PERMUTATIONS CONTAINING A PATTERN EXACTLY ONCE AND AVOIDING AT LEAST TWO PATTERNS OF THREE LETTERS

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ABSTRACT

In this paper, we find an explicit formulas, or recurrences, in terms of generating functions for the cardinalities of the sets $S_n(T;\tau)$ of all permutations in S_n that contain $\tau \in S_k$ exactly once and avoid a subset $T \subseteq S_3$ where $|T| \geq 2$.

1. INTRODUCTION

Let $[p] = \{1, \dots, p\}$ denote a totally ordered alphabet on p letters, and let $\alpha = (\alpha_1, \dots, \alpha_m) \in [p_1]^m$, $\beta = (\beta_1, \dots, \beta_m) \in [p_2]^m$. We say that α is *order-isomorphic* to β if for all $1 \leq i < j \leq m$ one has $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$. For two permutations $\pi \in S_n$ and $\tau \in S_k$, an *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\pi_{i_1}, \dots, \pi_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called the *pattern*. We say that π *avoids* (respectively, *contains* τ *exactly once*), if there is no occurrence of τ in π (respectively, if there is exactly one occurrence of τ in π). If π avoids τ then we shall often say that π is τ*-avoiding*. Pattern avoidance proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [Knu, Chapter 2.2.1] to singularities of Schubert varieties [LS]. A natural generalization of single pattern avoidance is *subset avoidance*; that is, we say that $\pi \in S_n$ avoids a subset $T \subset S_k$ if π avoids any $\tau \in T$. We denote The set of all permutations in S_n that avoid a set of patterns T by $S_n(T)$, and we denote the set of all permutations in $S_n(T)$ which contain τ exactly once by $S_n(T;\tau)$.

Two sets, T_1, T_2, are said to be *Wilf equivalent* (or to belong to the same *Wilf class*) if and only if $|S_n(T_1)| = |S_n(T_2)|$ for any $n \geq 0$. Furthermore,

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two pairs $(T_1;\tau^1)$ and $(T_2;\tau^2)$ are said to belong to the same *almost Wilf class* if and only if $|S_n(T_1;\tau^1)| = |S_n(T_2;\tau^2)|$ for any $n \geq 0$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns τ_1, τ_2. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SS]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]), and for $\tau_1, \tau_2 \in S_4$ (see [B, Kre] and references therein). Several recent papers [CW, MV1, Kra, MV2, MV3, MV3] deal with the case $\tau_1 \in S_3$, $\tau_2 \in S_k$ for various pairs τ_1, τ_2. Another natural question is to study permutations avoiding τ_1 and containing τ_2 exactly t times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $t = 1$ was investigated in [R], and for certain $\tau_1 \in S_3$, $\tau_2 \in S_k$ in [RWZ, MV1, Kra, MV2, MV3, MV3]. The tools involved in these papers include continued fractions, Chebyshev polynomials, and Dyck paths.

In the present paper, we find explicit formulas, or recurrences, for generating functions for the cardinalities of the sets $S_n(T;\tau)$ where $\tau \in S_k$ and $T \subseteq S_3$ together with $|T| \geq 2$. In particularly, we give a complete answer for the almost Wilf classes of $(T;\tau)$ where $\tau \in S_k$ and $T \subseteq S_3$ together with $|T| \geq 2$. Throughout the paper, we often make use of the following remark.

Remark 1.1. *In* [W] *observed that if* $\pi \in S_k$ *contains* $\tau \in T$, *then* $S_n(T,\pi) = S_n(T)$. *Besides,* $|S_n(T;\pi)| = 0$ *for all* $n < k$; *On the other hand, by* [ES] *we obtain that* $|S_n(T;\tau)| = 0$ *for all* $n \geq 5$ *where* $\{123, 321\} \subseteq T$.

Because of Remark 1.1, from now on suppose that π is T-avoiding, and $\{123, 321\} \not\subseteq T$.

The main body of the paper is divided into three sections corresponding to the cases $|T| = 2, 3$ and $|T| \geq 4$.

