On Graphs Whose Acyclic Graphoidal Covering Number Is One Less than Its Cyclomatic Number*

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Abstract. A *graphoidal cover* of a graph G is a collection ψ of (not necessarily open) paths in G such that every vertex of G is an internal vertex of atmost one path in ψ and every edge of G is an exactly one path in ψ. If further no member of ψ is a cycle, then ψ is called an *acyclic graphoidal cover* of G. The minimum cardinality of an acyclic graphoidal cover is called the *acyclic graphoidal covering number* of G is denoted by η_a. In this paper we characterize the class of graphs for which $\eta_a = q - p$ where p and q denote respectively the order and size of G.

1. Introduction

By a *graph* $G = (V, E)$ we mean a finite, undirected, connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [4].

If $P = (v_0, v_1, \ldots, v_n)$ is a path or a cycle in G, $v_1 \ldots, v_{n-1}$ are called *internal vertices* of P. The concept of graphoidal cover was introduced by Acharya and Sampathkumar [1] and that of acyclic graphoidal cover was introduced by Suresh Suseela [3].

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Definition 1.1 A *graphoidal cover* of a graph G is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

(i) Every path in ψ has atleast two vertices.
(ii) Every vertex of G is an internal vertex of atmost one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

ψ is called an *acyclic graphoidal cover* of G if no member of ψ is a cycle in G. The minimum cardinality of (an acyclic) a graphoidal cover of G is called the *(acyclic) graphoidal covering member* of G and is denoted by $(\eta_a)\eta$. The values of η and η_a for several families of graphs have been determined by Pakkiam and Arumugam [5, 6] and Suresh Suseela [3]. A comprehensive review of the progress made on graphoidal covers is given in Arumugam et. al. [2].

Definition 1.2 Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an *interior vertex* of ψ if it is an internal vertex of some path in ψ. Any vertex which is not an interior vertex of ψ is said to be an *exterior vertex* of ψ.

Theorem 1.3 [3] For any acyclic graphoidal cover ψ of G, let t_ψ denote the number of exterior vertices of ψ. Let $t = \min t_\psi$ where the minimum is taken over all acyclic graphoidal covers of G. Then $\eta_a = q - p + t$ where p and q denote the order and size respectively of G.

Corollary 1.4 For any graph G, $\eta_a \geq q - p$. Moreover the following are equivalent.

(i) $\eta_a = q - p$.
(ii) There exists an acyclic graphoidal cover without exterior vertices.
(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices.

In this paper we characterize the class of graphs G for which $\eta_a = q - p$, which is one less than the cyclomatic number of G. It follows from Corollary 1.4 that for such graphs $\delta \geq 2$. Moreover we have

Theorem 1.5 [3] For any graph with $\delta \geq 3$, $\eta_a = q - p$.

Hence we confine ourselves to graphs with $\delta = 2$. We need the following definition and theorem.

Definition 1.6 [4] A *theta graph* is a block with two non-adjacent vertices of degree 3 and all other vertices of degree 2.

Theorem 1.7 [4] Every non-hamiltonian 2-connected graph contains a theta graph.

2. Main Results

Let $\mathcal{F}_a$ denote the class of all connected graphs with $\eta_a = q - p$. The following is an immediate consequence of Corollary 1.4.

Lemma 2.1 If there exists a spanning subgraph H of G such that $H \in \mathcal{F}_a$, then $G \in \mathcal{F}_a$.

Theorem 2.2 Let $G_1 \in \mathcal{F}_a$. Let C be a cycle such that $|V(C) \cap V(G_1)| \geq 2$. Then $G_1 \cup C \in \mathcal{F}_a$.

Proof Let $C = (v_1, v_2, \ldots, v_n, v_1)$ and $G = G_1 \cup C$. Let ψ_1 be a minimum acyclic graphoidal cover of G_1 so that every vertex of G_1 is interior to ψ_1. Let $V(G_1) \cap V(C) = \{v_1, v_{i_1}, \ldots, v_{i_k}\}$ with $k \geq 1$ and $1 < i_1 < i_2 < \cdots < i_k \leq n$.

