On computing the dissociation number and the induced matching number of bipartite graphs

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Abstract

The dissociation number of a graph G is the number of vertices in a maximum size induced subgraph of G with vertex degrees at most 1. The problem of finding the dissociation number was introduced by Yannakakis who proved it is NP-hard on the class of bipartite graphs. In this paper, we analyze the dissociation number problem restricted to the class of bipartite graphs in more detail. We strengthen the result of Yannakakis by reducing the problem, in polynomial time, from general bipartite graphs to some particular classes such as bipartite graphs with maximum degree 3 or C_4-free bipartite graphs. Besides the negative results, we prove that finding the dissociation number is polynomially solvable for bipartite graphs containing no induced subgraph isomorphic to a tree with exactly three vertices of degree 1 of distances 1, 2, and 3 from the only vertex of degree 3.

The induced matching number of a graph G is the number of edges in a maximum size induced subgraph of G with vertex degrees all equal to 1. Analogous results hold for the induced matching number.

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1 Introduction

A set of vertices in a graph G is called a *dissociation set* if it induces a subgraph with maximum degree at most 1. The number of vertices in a maximum cardinality dissociation set in G is called the *dissociation number* of G and is denoted $diss(G)$.

The problem of finding a maximum cardinality dissociation set generalizes two other graph problems: maximum stable set and maximum induced matching. A stable set is an induced subgraph with all vertex degrees equal to 0. An induced matching can be thought of as an induced subgraph with all vertex degrees equal to 1. (Note that solving the maximum dissociation set problem does not solve either the maximum stable set problem or the maximum induced matching problem.) Both the maximum stable set problem and the maximum induced matching problem have received considerable attention in the literature. Both are known to be NP-hard in general graphs. However, in the class of bipartite graphs the complexity status of the problems is different. It is well-known that a maximum stable set in a bipartite graph can be found in polynomial time, while the maximum induced matching problem remains NP-hard for bipartite graphs [1, 9]. It is not surprising that the problem of computing $diss(G)$ (the dissociation number problem) is NP-hard on the class of bipartite graphs as well. This was proved by Yannakakis in [10]. In this paper we study the dissociation number problem restricted to bipartite graphs in more detail. Specifically, we strengthen the result of Yannakakis in the following way.

A graph G is called H*-free* if G does not contain graph H as an induced subgraph. We show in Section 2 that if H contains either a cycle or a vertex of degree more than 3 or two vertices of degree 3 in the same connected component, then the dissociation number problem is NP-hard in the class of H-free bipartite graphs. Moreover, it remains NP-hard even in the intersection of finitely many such classes. In particular, the problem is NP-hard in the classes of $K_{1,4}$-free or C_4-free bipartite graphs. Note that $K_{1,4}$-free bipartite graphs are exactly bipartite graphs with maximum degree at most 3. So, the problem is NP-hard in C_4-free bipartite graphs with maximum degree at most 3. Analogous results were proved for the induced matching problem in [7].

These results do more than establish the NP-hardness of the dissociation problem and the induced matching problem in a large family of graph classes. They characterize classes of graphs for which these problems could possibly be solved in polynomial time. Under the assumption that $P \neq NP$, our results imply that the dissociation number problem can be solved in polynomial time in the class of H-free bipartite graphs only if every connected component of H is a graph of the form $S_{i,j,k}$ as in Figure 1, and the same holds for the induced matching number problem.

In [6], the graph $S_{1,2,3}$ was called a skew star. In Section 3 of this paper, we use the structural characterization given in [6] for skew star-free bipartite graphs in order to derive a polynomial-time algorithm to solve the dissociation number problem in that class. In Section 4, we do the same for the induced matching problem.

All graphs we consider are undirected and simple, that is, without loops and multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex-set and

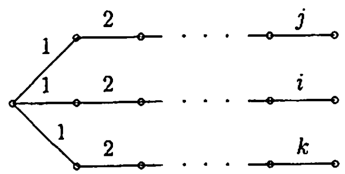

Figure 1: Graph $S_{i,j,k}$ $(i, j, k \geq 0)$

the edge-set of G, respectively. The neighborhood of a vertex $x \in V(G)$ is the set of vertices adjacent to x, and is denoted $N(x)$. The degree of $x \in V(G)$ is the number of vertices in $N(x)$. Given a set of vertices $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U, and $G - U = G[V(G) - U]$.