2. A pair

In this section we present, by explicit formulas or recurrences for generating functions, the cardinalities of the sets $S_n(\{\beta,\gamma\};\tau)$ where $\beta, \gamma \in S_3$, $\tau \in S_k$, $k \geq 3$. By the three natural operations, the complement, the reversal, the inverse (see Simion and Schmidt [SS, Lemma 1]), and Remark 1.1 we have to consider the following four possibilities:

$$\begin{array}{llll}
1) & S_n(\{132,123\};\tau), & \text{where} & \tau \in S_k(\{132,123\}),\\
2) & S_n(\{132,321\};\tau), & \text{where} & \tau \in S_k(\{132,321\}),\\
3) & S_n(\{132,213\};\tau), & \text{where} & \tau \in S_k(\{132,213\}),\\
4) & S_n(\{132,231\};\tau), & \text{where} & \tau \in S_k(\{132,231\}).
\end{array}$$

The main body of this section is divided into four subsections corresponding to the above four cases.

2.1. $\mathbf{T} = \{\mathbf{132}, \mathbf{123}\}$. We say that $\pi \in S_n$ is 123-*type permutation* if it avoids 132 and avoids 123. Let π be a 123-type permutation in S_n; so (see [M, Theorem 3(i)]) π can be presented as

$$\pi = (n-1, n-2, \dots, n-r+1, n, \pi'),$$

where $r \geq 1$ maximal and $\pi' \in S_{n-r}$.

Let $a_\tau(n)$ denote the number of 123-type permutations in S_n contain τ exactly once; that is, $a_\tau(n) = |S_n(\{132, 123\}; \tau)|$. Let $A_\tau(x) = \sum_{n\geq 0} a_\tau(n)x^n$ be the corresponding ordinary generating function.

Theorem 2.1. *Let $k \geq 2$.*

(i) *Let $\tau = (k-1, k-2, \dots, k-r+1, k, \tau')$ be a 123-type pattern in S_k such that $r \geq 2$; then*

$$A_{(k-1,k-2,\dots,k-r+1,k,\tau')}(x) = \begin{cases} \dfrac{x^r(1-x)}{1-2x+x^r} A_{\tau'}(x) & ;\tau' \neq \emptyset \\ \dfrac{x^r(1-x)}{1-2x+x^r} F_\tau(x) & ;\tau' = \emptyset, \end{cases}$$

where $F_\tau(x)$ is the generating function for the number of τ-avoiding 123-type permutations in S_n.

(ii) *Let $\tau = (k, k-1, \dots, k-m+1, \tau')$ such that $m \geq 1$ maximal; then*

$$A_\tau(x) = \begin{cases} xA_{(k-1,\dots,k-m+1,\tau')}(x) + \sum\limits_{j=2}^{m} x^{j+1} A_{(k-j,\dots,k-m+1,\tau')}(x) & ;m \geq 2 \\ (x - x^2 - x^3 - \cdots - x^{k-1}) A_{\tau'}(x) & ;m = 1 \end{cases}$$

together with $A_{(1)}(x) = x$.

Proof. Let τ be a 123-type pattern in S_k; so there exists r maximal such that $\tau = (k-1, k-2, \dots, k-r+1, k, \tau')$. Let α be a 123-type permutation in S_n; so there exists t maximal such that $\alpha = (n-1, \dots, n-t+1, n, \alpha')$.

(1) Let $r \geq 2$ and $\tau' \neq \emptyset$; so for any $n \geq k$,

$$a_\tau(n) = \sum_{t=1}^{r-1} a_\tau(n-t) + a_{\tau'}(n-r).$$

(2) Let $r \geq 2$ and $\tau' = \emptyset$; so for any $n \geq k$,

$$a_\tau(n) = \sum_{t=1}^{r-1} a_\tau(n-t) + a'_\tau(n-r),$$

where $f_\tau(n-r)$ is the number of τ-avoiding 123-type permutations in S_n.

(3) For $r = 1$, by use of [M, Theorem 3(i)] there exist m maximal such that $\tau = (k, k-1, \dots, k-m+1, \tau')$. Thus,

$$a_\tau(n) = a_{(k-1,k-2,\dots,k-m+1,\tau')}(n-1) + \\ + \sum_{j=2}^{m} a_{(k-j,k-j-1\dots,k-m+1,\tau')}(n-1-j),$$

for all $m \geq 2$, and

$$a_\tau(n) = a_{\tau'}(n-1) - \sum_{j=2}^{k-1} a_{\tau'}(n-j),$$

where $m = 1$.

