INDEPENDENT SETS AND MATCHINGS IN TENSOR PRODUCT OF GRAPHS

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ABSTRACT

Let $\alpha(G)$ and $\tau(G)$ denote the independence number and matching number of a graph G, respectively. The tensor product of graphs G and H is denoted by $G \times H$. Let $\underline{\alpha}(G \times H) = \max\{\alpha(G) \,.\, n(H), \alpha(H) \,.\, n(G)\}$ and $\underline{\tau}(G \times H) = 2\tau(G) \,.\, \tau(H)$, where $n(G)$ denotes the number of vertices of G. It is easy to see that $\alpha(G \times H) \geq \underline{\alpha}(G \times H)$ and $\tau(G \times H) \geq \underline{\tau}(G \times H)$. Several sufficient conditions for $\alpha(G \times H) > \underline{\alpha}(G \times H)$ are established. Further, a characterization is established for $\tau(G \times H) = \underline{\tau}(G \times H)$. We have also obtained a necessary condition for $\alpha(G \times H) = \underline{\alpha}(G \times H)$. Moreover, it is shown that neither the hamiltonicity of both G and H nor large connectivity of both G and H can guarantee the equality of $\alpha(G \times H)$ and $\underline{\alpha}(G \times H)$.

1. INTRODUCTION

By a graph we mean a finite, simple, undirected connected graph with at least two vertices. We denote the number of vertices of a graph G by $n(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S. The *tensor product* $G \times H$, of graphs G and H is the graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u,x)(v,y)\} \,/\, uv \in E(G)$ and $xy \in E(H)\}$. The *cartesian product*, $G \,\Box\, H$, of graphs G and H is the graph with $V(G \,\Box\, H) = V(G) \times V(H)$ and $E(G \,\Box\, H) = \{(u,x)(v,y) \mid$ either $u = v$ and $xy \in E(H)$ or $x = y$ and $uv \in E(G)\}$. The *strong product*, $G \boxtimes H$ of graphs G and H is the graph with $V(G \boxtimes H) = V(G) \times V(H)$ and $E(G \boxtimes H) = E(G \times H) \cup E(G \,\Box\, H)$. The *lexicographic product*, $G*H$, of graphs G and H is the graph with $V(G*H) = V(G) \times V(H)$ and $E(G*H) = \{(u,x)(v,y) \mid$ either $uv \in E(G)$ or $u = v$ and $xy \in E(H)\}$. The tensor product is also called as Kronecker product, direct product, categorical product and graph conjunction. It is well known that tensor product is commutative and associative.

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In general, if both G and H have some properties, then the product graphs are 'expected' to have the same property. Except the tensor product of graphs, when both G and H are connected, then the respective product graph is connected; however this is not true with tensor product of graphs [13]. Further, if both G and H are hamilton cycle decomposable, then the lexicographic product is hamilton cycle decomposable [2], and in many instances the cartesian product $G \square H$ is hamilton cycle decomposable [12]. However $G \times H$ is not necessarily hamilton cycle decomposable [1], when both G and H are hamilton cycle decomposable. Similarly Gravier and Khelladi conjectured that if G and H have domination number [4] $\gamma(G)$ and $\gamma(H)$ respectively, then $\gamma(G \times H) \geq \gamma(G) \cdot \gamma(H)$. However, this conjecture was disproved [7]. Hence, among the product graphs dealing with tensor product seems to be difficult in many respects. But tensor product of graphs have many applications [6] and [11].

The *independence number*, $\alpha(G)$, of the graph G is defined to be the maximum number of mutually nonadjacent vertices in G. The *matching number*, $\tau(G)$, of the graph G is defined to be the size of a maximum matching in G. Let M be a matching in G. A vertex v of G is said to be *M-saturated*, if some edge of M is incident with v, otherwise, v is *M-unsaturated*. A nontrivial walk W in G is called an *alternating walk* if any two consecutive edges of the walk are in M and $E(G) \setminus M$. An alternating walk of G is called an *M-augmenting walk* if the end vertices of the walk (not necessarily distinct) are M-unsaturated. A walk is said to be an *odd walk* if it has odd number of edges. For a vertex v of G, the *neighbour set* of v, $N_G(v)$, is the set of vertices which are adjacent to v. For $S \subseteq V(G)$ we define $N(S) = \bigcup_{v \in S} N(v)$.