As usual, P_n and C_n denote, respectively, the chordless path and the chordless cycle with n vertices. $K_{n,m}$ is the complete bipartite graph with parts of cardinality n and m. The graph $K_{1,3} = S_{1,1,1}$ is called a *claw*. We use H_n to denote the graph which can be obtained from two copies of P_3 by joining their central vertices by a chordless path with n edges (Figure 2).

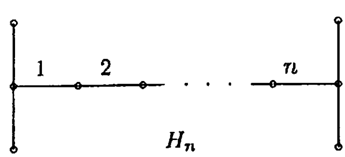

Figure 2: Graph H_n

In addition, we introduce notation for some particular classes of graphs:
X_k, the class of $(C_3, C_4, \ldots, C_k)$-free graphs,
Y_l, the class of $(H_1, H_2, \ldots, H_l)$-free graphs,
Z_3, the class of graphs with maximum degree at most 3.

2 NP-hardness of dissociation number on certain classes of bipartite graphs

Let G be a graph and x be a vertex in G. A *vertex stretching* with respect to x is defined to be the following transformation:

1) partition the neighborhood $N(x)$ of vertex x into two sets Y and Z in an arbitrary way;

2) delete vertex x from the graph together with incident edges;

3) add four vertices y, a, b, z and then a chordless path $P_4 = (y, a, b, z)$ to the remaining graph;

4) connect vertex y to each vertex in Y, and connect z to each vertex in Z.

Figure 3 illustrates the vertex stretching operation.

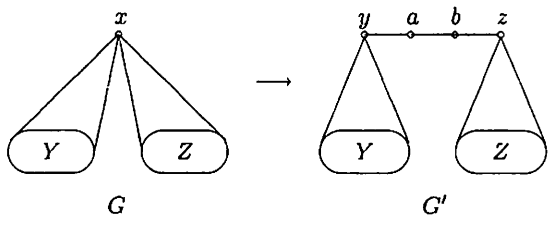

Figure 3: Vertex stretching operation

Lemma 1 *Let G' denote a graph obtained from G by the vertex stretching operation. Then*

$$diss(G') = diss(G) + 2.$$

Proof. First, let us consider a dissociation set D in G and show that graph G' contains a dissociation set D' of cardinality $|D| + 2$. If x does not belong to D, then $D' = D \cup \{(a, b)\}$ is the required set. If D contains vertex x, then clearly $|D \cap N(x)| \leq 1$. Assume without loss of generality that $D \cap Z = \emptyset$. Then $D' = (D - \{x\}) \cup \{y, b, z\}$ is the set we are looking for. Thus $diss(G') \geq diss(G) + 2$.

Conversely, let D' be a dissociation set in G' with at least two vertices. Our purpose now is to find in G a dissociation set D of cardinality $|D'| - 2$. The problem is trivial if D' contains at most two new vertices. Up to symmetry, the only possible case with more than two new vertices in D' is the following: $y, a, z \in D'$. In that case, $D' \cap Y = \emptyset$ and $|D' \cap Z| \leq 1$. Therefore, $D = (D' - \{y, a, z\}) \cup \{x\}$ is the desired set. Thus $diss(G) \geq diss(G') - 2$. ■

Lemma 2 *Any graph G can be transformed by a sequence of vertex stretching operations into a bipartite graph in the class $X_k \cap Y_l \cap Z_3$ for any integers $k \geq 3$ and $l \geq 1$.*

Proof. Assume first that graph G has a vertex x of degree at least 4. For a vertex stretching with respect to x, let us choose as Y any two vertices adjacent to x, and let Z contain all the remaining vertices in the neighborhood of x. Under this operation we obtain a graph with four new vertices y, a, b, z, where y is of degree 3, a and b are of degree 2, and z is of degree exactly one less than that of x. If the degree of z is still greater than 3, we can decrease it in a similar way by application of the vertex stretching operation with respect to z. Thus, repeatedly applying the operation we can obtain a graph in which every vertex has degree at most 3, i.e. a graph in Z_3.