Besides, in the above cases we have $a_\tau(n) = 0$ for all $n \leq k-1$ and $a_\tau(k) = 1$. Hence, if we convert the above recurrences into equations for generating functions, we obtain the claimed results. □

Corollary 2.2. *Let $r_i \geq 2$ such that $r_1 + \dots + r_m = k$, and let $\tau = (p_1, p_2, \dots, p_m) \in S_k$ where $p_i = (t_i - 1, t_i - 2, \dots, t_i - r_i + 1, t_i)$ together with $t_i = k - (r_1 + \dots + r_{i-1})$ for all $i = 1, 2, \dots, m$. Then*

$$A_\tau(x) = \frac{1-x}{1-2x+x^{r_m}} \prod_{i=1}^{m} \frac{x^{r_i}(1-x)}{1-2x+x^{r_i}}.$$

Proof. By induction and Theorem 2.1(i) we have that

$$A_\tau(x) = F_{p_m}(x) \prod_{i=1}^{m} \frac{x^{r_i}(1-x)}{1-2x+x^{r_i}},$$

where $F_\tau(x)$ is the generating function for the τ-avoiding 123-type permutations in S_n. The rest is easy to see by use of [M, Theorem 3(ii)]. □

Example 2.3. *For $k = 3$, Theorem 2.1 and* [M, Theorem 3(ii)] *yield* $A_{213}(x) = \frac{x^3}{(1-x-x^2)^2}$, $A_{231}(x) = A_{312}(x) = \frac{x^3}{1-x}$, *and* $A_{321}(x) = x^3 + 3x^4$.

2.2. **T = {132, 321}.** We say that $\pi \in S_n$ is 321-*type permutation* if it avoids 132 and avoids 321. Let π be a 321-type permutation in S_n; so (see [M, Theorem 6(i)]) π can be presented as

$$\pi = (d+1, d+2, \dots, m-1, 1, 2, \dots, d, m, m+1, \dots, n),$$

where $2 \leq m \leq n+1$ and $1 \leq d \leq m-2$.

Let $b_\tau(n)$ denote the number of 321-type permutations in S_n which contain τ exactly once; that is, $b_\tau(n) = |S_n(\{132, 321\}; \tau)|$. Let $B_\tau(x) = \sum_{n\geq 0} b_\tau(n)x^n$ be the corresponding ordinary generating function.

Theorem 2.4. *Let $k \geq 1$. Then*

(i) $B_{(1,2,\ldots,k)}(x) = x^k + 2\sum_{j=k+1}^{2k-1}(2k-j)x^j$;

(ii) $B_{(d+1,d+2,\ldots,k,1,2,\ldots,d)}(x) = \frac{x^k}{1-x}$, *for all* $1 \leq d \leq k-1$;

(iii) $B_{(d+1,d+2,\ldots,m-1,1,2,\ldots,d,m,m+1,\ldots,k)}(x) = x^k$, *for all* $1 \leq d \leq m-2 \leq k-2$.

Proof. Let τ be a 321-type pattern in S_k; so there exist m, $2 \leq m \leq k+1$, and d, $1 \leq d \leq m-2$ such that

$$\tau = (d+1, d+2, \ldots, m-1, 1, 2, \ldots, d, m, m+1, \ldots, k).$$

Let α be a 321-type permutation in S_n; so there exist r, $2 \leq r \leq n+1$, and t, $1 \leq t \leq r-2$ such that

$$\alpha = (t+1, t+2, \ldots, r-1, 1, 2, \ldots, t, r, r+1, \ldots, n).$$

Hence, the theorem holds by checking over all the possibilities of α contains τ exactly once. □

Example 2.5. *Theorem 2.4 yields* $B_{123}(x) = x^3+4x^4+2x^5$, $B_{213}(x) = x^3$, *and* $B_{231}(x) = B_{312}(x) = \frac{x^3}{1-x}$.

2.3. **T = {132, 213}.** We say that $\pi \in S_n$ is 213-*type permutation* if it avoids 132 and avoids 213. Let π be a 213-type permutation in S_n; so (see [M, Theorem 8(i)]) π can be presented as

$$\pi = (r_1, r_1+1, \ldots, k, r_2, r_2+1, \ldots, r_1-1, \ldots, r_m, r_m+1, \ldots, r_{m-1}-1);$$

where $n+1 = r_0 > r_1 > \cdots > r_m = 1$.

Let $c_\tau(n)$ denote the number of 213-type permutations in S_n which contain τ exactly once; that is, $c_\tau(n) = |S_n(\{132, 213\}; \tau)|$. Let $C_\tau(x) = \sum_{n \geq 0} c_\tau(n)x^n$ be the corresponding ordinary generating function.