A graph G is called a *split graph* if $V(G)$ can be partitioned as $S \cup K$ where S is an independent set of G and $G[K]$ is a complete subgraph of G.

A graph G is said to be H- free if G has no induced subgraph isomorphic to H. For two graphs G and H we shall denote $max\ \{\alpha(G) \cdot n(H), \alpha(H) \cdot n(G)\}$ by $\underline{\alpha}(G \times H)$ and $2\tau(G).\tau(H)$ by $\underline{\tau}(G \times H)$. From the definition of tensor product of graphs, it is easy to verify the following: $\alpha(G \times H) \geq \underline{\alpha}(G \times H)$ and $\tau(G \times H) \geq \underline{\tau}(G \times H)$.

Hedetniemi [5] conjectured that for all graphs G and H, the chromatic number of $G \times H$, $\chi(G \times H) = min\ \{\chi(G)\ , \chi(H)\}$, where $\chi(G)$ is the chromatic number of G. This conjecture is not yet settled. Similar to this conjecture one can raise if $\alpha(G \times H) = \underline{\alpha}(G \times H)$ holds. Jha and S. Klavzar [8] established that for any graph G and for any positive integer i, there exists a graph H such that $\alpha(G \times H) > \underline{\alpha}(G \times H) + i$. Further Jha has obtained some results on $\tau(G \times H)$ and $\alpha(G \times H)$ in [9].

One may believe that if both G and H have some special properties then it is likely that $\alpha(G \times H) = \underline{\alpha}(G \times H)$. Here we present graphs G and H, both having any one of the following properties, with $\alpha(G \times H) > \underline{\alpha}(G \times H)$:

(i) triangle free

(ii) $K_{1,3}$ – free

(iii) Hamiltonian split graph.

Also, we can find a bipartite graph G and a nonbipartite graph H so that $\alpha(G \times H) > \underline{\alpha}(G \times H)$.

Hence characterizing the class of graphs for which $\alpha(G \times H) = \underline{\alpha}(G \times H)$ seems to be a difficult problem. Here we have established some sufficient conditions for $\alpha(G \times H) > \underline{\alpha}(G \times H)$. These results are in Section 2.

In Section 3, we have established a characterization for $\tau(G \times H) = \underline{\tau}(G \times H)$; and also, using this characterization, we have given a necessary condition for $\alpha(G \times H) = \underline{\alpha}(G \times H)$.

Definitions that are not given here may be found in [3].

2. INDEPENDENCE NUMBER OF TENSOR PRODUCT OF GRAPHS

Theorem 2.1 Let G and H be two graphs such that $\underline{\alpha}(G \times H) = \alpha(H) \,.\, n(G)$. Let L be a maximum independent set in H. If there exists a subset S of L such that

$$(|N_H(S) \setminus N_H(L \setminus S)| + |S|) \,.\, \alpha(G) > |S| \,.\, n(G), \text{ then } \alpha(G \times H) > \underline{\alpha}(G \times H).$$