Now let us consider another instance of the vertex stretching operation in which set Y consists of a single vertex. In this case, the operation is equivalent to a triple subdivision of an edge in the graph. In other words, the edge is replaced by a chordless path of length 4. Let us call the application of this operation to each edge of the graph the total stretching of G. Under the total stretching, the length of every induced cycle increases four times, and therefore, the resulting graph contains no cycles of odd length, i.e. it is bipartite. Moreover, applying the total stretching sufficiently many times, we get rid of induced cycles C_i with $i \leq k$ and induced subgraphs of the form H_i with $i \leq l$. ■

It is not hard to see that the transformation described in Lemma 2 can be carried out in time bounded by a polynomial in the size of the input graph. In conjunction with Lemma 1 and the result in [10] this implies

Corollary 3 *For any integers $k \geq 3$ and $l \geq 1$, the dissociation number problem is NP-hard for bipartite graphs in the class $X_k \cap Y_l \cap Z_3$.*

Now we are ready to prove the main theorem of the section.

Theorem 4 *Let $\mathcal{K}$ be a class of graphs defined by a finite set $\mathcal{F}$ of forbidden induced subgraphs. If $\mathcal{F}$ does not contain a graph every connected component of which is of the form $S_{i,j,k}$ (Figure 1), then the dissociation number problem is NP-hard for bipartite graphs in the class $\mathcal{K}$.*

Proof. Let p be an integer greater than the number of vertices in a largest graph in $\mathcal{F}$. Suppose that a bipartite graph $G \in X_p \cap Y_p \cap Z_3$ does not belong to $\mathcal{K}$. Then G must contain a graph $A \in \mathcal{F}$ as an induced subgraph. Since $G \in X_p$, the graph A contains no induced cycles C_q of length $q \leq p$. Moreover, A can not contain a cycle C_q with $q > p$, because $|V(A)| < p$ due to the choice of p. Therefore, A contains no cycles, i.e., A is a forest. Analogously, since $G \in Y_p$ and $|V(A)| < p$, A contains no induced subgraphs of the form H_r, i.e., every connected component of A has at most one vertex of degree at least 3. Furthermore, since $G \in Z_3$, the graph A does not have vertex of degree more than 3. But then every connected component of A is of the form $S_{i,j,k}$ and this contradicts the hypothesis of the theorem. We thus have proved that every bipartite graph in $X_p \cap Y_p \cap Z_3$ belongs to $\mathcal{K}$. Now the conclusion of the theorem follows from Corollary 3. ■

As an immediate consequence of the theorem, we obtain NP-hardness of the dissociation number problem in a large family of subclasses of bipartite graphs such as $K_{1,4}$-free bipartite graphs or C_4-free bipartite graphs. On the other hand, Theorem 4 characterizes graph classes for which the dissociation problem could possibly be solved in polynomial time. In Section 3, we prove that the dissociation number problem can be solved in one such class in polynomial time.

3 A polynomially solvable case of the dissociation problem

A bipartite graph $G = (V_1, V_2, E)$ consists of a set of vertices $V_1 \cup V_2$ and a set of edges $E \subseteq V_1 \times V_2$. For a bipartite graph $G = (V_1, V_2, E)$, we denote by $\tilde{G}$ the bipartite complement of G, i.e., $\tilde{G} = (V_1, V_2, (V_1 \times V_2) - E)$. By mK_2 we denote the regular graph of degree 1 with $2m$ vertices. Clearly, mK_2 is a bipartite graph. Note that $\widetilde{3K_2} = C_6$.

Let us call two vertices of a graph *similar* if their neighborhoods are the same. Clearly, similarity is an equivalence relation, and for any two similarity classes (i.e. equivalence classes) M_i and M_j, either each pair of vertices $x \in M_i$ and $y \in M_j$ is adjacent or none of them are. We shall say that M_i and M_j are adjacent or non-adjacent, respectively. A graph every similarity class of which has size 1 will be called *prime*. It is not hard to see that any bipartite graph G has a unique (up to isomorphism) maximal prime induced subgraph that can be obtained by choosing exactly one vertex in each similarity class of G, and we call this the *prime graph* of G and denote it by $P(G)$. A graph G is called *H-like* if the prime graph of G is H. For instance, a cycle-like graph is a graph whose prime graph is a cycle; that is, a graph obtained from a cycle by duplicating some vertices (and not joining the duplicates).

