Note on acyclic colorings of graphs

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Abstract

A vertex k-coloring of a graph G is acyclic if no cycle is bichromatic. The minimum integer k such that G admits an acyclic k-coloring is called the acyclic chromatic number of G, denoted by $\chi_a(G)$. In this paper, we discuss some properties of maximal acyclic k-colorable graphs, prove a sharp lower bound of the $\chi_a(G)$ and get some results about the relation between $\chi(G)$ and $\chi_a(G)$. Furthermore, a conjecture of B. Grünbaum that $\chi_a(G) \leq \Delta+1$ is proved for maximal acyclic k-colorable graphs.

1 Introduction

In this paper, we consider only finite undirected simple graphs. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. We use $\delta(G)$ for the minimum degree of G and $\Delta(G)$ for the maximum degree. We denote the connectivity of G by $\kappa(G)$. As usual, we use $\chi(G)$ for the chromatic number of G. The concept of acyclic coloring of graphs, introduced by Grünbaum[6], is a generalization of vertex-arboricity. An acyclic coloring of G is a proper coloring of its vertices such that there is no two-colored cycle. The acyclic chromatic number of G, denoted by $\chi_a(G)$, is the minimum number of colors for an acyclic coloring of G.

A graph G is called maximal acyclic k-colorable if $\chi_a(G) = k$ and for any $e \in \bar{G}$, $\chi_a(G \cup \{e\}) > k$, where $\bar{G}$ is the complement of G. For all notation and terminology not defined here, see Bondy & Murty [2].

Grünbaum [6] conjectured $\chi_a(G) \leq \Delta(G) + 1$ for any graph G and proved the conjecture for $\Delta(G) = 3$. Burštein [5] proved the conjecture for arbitrary graphs of degree 4. There are many other results on acyclic coloring especially for planar graphs, see Alon et al. [1], Borodin et al. [3], [4].

2 Maximal acyclic colorable graphs

Theorem 1. Every maximal acyclic k-colorable graph with n vertices has exactly $(k-1)n - \binom{k}{2}$ edges.

Proof. Let G be a maximal acyclic k-colorable graph with n vertices. Then $V(G)$ has a partition into coloring classes $V_1, \cdots, V_k$. By the maximality of $E(G)$, we have that $G_{i,j} = G[V_i \cup V_j]$ is connected for any $1 \leq i < j \leq k$. Since the coloring is acyclic, the induced subgraph $G_{i,j}$ is acyclic and therefore $G_{i,j}$ is a tree. Thus, $|E(G_{i,j})| = |V_i| + |V_j| - 1$. So, $|E(G)| = \sum_{1 \leq i < j \leq k} |E(G_{i,j})| = \sum_{1 \leq i < j \leq k} (|V_i| + |V_j| - 1) = (k-1)|V(G)| - \binom{k}{2}$.

Theorem 2. Every maximal acyclic k-colorable graph is $(k-1)$-connected.
Proof. Let $V_1, \cdots, V_k$ be an acyclic k-coloring of G. For any $S \subset V(G)$ with $|S| \leq k-2$, there exist at least two of $V_1, \cdots, V_k$, say V_i and V_j, such that $|S \cap V_i| = |S \cap V_j| = 0$. Since G is a maximal acyclic k-colorable graph, the induced subgraph $G_{i,j} = G[V_i \cup V_j]$ is a tree and hence $G_{i,j}$ is a connected subgraph. So, for any $v \in V(G) \backslash S$, if $v \notin V_i \cup V_j$, then $v \in V_t$ for some $t \neq i, j$. Because G is a maximal k-acyclic colorable graph, then v is adjacent to some vertex of V_i. Therefore $G \backslash S$ is a connected graph. By the choice of S we know that G is $(k-1)$-connected.

Note. The lower bound of $\kappa(G) \geq k-1$ is sharp, since one can construct as follows a maximal acyclic k-colorable graph G such that $\delta(G) = k-1$ which means $\kappa(G) = k-1$. Let $V_1, V_2, \cdots, V_k$ be pairwise disjoint vertex sets with $|V_1| \geq 2$. For $v \in V_1$, we can construct a maximal acyclic k-colorable graph G with $V_1, V_2, \cdots, V_k$ an acyclic coloring and $d_{G_{1,j}}(v) = 1$ for $2 \leq j \leq k$ where $G_{i,j} = G[V_i \cup V_j]$ for $1 \leq i < j \leq k$. Then $\delta(G) = k-1$.

By Theorem 1, we can easily get the following result.

Corollary 1. Let G be a graph. For any positive integer k, if there exists a subgraph G^* of G such that $|E(G^*)| > (k-1)|V(G^*)| - \binom{k}{2}$, then $\chi_a(G) > k$.

Using the property of maximal acyclic k-colorable graph, we can get the following result.

Theorem 3. Grünbaum's Conjecture is true for maximal acyclic k-colorable graphs.

Proof. Let G be a maximal acyclic k-colorable graph. We use $d_{ave}(G)$ for the average degree of G. By Theorem 1 we have

$$d_{ave}(G)n = \sum_{v\in V(G)} d_G(v) = 2|E(G)| = 2(k-1)n - k(k-1) = (k-1)(2n-k).$$

So $d_{ave}(G) = (k-1)(2-\frac{k}{n}) \geq k-1$ since $k \leq n$, with equality holds iff G is complete. Therefore we have $k \leq d_{ave}(G) + 1 \leq \Delta(G) + 1$.