Proof. Let M be a largest independent set in G of cardinality $\alpha(G)$. Clearly, from the definition of tensor product of graphs,

$$\{M \times \{N_H(S) \setminus N_H(L \setminus S)\}\} \cup \{M \times L\} \cup \{\{V(G) \setminus M\} \times \{L \setminus S\}\}.$$

is an independent set in $G \times H$. Consequently,

$$\begin{aligned}\alpha(G\times H) &\geq |M|\,.\,|N_H(S)\setminus N_H(L\setminus S)| + |M|.\,|L| + |V(G)\setminus M|\,.\,|L\setminus S|\\ &= \alpha(G)\,.\,|N_H(S)\setminus N_H(L\setminus S)| + (|M|\,.\,|S| + |M|\,.\,|L\setminus S|) + |V(G)\setminus M|\,.\,|L\setminus S|\\ &= \alpha(G)\,.\,|N_H(S)\setminus N_H(L\setminus S)| + \alpha(G).\,|S| + |V(G)|\,.\,|L\setminus S|\\ &= \alpha(G)\,.\,(|N_H(S)\setminus N_H(L\setminus S)| + |S|) + |V(G)|\,.\,|L\setminus S|\\ &> |S|\,.\,n(G) + n(G)\,.\,|L\setminus S|, \text{ by hypothesis,}\\ &= n(G)\,.\,|L|\\ &= n(G)\,.\,\alpha(H)\\ &= \underline{\alpha}(G\times H) \quad \blacksquare\end{aligned}$$

Corollary 2.2 Let G and H be two graphs such that $\underline{\alpha}(G \times H) = \alpha(H)\,.\,n(G)$. Let M and L be maximum independent sets in G and H, respectively. If there exists a subset S of L such that

$$|N_H(S)\setminus N_H(L\setminus S)|\,.\,\alpha(G[V(G)\setminus M]) > |S|\,.\,(|V(G)| - \alpha(G[V(G)\setminus M])),$$

then $\alpha(G \times H) > \underline{\alpha}(G \times H)$.

Proof. Clearly, $|N_H(S)\setminus N_H(L\setminus S)|\,.\,\alpha(G) \geq |N_H(S)\setminus N_H(L\setminus S)|\,.\,\alpha(G[V(G)\setminus M])$
$> |S|\,.\,(|V(G)| - \alpha(G[V(G)\setminus M]))$ by

hypothesis,
$> |S|\,.\,(|V(G)| - \alpha(G))$

From this we have,

$$|N_H(S)\setminus N_H(L\setminus S)|\,.\,\alpha(G) + |S|.\,\alpha(G) > |S|\,.\,|V(G)|,$$

that is, $(|N_H(S)\setminus N_H(L\setminus S)| + |S|)\,.\,\alpha(G) > |S|\,.\,n(G)$ and hence

$\alpha(G \times H) > \underline{\alpha}(G \times H)$, by Theorem 2.1. $\blacksquare$

Theorem 2.3 Let G and H be two graphs such that $\underline{\alpha}(G\times H) = \alpha(H)\,.\,n(G)$. Let M and L be maximum independent sets in G and H, respectively. If there exists a subset S of M with $\alpha(G_1)\,.\,|V(H)\setminus L| > \alpha(H)\,.\,|N_G(M_1)|$ where $G_1 = G[N_G(S)\setminus N_G(M\setminus S)]$ and M_1 is the largest independent set in G_1, then $\alpha(G \times H) > \underline{\alpha}(G \times H)$.

Proof. By the definition of the tensor product of graphs, $\{(V(G) \setminus N_G(M_1)) \times L\} \cup \{M_1 \times (V(H) \setminus L)\}$ is an independent set of $G \times H$. Hence.

$$\begin{aligned} \alpha(G \times H) &\geq |V(G) \setminus N_G(M_1)| \,.\, |L| + \alpha(G_1) \,.\, |V(H) \setminus L| \\ &> |V(G) \setminus N_G(M_1)| \,.\, \alpha(H) + \alpha(H) \,.\, |N_G(M_1)|, \text{ by hypothesis,} \\ &= |V(G)| \,.\, \alpha(H) \\ &= \underline{\alpha}(G \times H) \quad \blacksquare \end{aligned}$$