It is well known and easy to see that for any graph G, induced matchings in G correspond precisely to stable sets in the square of the line-graph of G, denoted $(L(G))^2$. $(L(G))^2$ has a vertex for each edge of G and two vertices of $(L(G))^2$ are adjacent if the edges they correspond to in G either meet at a vertex of G or are joined by an edge of G. We now give a similar correspondence for dissociation sets. Given a graph G, construct a graph $W(G)$ as follows. $W(G)$ consists of a copy of G and a copy of $(L(G))^2$. So $W(G)$ has a vertex w_i for each vertex i of G, and a vertex w_{ij} for each edge ij of G (that is, for each vertex of $(L(G))^2$). Vertices w_i are called *white* and vertices w_{ij} are called *black*. A white vertex w_i is adjacent to a black vertex w_{jk} exactly when i is adjacent to either j or k in G.

Theorem 5 *For any graph G, where white vertices in $W(G)$ have weight 1 and black vertices have weight 2, diss(G)=the maximum weight of a stable set in $W(G)$.*

Proof. Let D be a dissociation set in graph G. Let S_D be the set consisting of the white vertices of $W(G)$ corresponding to isolated vertices of $G[D]$ and the black vertices of $W(G)$ corresponding to edges of $G[D]$. It can be easily verified that S_D is a stable set in $W(G)$ and clearly its weight is $|D|$. Similarly, a stable set S in $W(G)$ corresponds to a dissociation set D_S in G, and the size of D_S is the weight of S. ■

Alternatively, largest cardinality dissociation sets in G correspond to maximum weight stable sets in $W(P(G))$, where $P(G)$ is the prime graph of G, but the weights of white vertices need to be changed. In $P(G)$ a white vertex w_i corresponds to a similarity class M_i of G, so the weight of white vertex w_i in $W(P(G))$ is $|M_i|$.

Theorem 6 *For any graph G and its prime graph $P(G)$, consider $W(P(G))$ with weight $|M_i|$ for a white vertex corresponding to similarity class M_i and weight 2 for black vertices. Then diss(G)=the maximum weight of a stable set in $W(P(G))$.*

Proof. Let D be a dissociation set in graph G. We denote the set of vertices of degree 1 in $G[D]$ by D_1, and the set of isolated vertices in $G[D]$ by D_0. With the set D we associate a set of vertices S_D in $W(P(G))$ in the following way. We include w_i in S_D if and only if $M_i \cap D_0 \neq \emptyset$. We include w_{ij} in S_D if and only if $M_i \cap D_1 \neq \emptyset$ and $M_j \cap D_1 \neq \emptyset$ and M_i is adjacent to M_j in G. It is not hard to verify that S_D is a stable set in $W(P(G))$ and its weight is at least $|D|$. Hence $diss(G) \leq$ the maximum weight of a stable set in $W(P(G))$.

To prove the reverse inequality, let S be a stable set in the graph $W(P(G))$. We associate with S a set of vertices D_S in G in the following way. For each white vertex $w_i \in S$, D_S includes all vertices of the corresponding similarity class M_i. For each black vertex $w_{ij} \in S$, D_S includes a pair of adjacent vertices $x \in M_i$ and $y \in M_j$. Obviously D_S is a dissociation set in G and its cardinality is the weight of set S in $W(P(G))$. Hence $diss(G) \geq$ the maximum weight of a stable set in $W(P(G))$. ∎

We note that in many classes $\mathcal{G}$ of graphs where the induced matching problem has been shown to be solvable in polynomial time, this has been proved by showing that if $G \in \mathcal{G}$, then $(L(G))^2 \in \mathcal{H}$, where $\mathcal{H}$ is a class for which there is a polynomial-time algorithm for finding a maximum stable set. We expect the dissociation problem can be shown to be solvable in polynomial time for certain classes $\mathcal{G}$ by showing that either $W(G) \in \mathcal{H}$ or $W(P(G)) \in \mathcal{H}$, where $\mathcal{H}$ is a class for which there is a polynomial-time algorithm for finding a maximum weight stable set.

Throughout this section we denote by $\mathcal{C}$ the class of bipartite graphs containing no skew star (Figure 4) as an induced subgraph.

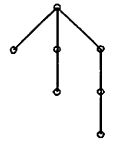

Figure 4: A skew star

We note that the class $\mathcal{C}$ includes several subclasses of bipartite graphs studied recently in the literature [2, 3, 4, 5]. The structure of graphs in $\mathcal{C}$ was characterized in [6] as follows.

