\ldots$, T_k, if $\cup_{i=1}^{k} V(T_i) = V(T)$ and $V(T_i) \cap V(T_j) = \emptyset$ $(1 \leq i < j \leq k)$.

[1] *Email:* li@lri.fr

[2] The research was partially supported by NNSF of China(19971027) and Shanghai Priority Academic Discipline. The research was done while the second author was visiting LRI. *Email:* jl_shu@yahoo.com

A tournament is an orientation of a complete graph. A tournament (a subtournament) with n vertices will be called an *n-tournament* (*n-subtournament* respectively). Let T be a tournament with vertex set $V(T)$ and arc set $E(T)$. $A = \{v_1, v_2, \ldots, v_m\}$ is a subset of $V(T)$, $< A >$ denotes the subtournament of T induced by A. We also write $T - A$ for $< V(T) - A >$. Specially, if $A = \{v\}$, we denote $< V(T) - v >$ by $T - v$. If no confusion arises, $< v_1, v_2, \ldots, v_m >$ will be used to denote $< A >$ which contains a directed hamiltonian path $v_1 v_2 \ldots v_m$. v_1 and v_m are called the first vertex and the last vertex of the path, respectively. If T is a 1-connected tournament, $v \in V(T)$ is *a cut vertex* of T if $T - v$ is a disconnected subtournament. v is a *non-cut vertex* if v is not a cut vertex.

Let T_1 be a subtournament of T and $v \in V(T)$, *the in-neighborhood of v in T_1* is $N^-_{T_1}(v) = \{u \in V(T_1) | (u, v) \in E(T)\}$, *the out-neighborhood of v in T_1* is $N^+_{T_1}(v) = \{u \in V(T_1) | (v, u) \in E(T)\}$. $d^-_{T_1}(v) = |N^-_{T_1}(v)|$ and $d^+_{T_1}(v) = |N^+_{T_1}(v)|$ are *in-degree* and *out-degree* of a vertex v in T_1, respectively. In the case $T_1 = T$, we use $N^+(v)$, $N^-(v)$, $d^+(v)$ and $d^-(v)$ instead of $N^+_{T_1}(v)$, $N^-_{T_1}(v)$, $d^+_{T_1}(v)$ and $d^-_{T_1}(v)$, respectively. In fact that $N^-_{T_1}(v) = N^-(v) \cap V(T_1)$ and $N^+_{T_1}(v) = N^+(v) \cap V(T_1)$. If T can be partitioned into k subgraphs, say as, T_1, T_2, $\ldots$, T_k, then $N^-(v) = \bigcup_{i=1}^k N^-_{T_i}(v)$, $N^+(v) = \bigcup_{i=1}^k N^+_{T_i}(v)$, $d^-(v) = \sum_{i=1}^k d^-_{T_i}(v)$ and $d^+(v) = \sum_{i=1}^k d^+_{T_i}(v)$.

We define $\delta^- = \min_{v \in V(T)} \{d^-(v)\}$, $\delta^+ = \min_{v \in V(T)} \{d^+(v)\}$, $\delta = \min\{\delta^+, \delta^-\}$, $\Delta^- = \max_{v \in V(T)} \{d^-(v)\}$, $\Delta^+ = \max_{v \in V(T)} \{d^+(v)\}$, and $\Delta = \max\{\Delta^+, \Delta^-\}$. A tournament is *regular* when $\delta = \Delta$ and *almost regular* when $\Delta - \delta \leq 1$.

If (u, v) is an arc in T, then *u dominates v*, we denote $u \Rightarrow v$ or $v \Leftarrow u$. A set $A \subseteq V(T)$ *dominates* a set $B \subseteq V(T)$ if every vertex of A dominates every vertex of B, we denote $A \Rightarrow B$ or $B \Leftarrow A$. If some vertex in A dominates some vertex in B and vice versa, this will be denoted by $A \leftrightarrow B$.

It is well-known that every tournament contains a directed hamiltonian path and every strong tournament contains a directed hamiltonian cycle. Conversely, a tournament is strong if it contains a directed hamiltonian cycle.