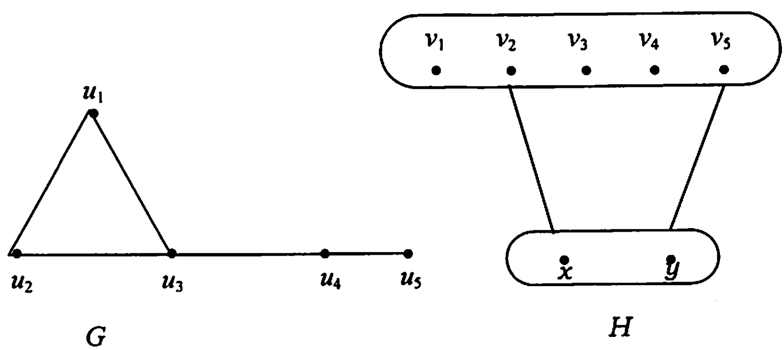


Figure 1

Let H be a graph having the vertex set $\{v_1, v_2, v_3, v_4, v_5, x, y\}$ where the set of vertices $\{v_1, v_2, v_3, v_4, v_5\}$ induces a clique and the pair $\{x, y\}$ of vertices are commonly adjacent to the vertices $\{v_2, v_3, v_4, v_5\}$. Let G and H be the two graphs of Figure 1. By setting $M = \{u_2, u_4\}$, $L = \{v_1, x, y\}$ and $S = \{u_4\}$ in Theoreom 2.3, we conclude that $\alpha(G \times H) > \underline{\alpha}(G \times H)$. However, this conclusion cannot be obtained using Theorem 2.1. In fact from the proof Theorem 2.3, we have $\alpha(G \times H) \geq 16$. But $\alpha(G \times H) = 16$ follows from Theorem 2.2 of [9].

Theorem 2.4 Let G and H be two graphs such that $\underline{\alpha}(G \times H) = \alpha(H) \,.\, n(G)$. Let M be a maximum independent set in G and let $G_1 = G[V(G) \setminus M]$. If there exists a subset S_1 of $V(G_1)$ with $|(N_G(S_1) \cap M) \setminus (N_G(V(G_1) \setminus S_1) \cap M)| \,.\, (|V(H)| \setminus \alpha(H)|) > |S_1| \,.\, \alpha(H)$, then $\alpha(G \times H) > \underline{\alpha}(G \times H)$.

Proof. Let L be a maximum independent set in H. Clearly, from the definition of tensor product of graphs, $\{(N_G(S_1) \cap M) \setminus (N_G(G_1 \setminus S_1) \cap M)\} \times \{V(H) \setminus L\} \cup \{L \times M\} \cup \{V(G_1) \setminus S_1) \times L\}$ is an independent set in $G \times H$. Hence,

$$\alpha(G \times H) \geq |(N_G(S_1) \cap M) \setminus (N_G(V(G_1) \setminus S_1) \cap M)| \,.\, |V(H) \setminus L| + |L| \,.\, |M| + |V(G_1) \setminus S_1| \,.\, |L|$$

$$> |S_1| \,.\, \alpha(H) + \alpha(H) \,.\, |M| + |V(G_1) \setminus S_1| \,.\, \alpha(H), \text{ by hypothesis}$$