Theorem 2.6. *Let τ be a 213-type pattern in S_k. Then, for all* $0 \leq r \leq k-1$,

$$C_{(r+1,\ldots,k,\tau')}(x) = \frac{x^{k-r}(1-x)}{1-2x+x^{k-r}} C_{\tau'}(x),$$

where $\tau' \neq \emptyset$, *and*

$$C_{(1,2,\ldots,k)}(x) = \frac{x^k(1-x)^2}{(1-2x+x^k)^2}.$$

Proof. Let $\tau = (r+1, r+2, \ldots, k, \tau')$, and let α be a 213-type permutation in S_n which contain τ exactly once. So there exist $n+1 = t_0 > t_1 > \cdots > t_m \geq 1$ such that

$$\alpha = (t_1, t_1+1, \ldots, t_0-1, t_2, t_2+1, \ldots, t_1-1, \ldots, t_m, t_m+1, \ldots, t_{m-1}-1),$$

therefore for any $\tau' \neq \emptyset$ we get

$$c_\tau(n) = \sum_{j=n-k+r_1+1}^{n} c_\tau(j-1) + c_{\tau'}(n-k+r_1-1).$$

If $\tau' = \emptyset$, which means that $\tau = (1, 2, \ldots, k)$, then we get

$$c_\tau(n) = \sum_{j=n-k+2}^{n} c_\tau(j-1) + c'_\tau(n),$$

where $c'_\tau(n)$ is the number of τ-avoiding 213-type permutations in S_n. Besides, $c_\tau(k) = 1$ and $c_\tau(n) = 0$ for all $n \leq k-1$. Hence, if we convert the above recurrences into equations for generating functions together with use of [M, Theorem 8(ii)], we obtain the claimed results. □

Corollary 2.7. *Let $k \geq 1$; then $C_{(k,k-1,\ldots,1)}(x) = x^k$.*

Example 2.8. *Theorem 2.6 yields $C_{123}(x) = \frac{x^3}{(1-x-x^2)^2}$, $C_{321}(x) = x^3$, and $C_{231}(x) = C_{312}(x) = \frac{x^3}{1-x}$.*

2.4. $\mathbf{T = \{132, 231\}}$. We say that $\pi \in S_n$ is 231-*type permutation* if it avoids 132 and avoids 231. Using [M, Theorem 11] we get that π is a 231-type permutation in S_n if and only if every element of π is either a right maximum or a right minimum; namely π can be presented either $\pi = (n, n-1, \ldots, n-r+1, \pi', n-r)$ or $\pi = (n-r, \pi', n-r+1, n-r+2, \ldots, n)$, where $1 \leq r \leq n-1$.

Let $d_\tau(n)$ denote the number of 231-type permutations in S_n which contain τ exactly once; that is $d_\tau(n) = |S_n(\{132, 231\}; \tau)|$. Let $D_\tau(x) = \sum_{n \geq 0} d_\tau(n) x^n$ be the corresponding ordinary generating function.

Theorem 2.9. *Let τ be a 231-type pattern in S_k. Then*

$$D_{(1,2,\ldots,k)}(x) = D_{(k,\ldots,2,1)}(x) = \frac{x^k}{(1-x)^{k-1}},$$

and

$$D_\tau(x) = \frac{x^{r+1}}{(1-x)^r} D_{\tau'}(x),$$

where either $\tau = (k, k-1, \ldots, k-r+1, \tau', k-r)$ or $\tau = (k-r, \tau', k-r+1, k-r+2, \ldots, k)$ together with $1 \leq r \leq k-1$ and $\tau' \neq \emptyset$.

Proof. Let α be a 231-type permutation in S_n; so there exists α' such that either $\alpha = (n, \ldots, n-t+1, \alpha', n-t)$ or $\alpha = (n-t, \alpha', n-t+1, n-t+2, \ldots, n)$, where $1 \leq t \leq n-1$. Therefore, for $0 \leq m \leq r-1$,

$$d_{(k-m,\ldots,k-r+1,\tau',k-r)}(n) = d_{(k-m,\ldots,k-r+1,\tau',k-r)}(n-1) \\ +d_{(k-1-m,\ldots,k-r+1,\tau',k-r)}(n-1),$$

and

$$d_{(k-r,\tau',k-r+1,\ldots,k-m)}(n) = d_{(k-r,\tau',k-r+1,\ldots,k-m)}(n-1) \\ +d_{(k-r,\tau',k-r+1,\ldots,k-1-m)}(n-1).$$

Besides $d_\tau(n) = 0$ for all $n \leq k-1$ and $d_\tau(k) = 1$. Hence, by convert the above recurrences into equations for generating functions we get

$$D_\tau(x) = \frac{x^{r+1}}{(1-x)^r} D_{\tau'}(x).$$

The rest is obtain immediately by the above recurrence. □

Let us denote the sequence $k \ldots (r+2)(r+1)$ by $\langle k, r \rangle$.