Let $P = v_1 v_2 v_3 \ldots v_n$ be a directed hamiltonian path of a n-tournament T. T is *a transitive tournament* if $v_i \Rightarrow v_j$ whenever $1 \leq i < j \leq n$. Clearly, a transitive tournament is not strong. T is *a quasi-transitive tournament* if $v_k \Rightarrow v_{k+1}$ and $v_j \Rightarrow v_i$ whenever $1 \leq k \leq n - 1$ and $1 \leq i < j - 1 \leq n - 1$. It is evident that a quasi-transitive tournament is strong when it contains at least three vertices. Let T be a quasi-transitive n-tournament. Specially, T contains an isolated vertex when $n = 1$. T only contains an arc when $n = 2$. T is a strong triangle when $n = 3$.

Let T_0 be a 7-tournament which contains no transitive 4-subtournament.

For a tournament, the following problem was posed by Bollobás[See [4]].

Problem A. If k is a positive integer, what is the least integer $g(k)$ so that all but a finite number of $g(k)$-connected tournaments contain k vertex-disjoint cycles that span $V(T)$?

Clearly, $g(1) = 1$. In 1985, Reid [4] showed Theorem 1 to answer that $g(2) = 2$.

Theorem 1 [4]. *If T is a 2-connected n-tournment, $n \geq 6$, and $T \ncong T_0$, then T contains two vertex-disjoint cycles that span $V(T)$. That is, T can be partitioned into two cycles. In particular, one of the two cycles can be a triangle.*

In 2001, Chen, Gould and Li [2] proved that $g(k) = k$. Theorem 2 was obtained.

Theorem 2 [2]. *Every k-connected n-tournament T with $n \geq 8k$ contains k vertex-disjoint cycles that span $V(T)$.*

So Problem A was solved completely.

Recently, we studied the structure of an arbitrary strong tournament which can not be partitioned into two cycles. A strong tournament without the condition of connectivity can be partitioned into two cycles. We obtain Theorem 3 and Corollary 4.

Theorem 3[3]. *Let T be a strong n-tournament with $n \geq 6$ and p cut vertices. If T can not be partitioned into two cycles and $T \ncong T_0$. Then*

(1) $\max\{\delta^+, \delta^-\} \leq 2$ *and*

(2) an arbitrary directed hamiltonian cycle of T can be partitioned into two consecutive segments Q and L, where Q is a transitive $(n-l)$-subtournament, L is a quasi-transitive l-subtournament and $l = p, p+1$ or $p+2$.

Moreover, all vertices in Q are non-cut vertices of T and $|V(Q)| \geq 3$. Let u and v are the first vertex and the last vertex of Q, respectively. Then $d_T^+(v) \leq 2$ and $d_T^-(u) \leq 2$. All cut vertices of T are consecutive on an arbitrary directed hamiltonian cycle and are included in L and $p \leq \frac{n+1}{2}$.

Corollary 4[3]. *Let T be a strong n-tournament with $n \geq 6$. If $T \ncong T_0$ and $\max\{\delta^+, \delta^-\} \geq 3$, then T can be partitioned into two cycles.*

Let d be the maximum number of vertices of a transitive subtournament in T. Theorem 5 give a sufficient condition of T such that T can be partitioned into k cycles.

Theorem 5[3]. *Let T be a strong n-tournament with $n \geq 6$. If $d \leq \frac{n-9k+8}{2}$, then T can be partitioned into k cycles.*

Indeed our research is motivated by these results above. Can we find some conditions about δ^+ or δ^- instead of "k-connected" or the condition in Theorem 5 such that T can be partitioned into k cycles? The answer is positive. The main result in this paper, Theorem 6 gives another sufficient condition about $\delta^+ + \delta^-$ such that T can be partitioned into k cycles.

Theorem 6. *Let T be a strong n-tournament and $T \ncong T_0$. k is a positive integer larger than one. If $\delta^+ + \delta^- \geq \frac{k-2}{k-1}n + 3k - 1$. Then T can be partitioned* *into k cycles.*

If T is a regular tournament, then $\delta^+ = \delta^- = \Delta^+ = \Delta^- = \delta = \Delta$ and $\delta^+ + \delta^- = n - 1$.