$$= \alpha(H).\,(|S_1| + |M| + |V(G_1) \setminus S_1|)$$

$$= \alpha(H) \,.\, n(G)$$

$$= \underline{\alpha}(G \times H) \quad \blacksquare$$

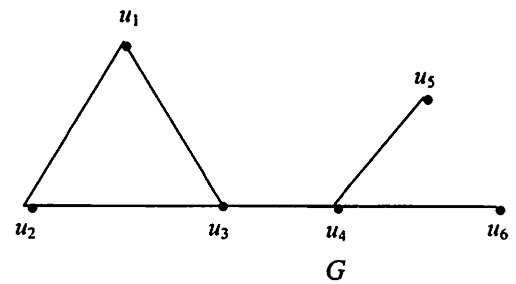
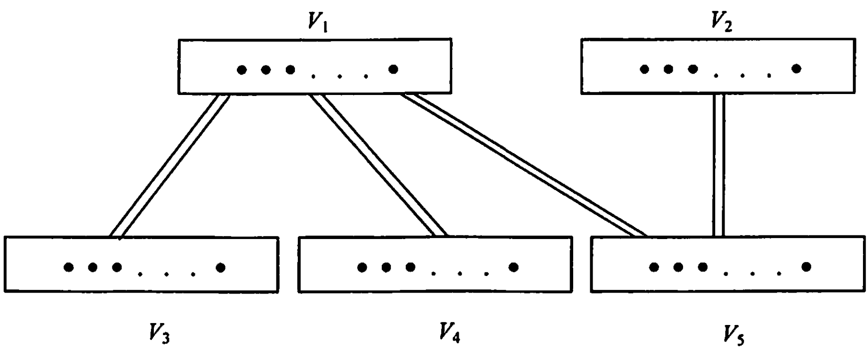


Figure 2

For the graph G of Figure 2 and $H = K_{3,4}$, by setting $M = \{u_1, u_5, u_6\}$ and $S_1 = \{u_4\}$, in Theorem 2.4, we conclude that $\alpha(G \times H) > \underline{\alpha}(G \times H)$. Further, neither Theorem 2.1 nor Theorem 2.3 can be applied to conclude that $\alpha(G \times H) > \underline{\alpha}(G \times H)$. It is also an example for K_4- e free graphs G and H having the property $\alpha(G \times H) > \underline{\alpha}(G \times H)$.

Here we give some classes of graphs having some special properties, with $\alpha(G \times H) > \underline{\alpha}(G \times H)$.

First we construct k-connected graphs G and H such that $\alpha(G \times H) > \underline{\alpha}(G \times H)$. Let G be the graph described in Figure 3, where $V(G) = \bigcup_{i=1}^{5} V_i$ and each V_i have k independent vertices. A pair of lines between V_i and V_j means $V_i \cup V_j$ induces a complete bipartite graph with bipartition (V_i, V_j).



Each V_i has k independent vertices and parallel lines between V_i and V_j means all possible edges between them.

Figure 3

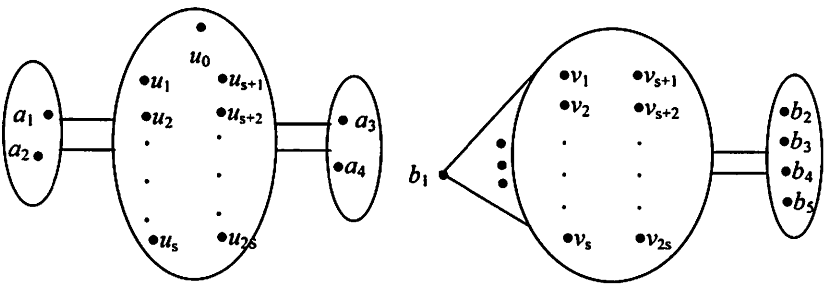
Set $G = H$. Then $\alpha(G \times H) > \underline{\alpha}(G \times H)$, follows from Theorem 2.1, by setting $S = V_5$ and $L = V_3 \cup V_4 \cup V_5$. Though the graph G has the following properties, we still have $\alpha(G \times H) > \underline{\alpha}(G \times H)$:

1. Both G and H are bipartite graphs.
2. $\alpha(G) > n(G) / 2$ and $\alpha(H) > n(H) / 2$.
3. Both G and H are k-connected graphs.

Remark 1. Infact, using Theorem 2.1, the above construction shows k-connected graphs G and H such that $\alpha(G \times H) \geq \underline{\alpha}(G \times H) + k^2$, for any $k > 1$. ■

Remark 2. If G and H satisfy the conditions of Theorem 2.1, we can immediately conclude that $G * \overline{K_m}$ and $H * \overline{K_m}$ also satisfy the conditions of Theorem 2.1 where $\overline{K_m}$ is the complement of K_m. But $G * \overline{K_m}$ is an m-connected graph. Hence constructing k-connected graphs G and H such that $\alpha(G \times H) > \underline{\alpha}(G \times H)$ is not difficult.