Theorem 7 *Let $G = (V_1, V_2, E)$ be a prime bipartite graph without an induced skew star. Then either G is disconnected, or G is the bipartite complement of a disconnected graph, or G can be partitioned into a stable set and a bi-clique, or G is a path or a cycle or the bipartite complement of a path or a cycle.*

Now we use this characterization in order to prove polynomial-time solvability of the dissociation number problem in the class $\mathcal{C}$. To this end, let us introduce the following notation: $\alpha_k(G)$ is the number of vertices in a maximum size dissociation set in G with at most k edges. With this notation, $\alpha_0(G)$ is the stability number of G, i.e. the number of vertices in a maximum size stable set. It is well known that for bipartite graphs the stability number can be computed in polynomial time. Consequently, for any fixed k, $\alpha_k(G)$ can be found in polynomial time if G is bipartite. Below we use this fact to derive a polynomial-time algorithm to compute the dissociation number of graphs in $\mathcal{C}$. Our algorithm is based on the following series of lemmas.

Lemma 8 *If $G_1, \dots, G_k$ are connected components of a graph G, then*

$$diss(G) = \sum_{i=1}^{k} diss(G_i).$$

For a bipartite graph G, where $U_1, \dots, U_k$ are the vertex-sets of the connected components of the bipartite complement of G, the induced subgraphs $B_i = G[U_i]$ are called the *co-components* of G.

Lemma 9 *If $B_1, \dots, B_k$ are the co-components of a bipartite graph $G = (V_1, V_2, E)$, then*

$$diss(G) = \max\{\alpha_2(G), diss(B_1), \dots, diss(B_k)\}.$$

Proof. Let D be a maximum cardinality dissociation set in G. We will show that if D is not contained in any co-component of G, then $G[D]$ contains at most two edges. To prove this, assume that $x \in U_i = V(B_i)$ and $y \in U_j = V(B_j)$ are two vertices of D in different co-components. Without loss of generality, we suppose that $x \in V_1$ and $y \in V_2$, since otherwise D is a stable set. This implies that xy is an edge in $G[D]$, and therefore $D - \{x, y\} \subseteq (V_1 \cap U_j) \cup (V_2 \cap U_i)$, otherwise either x or y has degree more than 1 in $G[D]$. Since the subgraph of G induced by $(V_1 \cap U_j) \cup (V_2 \cap U_i)$ is complete bipartite, there may be only one edge in $G[D - \{x, y\}]$. Hence the lemma. ■

In the next lemma we consider a bipartite graph $G = (V_1, V_2, E)$ whose vertices can be partitioned into a bi-clique (a complete bipartite subgraph with at least one edge) Q and a stable set S. Let $Q_i = Q \cap V_i$, $S_i = S \cap V_i$ $(i = 1, 2)$, $R_1 = G[Q_1 \cup S_2]$ and $R_2 = G[Q_2 \cup S_1]$. With this notation we prove the following lemma.

Lemma 10 *Let G be a bipartite graph whose vertices can be partitioned into a bi-clique Q and a stable set S, then*

$$diss(G) = \max\{\alpha_1(G), diss(R_1) + |S_1|, diss(R_2) + |S_2|\}.$$

Proof. Let D be a maximum dissociation set in G. If D is a stable set, then $diss(G) = \alpha_0(G) \leq \alpha_1(G)$. If D is not a stable set, it must contain a vertex x in Q. Assume $x \in Q_1$. First suppose $D \cap Q_2 \neq \emptyset$, say $y \in D \cap Q_2$. Of course, y is a neighbor of x in $D \cap Q_2$. Then clearly x and y do not have other neighbors in D. In other words, $D - \{x, y\} \subseteq \widetilde{N}(x) \cup \widetilde{N}(y) \subseteq S$. Consequently $diss(G) = \alpha_1(G)$. If $D \cap Q_2 = \emptyset$, then obviously $S_1 \subseteq D$ and $diss(G) = diss(R_1) + |S_1|$. Similarly, if $D \cap Q_1 = \emptyset$ and $D \cap Q_2 \neq \emptyset$, then $diss(G) = diss(R_2) + |S_2|$. ■

Lemmas 8, 9 and 10 together with Theorem 7 allow us to reduce the problem in question from the class $\mathcal{C}$ to path-like or cycle-like graphs or their bipartite complements. Moreover, if G is the bipartite complement of a P_k-like or C_k-like graph, then $k \geq 7$ or else G is disconnected.