Let T be a almost regular tournament. By the definition, $\Delta - \delta \leq 1$. We obtain $\delta^+ + \delta^- \geq n - 2$ since $\Delta^- - \delta^- \leq 1$ and $\Delta^- = n - 1 - \delta^+$.

By Theorem 6, it is easy to get the following Corollary 7.

Corollary 7. *Let T be a strong n-tournament and $T \ncong T_0$. k is a positive integer larger than one.*

(1) If T is a regular tournament and $n \geq 3k(k-1)$, then T can be partitioned into k cycles;

(2) If T is a almost regular tournament and $n \geq (3k+1)(k-1)$, then T can be partitioned into k cycles;

We consider the union of vertex-disjoint cycles.

Theorem 8. *If a strong tournament T can be partitioned into k cycles. For any positive integer q satisfying $1 \leq q \leq k$, then T can be partitioned into q cycles.*

Now we give some notation.

Let C be a cycle in T. For every vertex $v \in V(C)$, v_C^+ denote the successor of v on C and let v_C^- denote the predecessor of v on C. If no confusion arises, v^+ and v^- will be used to denote v_C^+ and v_C^-, respectively. Let X be a cycle or a path of T and let u and v be two vertices on X (u, v are in that order along X if X is a path). We define $X[u, v]$ as *the subpath*(or *the consecutive segment*) of X from u to v. For any $u \notin V(C)$, if u is dominated by a vertex $v \in V(C)$ and u dominates x^+, then $ux^+C[x^+, x]xu$ is a cycle longer than C. In this case, we say that *u can be inserted into C*. Let S be a vertex subset of T and $S \cap V(C) = \emptyset$. If $< S \cup V(C) >$ is a strong subtournament, i.e., $< S \cup V(C) >$ contains a directed hamiltonian cycle, we call *S can be inserted into C*. If $u \notin V(C)$ and $u \not\leftrightarrow V(C)$, then u can be inserted into C. Moreover, if C_1 and C_2 are two vertex-disjoint cycles of a tournament T and $V(C_1) \not\leftrightarrow V(C_2)$, it is clearly that $< V(C_1) \cup V(C_2) >$ is a strong subtournament, i.e., C_1 and C_2 can be united into a cycle which contains all vertices of them. So if u cannot be inserted into a cycle C, then $u \Rightarrow V(C)$ or $V(C) \Rightarrow u$. If C_1 and C_2 can not be united into a cycle, then $V(C_1) \Rightarrow V(C_2)$ or $V(C_2) \Rightarrow V(C_1)$.

Other notation and terminology not defined here can be found in [1].

2 Proofs of Main Results

Proof of Theorem 6. As $\delta^+ + \delta^- \leq n - 1$ and $\delta^+ + \delta^- \geq \frac{k-2}{k-1}n + 3k - 1$, $n \geq 3k(k-1)$.

If $k = 2$, then $\delta^+ + \delta^- \geq 5$ and $n \geq 6$. So $\max\{\delta^+, \delta^-\} \geq 3$. By $T \ncong T_0$ and Corollary 4, T can be partitioned into two cycles.

Now, we assume $k \geq 3$. Then $n \geq 9(k-1)$.

Suppose, to the contrary, k is the smallest integer such that Theorem 6 fails. Then, $k \geq 3$ and T can be partitioned into $k-1$ cycles, say as, $C_1, C_2, \ldots, C_{k-1}$. So C_i can not be partitioned into two cycles for every $1 \leq i \leq k-1$.

We choose a partition of T such that C_1 is the longest cycle and any vertex subset $S \subset V(C_i)$ cannot be inserted into C_1 if $< V(C_i) - S >$ is strong whenever $2 \leq i \leq k-1$.

Thus, $|V(C_1)| \geq \frac{n}{k-1} \geq 9$. By Theorem 3, C_1 can be partitioned into a transitive subtournament Q_1 and a quasi-transitive subtournament L_1. Let u and v are the first vertex and the last vertex of Q_1, respectively. Moreover, we have $d^+_{C_1}(v) \leq 2$ and $d^-_{C_1}(u) \leq 2$. So

$$d^-_T(u) = \sum_{i=1}^{k-1} d^-_{C_i}(u) \leq 2 + \sum_{i=2}^{k-1} d^-_{C_i}(u)$$

and

$$d^+_T(v) = \sum_{i=1}^{k-1} d^+_{C_i}(v) \leq 2 + \sum_{i=2}^{k-1} d^+_{C_i}(v).$$

We will show that $d^-_T(u) + d^+_T(v) \leq \frac{k-2}{k-1}n + 3k - 2$, which produces a contradiction.