Case i $E(G_1) \cap E(C) = \emptyset$.

$$\begin{aligned} \text{Let } P_1 &= (v_1, v_2, \cdots, v_{i_1}), \\ P_2 &= (v_{i_1}, \ldots, v_{i_2}), \\ &\vdots \\ P_k &= (v_{i_{k-1}}, \ldots, v_{i_k}) \\ \text{and } P_{k+1} &= (v_{i_k}, \ldots, v_n, v_1). \end{aligned}$$

Then $\psi = \psi_1 \cup \{P_1, P_2, \ldots, P_{k+1}\}$ is an acyclic graphoidal cover of G and every vertex of G is interior to ψ.

Case ii $E(G_1) \cap E(C) \neq \emptyset$.

Let $E(G_1) \cap E(C) = \{e_1, e_2, \ldots, e_m\}$. Then each nontrivial component of $C - \{e_1, e_2, \ldots, e_m\}$ is a path. Let $P = (w_1, w_2, \ldots, w_k)$ be a nontrivial component of $C - \{e_1, e_2, \ldots, e_m\}$. If an internal vertex w_i of P belongs to $V(G_1)$, we replace P by $P_1 = (w_1, w_2, \ldots, w_i)$ and $P_2 = (w_i, w_{i+1}, \ldots, w_k)$. By repeating this process we obtain a collection $\mathcal{P}$ of edge-disjoint and internally disjoint paths covering all the edges of $C - \{e_1, e_2, \ldots, e_m\}$ and $\psi = \psi_1 \cup \mathcal{P}$ is a graphoidal cover of G with no exterior vertices. Thus $G \in \mathcal{F}_a$. □

Corollary 2.3 Let G be a block with $p \geq 3$. If there exists a connected subgraph H of G such that $H \in \mathcal{F}_a$, then $G \in \mathcal{F}_a$.

Proof Let $e \in E(H)$ and $v \in V(G)\backslash V(H)$. Since G is a block, e and v lie on a common cycle in G. By Theorem 2.2, $H \cup C \in \mathcal{F}_a$. Repeating this process we obtain a spanning subgraph H_1 of G such that $H_1 \in \mathcal{F}_a$ and by Lemma 2.1, $G \in \mathcal{F}_a$. □

Theorem 2.4 Let $G_1, G_2 \in \mathcal{F}_a$ and $V(G_1) \cap V(G_2) \neq \emptyset$. Then $G_1 \cup G_2 \in \mathcal{F}_a$.

Proof Let $V(G_1) \cap V(G_2) = \{v_1, v_2, \dots v_n\}$. Let ψ_1 and ψ_2 be minimum acyclic graphoidal covers of G_1 and G_2 respectively so that all the vertices of G_1 are interior to ψ_1 and all the vertices of G_2 are interior to ψ_2.
Case i $E(G_1) \cap E(G_2) = \emptyset$.

Let P be the u-v path in ψ_2 having v_1 as an internal vertex. Let P_1 and P_2 denote respectively the u-v_1 section and v_1-v section of P and replace ψ_2 by $(\psi_2 - \{P\}) \cup \{P_1, P_2\}$. Repeating this process for each vertex v_i we obtain an acyclic graphoidal cover ψ_3 of G_2. Now $\psi_1 \cup \psi_3$ is an acyclic graphoidal cover of $G_1 \cup G_2$ with no exterior vertices.

Case ii $E(G_1) \cap E(G_2) \neq \emptyset$.