We can also construct k-connected hamiltonian split graphs G and H such that $\alpha(G \times H) > \underline{\alpha}(G \times H)$.



G H

The parallel lines represent all possible edges between $\{a_1,a_2\}$ and $\{u_1,u_2, \ldots ,u_s\}$ and $\{a_3,a_4\}$ and $\{u_{s+1},u_{s+2}, \ldots , u_{2s}\}$

The parallel lines represent all possible edges between $\{b_2,b_3,b_4,b_5\}$ and $\{v_{s+1},v_{s+2}, \ldots ,v_{2s})$.

Figure 4

Let G be the graph having the vertex set $\{u_0, u_1, u_2, \ldots, u_{2s}, a_1, a_2, a_3, a_4\}$ where the set of vertices $\{u_0, u_1, u_2, \ldots, u_{2s}\}$ induces a clique and the pair $\{a_1, a_2\}$ (resp. $\{a_3, a_4\}$) of vertices are commonly adjacent to the vertices $\{u_1, u_2, \ldots, u_s\}$ (resp. $\{u_{s+1}, u_{s+2}, \ldots, u_{2s}\}$), see Figure 4. Clearly it is a split graph. Similarly we define another split graph H as follows: Let $V(H) = \{v_1,v_2, \ldots ,v_{2S}, b_1, b_2, b_3, b_4, b_5\}$. Here the set of vertices $\{v_1,v_2, \ldots ,v_{2S}\}$ induces a clique. Further, the vertex b_1 is adjacent to $v_1,v_2, \ldots , v_s$, and each b_i, $2 \leq i \leq 5$, is adjacent to $v_{s+1}, v_{s+2}, \ldots , v_{2S}$ see Figure 4. Clearly H is a split graph and $\alpha(G \times H) > \underline{\alpha}(G \times H)$: this inequlity can be verfied by setting $L = \{b_1, b_2, b_3, b_4, b_5\}$ and $S = \{b_1\}$ in Theorem 2.1.

In the above example, if s is large enough, then both G and H are s-connected hamiltonian graphs with $\alpha(G \times H) > \underline{\alpha}(G \times H)$. In fact, it can be shown that $\alpha(G \times H) \geq \underline{\alpha}(G \times H) + 3s$. ■

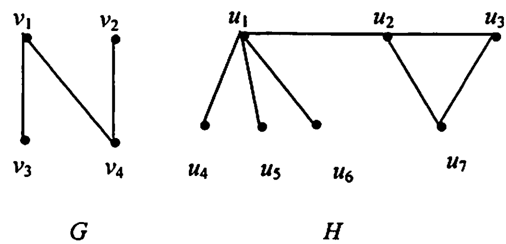


Figure 5

In the graphs of Figure 5, G is a bipartite split graph and H is a non bipartite graph and $\alpha(G \times H) > \underline{\alpha}(G \times H)$; this can be obtained by setting $L = \{u_4, u_5, u_6, u_7\}$ and $S = \{u_7\}$ in Theorem 2.1.

3. MATCHING NUMBER IN TENSOR PRODUCT OF GRAPHS

Let G and H be two simple graphs. It is clear that if $e = uv \in E(G)$ and $f = xy \in E(H)$ are edges in the graphs G and H respectively, then these two edges give two independent edges, namely, $(u,x)(v,y)$ and $(v,x)(u,y)$ in $G \times H$. In general, the maximum matchings M and M_1 of G and H, respectively, give a matching, say, M^*, of $G \times H$ with $2\,|M|\,.\,|M_1|$ edges, that is $|M^*| = 2\,\tau(G)\,.\,\tau(H)$. In the sequel, M^* always represents the matching in $G \times H$ obtained as above.