It is evident that we can obtain Proposition A.

Proposition A. *Let $w_1 w_2 \ldots w_t w_1$ be a directed hamiltonian cycle of C_i where $2 \leq i \leq k-1$. If $V(C_1) \Rightarrow w_a$, $w_b \Rightarrow V(C_1)$ and $w_a^- \Rightarrow w_b^+$ where $1 \leq a \leq b \leq t$, $w_1^- = w_t$, $w_t^+ = w_1$. Then the segment $C_i[w_a, w_b]$ can be inserted into C_1 and the other vertices of C_i form a cycle*

$$w_1 C_i[w_1, w_a^-] w_a^- w_b^+ C_i[w_b^+, w_t] w_t w_1.$$

In the proof, we will often use Proposition A to produce a contradiction of the choice of C_1.

Let

$$\mathcal{F} = \{C_i \mid |V(C_i)| \geq 6 \text{ and } < V(C_i) > \ncong T_0 \text{ where } 2 \leq i \leq k-1\}$$

and

$$\mathcal{H} = \{C_i \mid |V(C_i)| \leq 5 \text{ or } < V(C_i) > \cong T_0 \text{ where } 2 \leq i \leq k-1\}.$$

So

$$\mathcal{F} \cup \mathcal{H} = \{C_2, C_3, \ldots, C_{k-1}\} \text{ and } \mathcal{F} \cap \mathcal{H} = \emptyset.$$

We distinguish two cases.

Case 1. $C_i \in \mathcal{F}$.

By Theorem 3, C_i can be partitioned into a transitive subtournament Q_i and a quasi-transitive subtournament L_i. All vertices of Q_i are the non-cut vertices of C_i. Let $|V(C_i)| = n_i$, $|V(L_i)| = l_i$ and $|V(Q_i)| = q_i \geq 3$. So $n_i = l_i + q_i$.

CLAIM 1. *If w is an arbitrary vertex of Q_i, then $w \Rightarrow V(C_1)$ or $V(C_1) \Rightarrow w$.*

Proof of Claim 1. As w is a non-cut vertex of C_i, $C_i - w$ is a strong. If $w \not\leftrightarrow V(C_1)$, then w can be inserted into C_1. $< C_1 \cup \{w\} >$ is a new cycle longer than C_1. It contradicts the choice of C_1. So $w \Rightarrow V(C_1)$ or $V(C_1) \Rightarrow w$. □

By Claim 1, we have $d^-_{Q_i}(u) + d^+_{Q_i}(v) = |V(Q_i)| = q_i$. We assume $Q_i =< v_1, v_2, \ldots, v_{q_i} >$, $L_i =< u_1, u_2, \ldots, u_{l_i} >$ and $v_1 v_2 \ldots v_{q_i} u_1 u_2 \ldots u_{l_i} v_1$ is a directed hamiltonian cycle of $< C_i >$.

CLAIM 2. *One of the following cases holds.*
(1) $V(Q_i) \Rightarrow V(C_1)$ for $1 \leq i \leq q_i$;
(2) $V(C_1) \Rightarrow V(Q_i)$ for $1 \leq i \leq q_i$;
(3) There is a integer m such that

$$v_j \Rightarrow V(C_1) \quad \textit{for every} \quad 1 \leq j \leq m$$

and

$$V(C_1) \Rightarrow v_j \quad \textit{for every} \quad m+1 \leq j \leq q_i,$$

where $1 \leq m \leq q_i - 1$;
(4) $V(C_1) \Rightarrow v_1$ and $Q_i - v_1 \Rightarrow V(C_1)$;
(5) $V(C_1) \Rightarrow Q_i - v_{q_i}$ and $v_{q_i} \Rightarrow V(C_1)$.