Let $E(G_1) \cap E(G_2) = \{e_1, e_2, \dots, e_m\}$. For each e_i, we replace the path in ψ_2 which contains e_i by the non trivial components of $P - e_i$. If a vertex of $V(G_1)$ is an internal vertex of some path in ψ_2 we proceed as in Case i to obtain a collection ψ_3 of paths such that $\psi_1 \cup \psi_3$ is an acyclic graphoidal cover of $G_1 \cup G_2$ with no exterior vertices. Also since G_1 and G_2 are connected and $V(G_1) \cap V(G_2) \neq \emptyset$, $G_1 \cup G_2$ is also connected and hence $G_1 \cup G_2 \in \mathcal{F}_a$. □

We now proceed to characterize graphs with $\eta_a = q - p$. We first introduce a family of graphs.

Let $\mathcal{G}(f)$ denote the collection of all blocks whose edge set can be decomposed into a cycle C and a collection $\mathcal{P}$ of internally dispoint paths such that each path P in $\mathcal{P}$ has $f \in V(C)$ as its origin and $|V(P) \cap V(C)| \leq 2$ (The collection $\mathcal{P}$ may be empty in which case the corresponding member of $\mathcal{G}(f)$ is a cycle)

We observe that if $G \in \mathcal{G}(f)$ and G is not a cycle, then $\deg f = |\mathcal{P}|+2 = \Delta$. A member of $\mathcal{G}(f)$ is given in Figure 1.

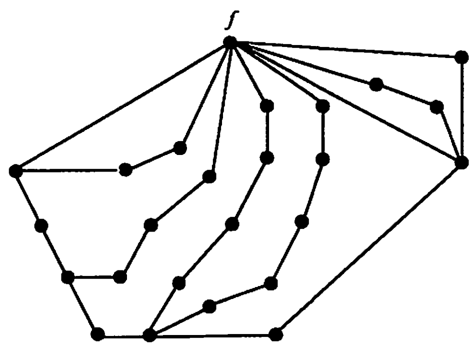

Figure 1.

Theorem 2.5 Let G be a 2-connected graph with $p \geq 3$. Then $\eta_a = q - p$ if and only if $G \notin \mathcal{G}(f)$.

Proof If $G \in \mathcal{G}(f)$ then $\eta_a = d - 1$ and $q - p = d - 2$ where d is the degree of f and hence $\eta_a \neq q - p$.

Conversely suppose $\eta_a \neq q - p$.

Case i G is hamiltonian.

Since any graph consisting of a cycle with two non-adjacent chords is a member of $\mathcal{F}_a$ it follows from Lemma 2.1 that $G \in \mathcal{G}(f)$.

Case ii G is non-hamiltonian.

By Theorem 1.7, G has a theta graph H.

$$\text{Let } P_1 = (u, u_1 \ldots, u_n = v),$$
$$P_2 = (u, v_1, \ldots, v_m = v)$$
$$\text{and } P_3 = (u, w_1, \ldots w_s = v)$$

be the three paths of H. If there exists a path $P = (x_1, x_2, \ldots x_r)$ in G such that $x_1, x_r \notin \{u, v\}$ and $x_1, x_r \in V(H)$ then $H \cup P \in \mathcal{F}_a$ and hence by Corollary 2.3 $G \in \mathcal{F}_a$ which is contradiction. Suppose Q_1 is a u-y_1 path and Q_2 is a v-y_2 path. If $y_1 \neq v$ and $y_2 \neq u$ then $H \cup Q_1 \cup Q_2 \in \mathcal{F}_a$ and hence by corollary 2.3 $G \in \mathcal{F}_a$ which is a contradiction. On the other hand if $y_1 = v$ then $H \cup Q_1 \cup Q_2 \in \mathcal{G}(u)$ or $H \cup Q_1 \cup Q_2 \in \mathcal{G}(v)$. Thus we see that $G \in \mathcal{G}(u)$ or $G \in \mathcal{G}(v)$. □

Theorem 2.6 Let G be a graph with $\delta > 1$ and $\kappa(G) = 1$. Then $G \notin \mathcal{F}_a$ if and only if atleast one end block of G is a member of $\mathcal{G}(f)$ with f as a cut-vertex of G.