Lemma 11 *Let G be the bipartite complement of a P_k-like or a C_k-like graph with $k \geq 7$. Then*

$$diss(G) = \alpha_2(G).$$

Proof. Obviously, any P_k-like or C_k-like graph with $k \geq 7$ is C_6-free. Since the bipartite complement of C_6 is $3K_2$, the graph G is $3K_2$-free. Consequently, any dissociation set in G contains at most two edges. Hence $diss(G) \leq \alpha_2(G)$. The reverse inequality is obvious. ■

We use Theorem 6 in order to reduce the dissociation number problem in path- or cycle-like graphs to the weighted version of the stable set problem in claw-free graphs.

Theorem 12 *If G is a path- or cycle-like graph, then $W(P(G))$ is a claw-free graph.*

Proof. To prove the theorem, it is sufficient to show that the neighborhood of each vertex in $W(P(G))$ does not contain a stable set of size 3 (an antitriangle).

The neighborhood of a white vertex w_i in $W(P(G))$ contains at most six vertices: two white and four black. If $P(G) = C_k$ where $k \geq 6$, or if $P(G)$ is a path and i is not an endpoint or next-to-endpoint of the path, then the neighborhood of w_i in $W(P(G))$ induces the subgraph F_1 in Figure 5. It is easy to see that F_1 does not contain an antitriangle. It can be verified that in the remaining cases (when $P(G) = C_k$, $3 \leq k \leq 5$, or if $P(G)$ is a path and i is an endpoint or next-to-endpoint of the path), the neighborhood of w_i in $W(P(G))$ is also antitriangle-free.

The neighborhood of a black vertex w_{ij} in $W(P(G))$ contains at most eight vertices: four white and four black. If $P(G) = C_k$ where $k \geq 7$, or if $P(G)$ is a path and ij is not the end edge or next-to-end edge of the path, then the neighborhood of w_{ij} in $W(P(G))$ induces the subgraph F_2 in Figure 5. It is easy to see that F_2 is antitriangle-free. If $P(G)$ is a C_6-like graph, then the neighborhood of w_{ij} is F_2 together with an edge joining the two white vertices of F_2 of degree 3, and hence is antitriangle-free as well. It can be verified that in the remaining cases (when $P(G) = C_k$, $3 \leq k \leq 5$, or if $P(G)$ is a path and ij is an end edge or next-to-end edge of the path), the neighborhood of w_{ij} in $W(P(G))$ is also antitriangle-free. ■

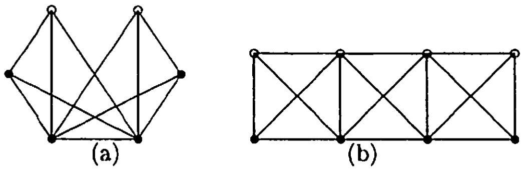

Figure 5: (a) subgraph F_1; (b) subgraph F_2

The result of Theorem 12 in conjunction with Theorem 6 and Minty's algorithm to find a maximum weight stable set in a claw-free graph [8] lead to a polynomial-time algorithm to compute the dissociation number of path- or cycle-like graphs.

We summarize the above arguments in the following algorithm to compute the dissociation number of a graph $G \in C$.

DISS(G)

Input: A bipartite graph $G = (V_1, V_2, E)$ without a skew star.
Output: The dissociation number of G.

1. If G is disconnected, decompose it into connected components $G_1, \dots, G_k$ and set
$$\mathrm{DISS}(G) = \sum_{i=1}^{k} \mathrm{DISS}(G_i).$$
2. If G is the bipartite complement of a disconnected graph, then decompose G into co-components $B_1, \dots, B_k$ and set
$$\mathrm{DISS}(G) = \max\{\alpha_2(G), \mathrm{DISS}(B_1), \dots, \mathrm{DISS}(B_k)\}.$$
3. If G can be partitioned into a bi-clique Q and a stable set S, then set
$$\mathrm{DISS}(G) = \max\{\mathrm{DISS}(R_1) + |S_1|, \mathrm{DISS}(R_2) + |S_2|, \alpha_1(G)\},$$
where $Q_i = Q \cap V_i$, $S_i = S \cap V_i$ $(i = 1, 2)$, $R_1 = G[Q_1 \cup S_2]$ and $R_2 = G[Q_2 \cup S_1]$.
4. If the bipartite complement of G is a cycle-like or path-like graph, then set
$$\mathrm{DISS}(G) = \alpha_2(G).$$
5. If G is a cycle-like or path-like graph, then construct the auxiliary weighted graph $W(P(G))$, apply Minty's algorithm to find the maximum weight of a stable set in $W(P(G))$, and set
$$\mathrm{DISS}(G) = \text{maximum weight of a stable set in } W(P(G)).$$