Proof of Claim 2. If all these cases do not occur, we must have v_j and v_{j+1} such that $v_{j+1} \Rightarrow V(C_1)$, $V(C_1) \Rightarrow v_j$ and $2 \leq j, j+1 \leq q_i - 1$. It is evident that v_j and v_{j+1} can be inserted into C_1 and $< C_i - \{v_j, v_{j+1}\} >$ is strong. It is a contradiction. □

Now, we consider $d^-_{L_i}(u) + d^+_{L_i}(v)$.

CLAIM 3. *Let $v \Rightarrow u_j$ and $u_r \Rightarrow u$. If $j \geq 5$ then $r \geq j - 3$.*

Proof of Claim 3. To the contrary, $r \leq j - 4$. Then

$$u_j L_i[u_j, u_{l_i}] u_{l_i} v_1 Q_i[v_1, v_{q_i}] v_{q_i} u_1 L_i[u_1, u_r] u_r$$

can be inserted into C_1 and $< u_{r+1}, u_{r+2}, \ldots, u_{j-1} >$ is strong. It is a contradiction. □

Let $j_0 = \max\{j \mid v \Rightarrow u_j, u_j \in V(L_j)\}$.

CLAIM 4. $d^-_{L_i}(u) + d^+_{L_i}(v) \leq l_i + 4$.

Proof of Claim 4. If $j_0 \geq 5$, clearly, $d^+_{L_i}(v) \leq j_0$. By Claim 3, $u_1, u_2, \ldots, u_{j_0-4}$ can not dominate u. So $d^-_{L_i}(u) \leq l_i - (j_0 - 4) = l_i - j_0 + 4$. Thus $d^-_{L_i}(u) + d^+_{L_i}(v) \leq l_i + 4$.

If $j_0 \leq 4$, clearly, $d^+_{L_i}(v) \leq j_0 \leq 4$ and $d^-_{L_i}(u) \leq l_i$. We have $d^-_{L_i}(u)+d^+_{L_i}(v) \leq l_i + 4$. □

CLAIM 5. $d^-_{L_i}(u) + d^+_{L_i}(v) \leq l_i + 3$.

Proof of Claim 5. By Claim 4, we have $d^-_{L_i}(u) + d^+_{L_i}(v) \leq l_i + 4$. When the equality holds, we obtain $j_0 \geq 4$,

$$v \Rightarrow u_j \quad \text{for every} \quad 1 \leq j \leq j_0$$

and

$$u_j \Rightarrow u \quad \text{for every} \quad j_0 - 3 \leq j \leq l_i.$$

(1) If $V(Q_i) \Rightarrow u$, then $V(Q_i) \Rightarrow V(C_1)$. So $u_{j_0}u_{j_0+1} \ldots u_{l_i}v_1v_2 \ldots v_{q_i}$ can be inserted into C_1 and the other vertices of C_i contains a cycle $u_1u_2 \ldots u_{j_0-1}u_1$. It is a contradiction.

(2) If $v \Rightarrow V(Q_i)$, then $V(C_1) \Rightarrow V(Q_i)$. Similarly, $v_1v_2 \ldots v_{q_i}u_1u_2 \ldots u_{j_0-3}$ can be inserted into C_1 and the other vertices of C_i contains a cycle $u_{j_0-2}u_{j_0-1} \ldots u_{l_i}u_{j_0-2}$. It is a contradiction.

(3) By Claim 2, if there is a integer m such that

$$v_j \Rightarrow V(C_1) \quad \text{for every} \quad 1 \leq j \leq m$$

and

$$V(C_1) \Rightarrow v_j \quad \text{for every} \quad m+1 \leq j \leq q_i,$$

where $1 \leq m \leq q_i - 1$. We have $v_1 \Rightarrow V(C_1)$ and $V(C_1) \Rightarrow v_{q_i}$. As $q_i \geq 3$, $m \geq 2$ or $q_i - m \geq 2$. Without loss of generality, we assume $m \geq 2$.

If $v_m \Rightarrow u_{j_0-1}$, then $v_mu_{j_0-1}u_{j_0} \ldots u_{l_i}v_1v_2 \ldots v_m$ is a cycle and

$$v_{m+1}v_{m+2} \ldots v_{q_i}u_1u_2 \ldots u_{j_0-2}$$

can be inserted into C_1. It is a contradiction.