Corollary 2.10. *Let $k \geq 1$, and let*

$$\tau = (\langle k, r_1 \rangle, \langle r_1 - 1, r_2 \rangle, \ldots, \langle r_{m-1} - 1, r_m \rangle, r_{m-1}, r_{m-2}, \ldots, r_1)$$

be any 231-type pattern in S_k. Then

$$D_\tau(x) = \frac{x^k}{(1-x)^{k-m}}.$$

Proof. By Theorem 2.9 we get

$$D_\tau(x) = \prod_{i=1}^{m-1} \frac{x^{r_{i-1}-r_i} + 1}{(1-x)^{r_{i-1}-r_i}} D_{(r_m,\ldots,2,1)}(x),$$

equivalently,

$$D_\tau(x) = \prod_{i=1}^{m-1} \frac{x^{r_{i-1}-r_i} + 1}{(1-x)^{r_{i-1}-r_i}} \cdot \frac{x^{r_m}}{(1-x)^{r_m-1}}.$$

□

Example 2.11. *Theorem 2.9 yields, $D_{123}(x) = D_{321}(x) = \frac{x^3}{(1-x)^2}$, and $D_{213}(x) = D_{312}(x) = \frac{x^3}{1-x}$.*

3. A TRIPLET

In this section we present, by explicit formulas or recurrences for generating functions, the cardinalities of the sets $S_n(T;\tau)$ where $T \subset S_3$, $|T| = 3$, and $\tau \in S_k(T)$ for $k \geq 3$. By the three natural operations the complement, the reversal, the inverse (see Simion and Schmidt [SS, Lemma 1]), and Remark 1.1 we have to consider the following four possibilities:

1) $S_n(\{123,132,213\};\tau)$, where $\tau \in S_k(\{123,132,213\})$,
2) $S_n(\{123,132,231\};\tau)$, where $\tau \in S_k(\{123,132,231\})$,
3) $S_n(\{123,231,312\};\tau)$, where $\tau \in S_k(\{123,231,312\})$,
4) $S_n(\{132,213,231\};\tau)$, where $\tau \in S_k(\{132,213,231\})$.

The main body of this section is divided into four subsections corresponding to the above four cases.

3.1. T = {123, 132, 213}. Let $e_\tau(n) = |S_n(\{123,132,213\};\tau)|$, and let $E_\tau(x) = \sum_{n\geq 0} e_\tau(n)x^n$ be the corresponding ordinary generating function.

Let $\pi \in S_n(\{123,132,213\})$; by [M, Lemma 14] we can present π as either $\pi = (n-1,n,\pi')$ where $\pi' \in S_{n-2}(\{123,132,213\})$, or $\pi = (n,\pi')$ where $\pi' \in S_{n-1}(\{123,132,213\}$. Using this fact we get

Theorem 3.1. *Let $k \geq 4$, $\tau \in S_k(\{123,132,213\})$. Then*

$$E_{(k-1,k,\tau')}(x) = \frac{x^2}{1-x}E_{\tau'}(x) \quad \textit{and} \quad E_{(k,\tau')}(x) = xE_{\tau'}(x).$$

Besides, $E_\tau(x)$ is given by x^4, $\frac{x^3}{1-x}$, $\frac{x^3}{1-x}$, x^3, x^2, $\frac{x^2}{(1-x)^2}$, x, where $\tau =$ 4231, 231, 312, 321, 21, 12, 1; respectively.

Proof. Let $\alpha \in S_n(\{123,132,213\};\tau)$; so π can be presented as either $\alpha = (n-1,n,\alpha')$ or $\alpha = (n,\alpha')$.

If $\tau = (k-1,k,\tau')$, then $e_\tau(n) = e_\tau(n-1) + e_{\tau'}(n-2)$ for all $n \geq k$. If $\tau = (k,\tau') \neq 4231$, then $e_\tau(n) = e_{\tau'}(n-1)$ for qll $n \geq l$. Hence, if we convert the above recurrences into equations for generating functions we obtain the claimed recurrences. The rest is easy to check. □

Example 3.2. *Theorem 3.1 yields for all $k \geq 4$,*

$$E_{(k,\dots,2,1)}(x) = E_{(k,\dots,4,2,3,1)}(x) = x^k.$$

Another example, for $k \geq 2$, is

$$E_{(k-1,k,k-3,k-2,\dots,1,2)}(x) = \begin{cases} \frac{x^k}{(1-x)^{k/2+1}}, & k = 2,4,6,\dots \\ \frac{x^k}{(1-x)^{(k-1)/2}}, & k = 3,5,7,\dots \end{cases}.$$

3.2. $\mathbf{T} = \{\mathbf{123},\mathbf{132},\mathbf{231}\}$. Let $f_\tau(n) = |S_n(\{123,132,231\};\tau)|$, and let $F_\tau(x) = \sum_{n\geq 0} f_\tau(n)x^n$ be the corresponding ordinary generating function.