Proof Suppose there exists an end block B in G such that $B \in \mathcal{G}(f)$ and f is a cut-vertex of G. Then for any acyclic graphoidal cover ψ of G atleast one vertex of $V(B) - \{f\}$ is exterior to ψ and hence $G \notin \mathcal{F}_a$.

Conversely suppose $G \notin \mathcal{F}_a$. Let m denote the number of blocks in G. We prove by induction on m that atleast one end block of G is a member of $\mathcal{G}(f)$ with f as a cut-vertex of G.

Suppose $m = 2$. Let B_1 and B_2 be the blocks of G. Since $G \notin \mathcal{F}_a$ it follows from Theorem 2.4 that atleast one of B_1, B_2 is not in $\mathcal{F}_a$.

Case i $B_1 \notin \mathcal{F}_a$, $B_2 \in \mathcal{F}_a$.

It follows from Theorem 2.5 that $B_1 \in \mathcal{G}(f)$ and since $G = B_1 \cup B_2 \notin \mathcal{F}_a$, f is a cut-vertex of G.

Case ii $B_1 \notin \mathcal{F}_a$, $B_2 \notin \mathcal{F}_a$.

Then $B_1 \in \mathcal{G}(f_1)$ and $B_2 \in \mathcal{G}(f_2)$ and since $B_1 \cup B_2 \notin \mathcal{F}_a$, either f_1 or f_2 is a cut-vertex of $B_1 \cup B_2$. Hence the result is true for $m = 2$.

We now assume that the result is true for any graph with $m - 1$ blocks. Let G be a graph with m blocks, $m > 2$ and $G \notin \mathcal{F}_a$. Let B be an end block of G. If $B \in \mathcal{G}(f)$ where f is a cut-vertex of G, then the proof is complete. Otherwise $B \in \mathcal{G}(f)$ where f is not a cut-vertex of G, or $B \in \mathcal{F}a$. Let $f_1 \in V(B)$ be a cut-vertex of G. Let G_1 be the subgraph of G, obtained by removing all the vertices of $B - f_1$ then $G_1 \notin \mathcal{F}a$. By induction hypothesis there exists an end block B_1 of G_1 such that $B_1 \in \mathcal{G}(f_2)$ and f_2 is a cut-vertex of G_1. If B_1 is an end block of G, the proof is complete. Otherwise the blocks B and B_1 have f_1 as the common vertex. Let G_2 be the graph obtained by deleting all the vertices of $B_1 - f_2$ from G_1. Since $B_1 \in \mathcal{G}(f_2)$ and $B \in \mathcal{G}(f)$ where f is not a cut-vertex of G or $B \in \mathcal{F}a$, there exists a graphoidal cover ψ of $B \cup B_1$ such that f_2 is the only vertex which is exterior to ψ and hence $G_2 \notin \mathcal{F}a$. Continuing this process, we obtain an end block of G of the required type. □

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References

[1] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure appl. Math.* **18** (10) (1987), 882–890.

[2] S. Arumugam B. D. Acharya and E. Sampathkumar, *Graphoidal covers of a graph: a creative review*, Graph Theory and Its Applications, Proceedings of the National Workshop, Manonmaniam Sundaranar University, Tirunelveli, Ed. S. Arumugam, B. D. Acharya and E. Sampathkumar, Tata-McGraw Hill (1997), 1–28.

[3] S. Arumugam, J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, Discrete Math. 190 (1998) 67–77.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass. (1969).

[5] C. Pakkiam and S. Arumugam, On the graphoidal covering number of a graph, *Indian J. pure appl. Math.* **20** (4) (1989), 330– 333.

[6] C. Pakkiam and S. Arumugam, The graphoidal covering number of unicyclic graphs, *Indian J. pure appl. Math.* **23** (2) (1992), 141–143.