If $u_{j_0-1} \Rightarrow v_m$, then $v_mv_{m+1} \ldots v_{q_i}u_1u_2 \ldots u_{j_0-1}v_m$ is a cycle and

$$u_{j_0}u_{j_0+1} \ldots u_{l_i}v_1v_2 \ldots v_{m-1}$$

can be inserted into C_1. It is a contradiction.

(4) If $V(C_1) \Rightarrow v_1$ and $Q_i - v_1 \Rightarrow V(C_1)$. Then $u_{j_0}u_{j_0+1} \ldots u_{l_i}v_1v_2 \ldots v_{q_i}$ can be inserted into C_1, $u_1u_2 \ldots u_{j_0-1}u_1$ is a cycle. It is a contradiction.

(5) If $V(C_1) \Rightarrow Q_i - v_{q_i}$ and $q_i \Rightarrow V(C_1)$. Similarly, we have $u_{j_0}u_{j_0+1} \ldots u_{l_i}v_1 v_2 \ldots v_{q_i}$ can be inserted into C_1, $u_1u_2 \ldots u_{j_0-1}u_1$ is a cycle. It is a contradiction.

By these contradictions, we have $d^-_{L_i}(u) + d^+_{L_i}(v) \leq l_i + 3$. □

From $d^-_{Q_i}(u) + d^+_{Q_i}(v) = |V(Q_i)| = q_i$ and $d^-_{L_i}(u) + d^+_{L_i}(v) \leq l_i + 3$, we have $d^-_{C_i}(u) + d^+_{C_i}(v) \leq n_i + 3$ whenever $C_i \in \mathcal{F}$.

Case 2. $C_i \in \mathcal{H}$.

Define $R_i = \{w \in V(C_i) \mid w \Rightarrow u \text{ and } v \Rightarrow w\}$. Clearly $|R_i| \leq |V(C_i)| = n_i$.

We will prove $d^-_{C_i}(u) + d^+_{C_i}(v) \leq n_i + 3$ whenever $C_i \in \mathcal{H}$. Clearly, $d^-_{C_i}(u) + d^+_{C_i}(v) \leq n_i + |R_i|$, so we will only prove $|R_i| \leq 3$.

(1) If $n_i = 3$. It is evident that $|R_i| \leq 3$ since $|R_i| \leq n_1 \leq 3$.

(2) If $n_i = 4$. As C_i contains a directed triangle. Let v_0 be a vertex of C_i and not belong to the triangle. So v_0 is a non-cut vertex of C_i. Clearly, if $v_0 \in R_i$, then $v_0 \Rightarrow u, v \Rightarrow v_0$, v_0 can be inserted into C_i. It is a contradiction. Thus $|R_i| \leq 3$.

(3) If $n_i = 5$. Let $w_1w_2w_3w_4w_5w_1$ be a directed hamiltonian cycle of C_i. If $|R_i| \geq 4$, without loss of generality, we can assume $\{w_1, w_2, w_3, w_4\} \subseteq R_i$.
If $w_2 \Rightarrow w_5$, then $w_1w_2w_5w_1$ is a cycle of C_i. Clearly, the segment $C_i[w_3, w_4]$ can be inserted into C_1. It is a contradiction.
If $w_5 \Rightarrow w_2$, then $w_2w_3w_4w_5w_2$ is a cycle of C_i. Clearly, w_1 can be inserted into C_1. It is a contradiction.
Thus $|R_i| \leq 3$.

(4) If $< V(C_i) > \cong T_0$. Clearly, T_0 is 2-connected. All vertices of T_0 are non-cut vertex and $R_i = \emptyset$. So $|R_i| \leq 3$.