3 Lower bound for $\chi_a(G)$ and relation with $\chi(G)$.

In general, large $\delta(G)$ does not imply large $\chi(G)$ since a bipartite graph can have large minimum degree and its chromatic number is at most 2, but for acyclic coloring, it's different. We have the following result.

Theorem 4. For any connected graph G with $|V(G)| \geq 2$, $\chi_a(G) \geq f(\delta(G))$, where

$$f(t) = \lfloor \frac{t+4}{2} \rfloor.$$

Proof. Suppose $\chi_a(G) = k$. If G is a maximal acyclic k-colorable graph, then by Theorem 1, $|E(G)| = (k-1)|V(G)| - \binom{k}{2}$. Thus

$$\delta(G)|V(G)| \leq \sum_{v\in V(G)} d_G(v) = 2|E(G)| = 2(k-1)|V(G)| - k(k-1)$$

which implies $k \geq \frac{\delta(G)}{2} + \frac{k(k-1)}{2|V(G)|} + 1$. Since G is a connected graph with $|V(G)| \geq 2$, then $k \geq 2$. Therefore Theorem 4 follows easily for this case. If G is not a maximal acyclic k-colorable graph, then there is a maximal acyclic k-colorable graph G^* with $V(G^*) = V(G)$ such that G is a subgraph of G^*. Similarly to the above discussion, we can get $k \geq \frac{\delta(G^*)}{2} + \frac{k(k-1)}{(2|V(G^*)|} + 1$. Since $\delta(G^*) \geq \delta(G)$, we have $k \geq \frac{\delta(G)}{2} + \frac{k(k-1)}{2|V(G^*)|} + 1$ and Theorem 4 is true.

Note. The lower bound in Theorem 4 is sharp since equality holds for any graph which is a cycle. But we can get a better lower bound for $\chi_a(G)$ as follows:

Corralary 2. Let G be a graph. We define $\delta^*(G) = \text{Max}\{\delta(G')|G'$ is a connected subgraph of G with $|V(G')| \geq 2\}$. Then $\chi_a(G) \geq f(\delta^*(G))$, where $f(t)$ is defined as in Theorem 4.

Clearly, for any graph G, $\chi(G) \leq \chi_a(G)$, so it seems interesting to find when the equality holds. The following theorem gives a sufficient condition for equality.

A graph G is called uniquely k-colorable if G has only one vertex k-coloring up to isomorphism.

Theorem 5. Let G be a uniquely k-colorable graph with at most $(k-1)|V(G)| - \binom{k}{2}$ edges, then $\chi_a(G) = \chi(G)$.

Proof. Let $V_1, \cdots, V_k$ be a normal coloring of G. We claim that the induced subgraph $G_{i,j} = G[V_i \cup V_j]$ is connected for $1 \leq i < j \leq k$. Assume that there exist V_i, V_j such that $G_{i,j}$ is not connected, since $|E(G_{i,j})| \geq 1$, then $G_{i,j}$ has a component C with $|V(C)| \geq 2$, switch colors i and j in C, we get another k-coloring of G, a contradiction. So $|E(G_{i,j})| \geq |V_i|+|V_j|-1$ for any $1 \leq i < j \leq k$ with equality holds only when $G_{i,j}$ is a tree. Then $|E(G)| = \sum_{1\leq i<j\leq k} |E(G_{i,j})| \geq (k-1)n - \binom{k}{2}$, with equality holds only when each $G_{i,j}$ is a tree. But by the condition, $|E(G)| \leq (k-1)|V(G)| - \binom{k}{2}$. So $|E(G)| = (k-1)|V(G)| - \frac{k(k-1)}{2}$ and each $G_{i,j}$ is a tree. So G is a maximal acyclic k-colorable graph and $\chi_a(G) = k = \chi(G)$.

In general, $\chi(G)$ and $\chi_a(G)$ can be very different, there exists graph G with small $\chi(G)$ but $\chi_a(G)$ is arbitrary large. In fact we have the following theorem.

Theorem 6. There's no function $f(x) : Z^+ \longrightarrow Z^+$ such that for any graph G, $\chi_a(G) \leq f(\chi(G))$.

Proof. We prove this by constructing a graph G with the following property:

For any $3 \leq k \leq t$, there exist a graph G with $\chi(G) \leq k$ and $\chi_a(G) = t$.

Let $A_1, \cdots, A_t$ be the pairwise disjoint vertex sets with $A_i = \{a_i^1, a_i^2, \cdots, a_i^k\}$. We first construct a graph G^*. Let $V(G^*) = \bigcup_{i=1}^{t} A_i$ and $E(G^*) = \{a_i^m a_j^n | 1 \leq i < j \leq t, 1 \leq m, n \leq k$ and $m \neq n\}$. Then let $B_j = \bigcup_{1\leq i\leq t} \{a_i^j\}$ for $1 \leq j \leq k$, we can easily see that $B_1, \cdots, B_k$ is a normal coloring of G^*, so $\chi(G^*) \leq k$. Since $|A_i| \geq 3$, so $G^*_{i,j} = G^*[A_i \cup A_j]$

is connected. Then $G^*_{i,j}$ contains a spanning tree $G_{i,j}$ for $1 \leq i < j \leq t$. Let $G = \bigcup_{1 \leq i < j \leq t} G_{i,j}$. Then G is a subgraph of G^* and G is a maximal acyclic t-colorable graph. So $\chi(G) \leq \chi(G^*) \leq k$ and $\chi_a(G) = t$.

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