Concerning step (3) of DISS(G), we actually need to find the biclique Q and the stable set S that partition G. Here is an algorithm to determine if the vertex-set of a connected bipartite graph $G = (U, V, E)$ can be partitioned into a biclique Q and a stable set S where edge uv has its ends in Q. The algorithm puts a vertex in Q or S when it is "forced" to. To start, $u, v \in Q$. U and V are called the color-classes.

When a vertex enters Q, all of its non-neighbors in the other color-class enter S. When a vertex enters S, all of its neighbors enter Q. As soon as a vertex enters Q, we check that it is adjacent to all of Q in the other color-class, and as soon as a vertex enters S, we check that it is non-adjacent to all of S; if this does not hold, we stop: the required partition does not exist. If at any time, $Q \cap S \neq \emptyset$, stop: the required partition does not exist. Otherwise, when the procedure stops, if $Q \cup S = U \cup V$, we are finished - the required partition has been found; if not, every vertex not in $Q \cup S$ is adjacent to all of Q in the other color class, and is non-adjacent to all of S.Then $Q' = Q \cup (U - (Q \cup S))$, $S' = S \cup (V - (Q \cup S))$ is the required bipartition.

Taking into account the observation that $\alpha_k(G)$, for a fixed k, can be computed in polynomial time for a bipartite graph G, we conclude that procedure **DISS** has polynomial time complexity.

Theorem 13 *The dissociation number of bipartite $S_{1,2,3}$-free graphs can be computed in polynomial time.*

4 A polynomially solvable case of the induced matching problem

A set M of edges in a graph G is called an *induced matching* if no two edges of M meet at a vertex or are joined by an edge of G. Equivalently, an induced matching is a matching which forms an induced subgraph. The number of edges in a maximum cardinality induced matching in G is called the *induced matching number* of G and is denoted $im(G)$.

An approach analogous to that used in Section 3 for the dissociation problem can be used to give a polynomial-time algorithm for the induced matching problem in bipartite graphs with no induced skew star. The proofs are similar to those of Section 3, so they are omitted.

Lemma 14 *If $G_1, \ldots, G_k$ are connected components of a graph G, then*

$$im(G) = \sum_{i=1}^{k} im(G_i).$$

Lemma 15 *If $B_1, \ldots, B_k$ are the co-components of a bipartite graph $G = (V_1, V_2, E)$, then either $im(G) = 1$ or*

$$im(G) = \max\{2, im(B_1), \ldots, im(B_k)\}.$$

In the next lemma we consider a bipartite graph $G = (V_1, V_2, E)$ whose vertices can be partitioned into a bi-clique Q and a stable set S. As in Section 3, let $Q_i = Q \cap V_i$, $S_i = S \cap V_i$ $(i = 1, 2)$, $R_1 = G[Q_1 \cup S_2]$ and $R_2 = G[Q_2 \cup S_1]$.

Lemma 16 *Let G be a bipartite graph whose vertices can be partitioned into a bi-clique Q and a stable set S, then*

$$im(G) = \max\{1, im(R_1), im(R_2)\}.$$

Lemmas 14, 15 and 16 together with Theorem 7 allow us to reduce the problem in question from the class C to path-like or cycle-like graphs or their bipartite complements. Moreover, if G is the bipartite complement of a P_k-like or C_k-like graph, then $k \geq 7$ or else G is disconnected.

Lemma 17 *Let G be the bipartite complement of a P_k-like or a C_k-like graph with $k \geq 7$. Then*

$$im(G) = 2.$$

Lemma 18 *Let G be a P_k-like graph. Then*

$$im(G) = \lceil (k-1)/3 \rceil.$$

Lemma 19 *Let G be a C_k-like graph. Then*

$$im(G) = \lfloor k/3 \rfloor.$$

Theorem 20 *The induced matching number of bipartite $S_{1,2,3}$-free graphs can be computed in polynomial time.*

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