Let $\pi \in S_n(\{123,132,231\})$; by [M, Theorem 17] we can present π as $\pi = (n, n-1,\dots,n-r+1,n-r-1,\dots,1,n-r)$, where $1 \leq r \leq n$. Using this fact we get

Theorem 3.3. *Let $k \geq 3$, and let $k-2 \geq r \geq 1$; then*

$$F_{(k,\dots,2,1)}(x) = x^k + (k-1)x^{k+1};$$
$$F_{(k-1,\dots,2,1,k)}(x) = \tfrac{x^k}{1-x};$$
$$F_{(k,\dots,k-r+1,k-r-1,\dots,1,k-r)}(x) = x^k.$$

3.3. $\mathbf{T} = \{\mathbf{123},\mathbf{231},\mathbf{312}\}$. Let $g_\tau(n) = |S_n(\{123,231,312\};\tau)|$, and let $G_\tau(x) = \sum_{n\geq 0} g_\tau(n)x^n$ be the corresponding ordinary generating function.

Let $\pi \in S_n(\{123,231,312\})$; by [M, Theorem 21] we can present π as $\pi = (r, r-1,\dots,1,n,n-1,\dots,r+1)$, where $1 \leq r \leq n$. Using this fact we get

Theorem 3.4. *Let $k \geq 3$, and let $k-1 \geq r \geq 1$; then*

$$G_{(k,\dots,2,1)}(x) = \tfrac{x^k(1+x)}{1-x};$$
$$G_{(r,r-1,\dots,1,k,k-1,\dots,r+1)}(x) = x^k.$$

3.4. $\mathbf{T} = \{\mathbf{132},\mathbf{213},\mathbf{231}\}$. Let $h_\tau(n) = |S_n(\{132,213,231\};\tau)|$, and let $H_\tau(x) = \sum_{n\geq 0} h_\tau(n)x^n$ be the ordinary generating function.

Let $\pi \in S_k(\{132,213,231\})$; by [M, Theorem 23] we can present π as $\tau = (n, n-1,\dots,r+1,1,2,\dots,r)$, where $1 \leq r \leq n$. Using this fact we get

Theorem 3.5. *Let $k \geq 3$, and let $k-1 \geq r \geq 1$; then*

$$H_{(1,2,\dots,k)}(x) = \tfrac{x^k}{1-x};$$
$$H_{(k,k-1,\dots,r+1,1,2,\dots,r)}(x) = x^k.$$

4. A quartet and a quintet

By Simion and Schmidt [SS, Proposition 17] we have that $|S_n(T)| = 0$ for all $T \subseteq S_3$ such that $\{123,321\} \subseteq T$, and $|S_n(T)| = 2,1$ for all $\{123,321\} \not\subseteq T \subseteq S_3$ such that $|T| = 4,5$. These facts yield the following theorem.

Theorem 4.1. *Let $\tau \in S_k$. Then*

(i) $|S_n(\{123,132,213,231\};\tau)| = \begin{cases} 1, & n=k,\ \tau=(k,\dots,3,2,1),(k,\dots,3,1,2) \\ 0, & \textit{otherwise} \end{cases}$;

(ii) $|S_n(\{123,132,231,312\};\tau)| = \begin{cases} 1, & n=k,k+1,\ \tau=(k,\dots,2,1) \\ 1, & n=k,\ \tau=(k-1,\dots,2,1,k) \\ 0, & \textit{otherwise} \end{cases}$;

(iii) $|S_n(\{132,213,231,312\};\tau)| = \begin{cases} 1, & n=k,\ \tau=(k,\dots,2,1),(1,2,\dots,k) \\ 0, & otherwise \end{cases}$

(iv) $|S_n(S_3\backslash\{123\};\tau)| = \delta_{n,k}\delta_{\tau,(1,2,\dots,k)}$;

(v) $|S_n(S_3;\tau)| = 0$.

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