By Case 1 and Case 2, we have
$d^-_{C_i}(u) + d^+_{C_i}(v) \leq n_i + 3$ whenever $2 \leq i \leq k-1$. Thus

$$
\begin{aligned}
\delta^- + \delta^+ &\leq d^-_T(u) + d^+_T(v) \\
&= \sum_{i=1}^{k-1} \left(d^-_{C_i}(u) + d^+_{C_i}(v)\right) \\
&\leq 4 + \sum_{i=2}^{k-1} \left(d^-_{C_i}(u) + d^+_{C_i}(v)\right) \\
&\leq 4 + \sum_{i=2}^{k-1} (n_i + 3) \\
&= 4 + (n - n_1) + 3(k-2) \\
&\leq 3k - 2 + n - \tfrac{n}{k-1} \\
&= \tfrac{k-2}{k-1} n + 3k - 2.
\end{aligned}
$$

These contradict the hypothesis of Theorem 6 and complete the proof. □

Proof of Theorem 8. T can be partitioned into k cycles, say so C_1, C_2, $\ldots$, C_k, where $k \geq 2$. We need only to prove that T can be partitioned into $k-1$ cycles, i.e., C_1, C_2, $\ldots$, C_k can be combined into $k-1$ vertex-disjoint cycles.

To the contrary, for two arbitrary vertex-disjoint cycles C_i and C_j, as C_i and C_j can not be combined into one cycle, we have $V(C_i) \Rightarrow V(C_j)$ or $V(C_j) \Rightarrow V(C_i)$.

Define

$$\Im_1^{\text{in}} = \{C_i \mid V(C_i) \Rightarrow V(C_1)\}$$

and

$$\Im_1^{\text{out}} = \{C_j \mid V(C_1) \Rightarrow V(C_j)\}.$$

So

$$\Big(\bigcup_{C_i \in \Im_1^{\text{in}}} V(C_i)\Big) \bigcup \Big(\bigcup_{C_j \in \Im_1^{\text{out}}} V(C_j)\Big) \bigcup V(C_1) = V(T).$$

CLAIM 6. *For an arbitrary cycle $C_i \in \Im_1^{in}$ and an arbitrary cycle $C_j \in \Im_1^{out}$, then $V(C_i) \Rightarrow V(C_j)$.*

Proof of Claim 6. To the contrary, thus $V(C_i) \Rightarrow V(C_1) \Rightarrow V(C_j) \Rightarrow V(C_i)$. We assume $x \in V(C_i)$, $y \in V(C_1)$ and $z \in V(C_j)$. So $xyzx$ and

$$x^+C_i[x^+, x^-]x^-y^+C_1[y^+, y^-]y^-z^+C_j[z^+, z^-]x^+$$

are two vertex-disjoint cycles. Thus we can combine C_i, C_1 and C_j into two vertex-disjoint cycles. With the other $k-3$ unused cycles, T can be partitioned into $k-1$ cycles, a contradiction. □

Without loss of generality, we may assume $C_2 \in \Im_1^{\text{in}}$. We define

$$\Im_2^{\text{in}} = \{C_i \mid V(C_i) \Rightarrow V(C_2), C_i \in \Im_1^{\text{in}}\}$$

and

$$\Im_2^{\text{out}} = \{C_j \mid V(C_2) \Rightarrow V(C_j), C_j \in \Im_1^{\text{in}}\}.$$

Similarly, we know $V(C_i) \Rightarrow V(C_2) \Rightarrow V(C_j)$ and $V(C_i) \Rightarrow V(C_j)$ for an arbitrary cycle $C_i \in \Im_2^{\text{in}}$ and an arbitrary cycle $C_j \in \Im_2^{\text{out}}$.

Repeating this course, we can order all of cycles as

$$V(C_{i_1}) \Rightarrow V(C_{i_2}) \Rightarrow \ldots \Rightarrow V(C_{i_k})$$

and $V(C_{i_j}) \Rightarrow V(C_{i_l})$ whenever $1 \leq j < l \leq k$. So T has k strong components. It contradicts that T is strong. This completes the proof. □

Acknowledgments

We would like to thank one referee for many useful suggestions, which have considerably improved the presentation of the article.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan, London and New York, 1976.

[2] Guantao Chen, Ronald J. Gould and Hao Li, Partitioning vertices of a tournament into independent cycles, *J. Combin. Theory (B)* **83** (2001), 213-220.

[3] Hao Li and Jinlong Shu, The partition of a strong tournament, Preprint, 2002.

[4] K. B. Reid, Two complementary circuits in two-connected tournaments. In "Cycles in Graphs". B.R. Alspach and C.D. Godsil, Eds. *Annal of Discrete Math., Vol.* **27** (1985), 321-334.