We need the following Theorem to prove our result.

Theorem A [3,p 70] A matching M in G is a maximum matching if and only if G contains no M– augmenting path. ■

Theorem 3.1 Let G and H be two graphs with maximum matchings M and M_1, respectively. Then $\tau(G \times H) > \underline{\tau}(G \times H)$ if and only if any one of the following is true:

(i) Either G has an M- augmenting walk W (not necessarily open) in which each edge of G in W is repeated at most twice in the walk or H has an M_1- augmenting walk W_1 (not necessarily open) in which each edge of H in W_1 is repeated at most twice in the walk.

(ii) Both G and H do not have perfect matchings.

Proof. Let M and M_1 be maximum matchings in G and H, respectively. By definition, $\underline{\tau}(G \times H) = |M^*|$. First we shall prove that if there is an augmenting walk as defined in (i) of the theorem, then $\tau(G \times H) > \underline{\tau}(G \times H)$.

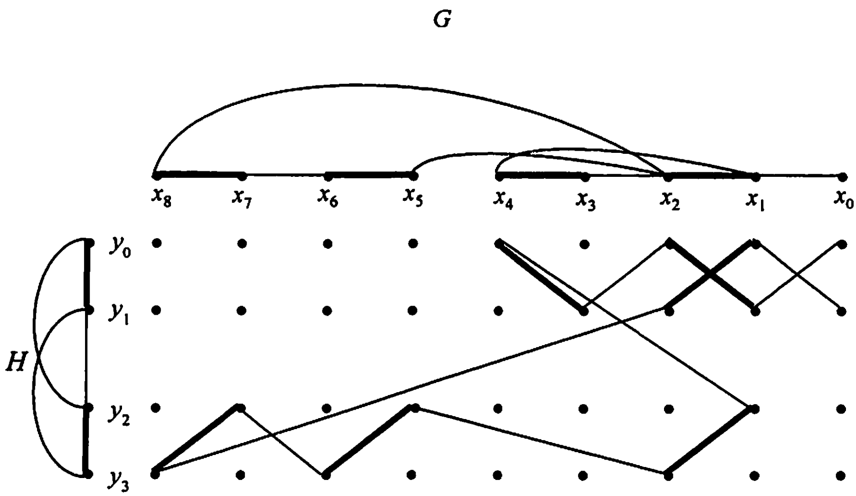
Without loss of generality let us assume that $W_1 = x_0, x_1, \ldots, x_{2n-1}$ be a shortest M_1- augmenting walk in H and hence W_1 is of odd length satisfying the conditions of (i) of the theorem. Let $e = ab$ be an edge of M in G. Then, $W_0 = (a,x_0)(b,x_1)(a,x_2)(b,x_3)\ldots(b,x_{2n-1})$ is an M^*- augmenting path in $G \times H$, since M^* cannot saturate a vertex whose second coordinate is x_0 or x_{2n-1} and W_0 is a path is guaranteed by the fact that each edge of G in W_1 is repeated atmost twice in W_1. Therefore M^* is not a maximum matching of $G \times H$, by Theorem A. Hence $\tau(G \times H) > \underline{\tau}(G \times H)$. Similar argument also holds if G has an M-augmenting walk of odd length satisfying the conditions of (i) of the theorem.

Next we shall prove that, if both G and H do not admit perfect matchings, then $\tau(G \times H) > \underline{\tau}(G \times H)$. Let x_o (resp. y_0) be an M- unsaturated (resp. M_1- unsaturated) vertex in G (resp. H). Let x_0x_1 and y_0y_1 be edges in G and H respectively. Then $(x_0, y_1)(x_1, y_0) \cup M^*$ is a matching in $G \times H$ and hence $\tau(G \times H) > \underline{\tau}(G \times H)$.

Conversely assume that $\tau(G \times H) > \underline{\tau}(G \times H)$. Let M^* be a matching in $G \times H$ obtained by the maximum matching M of G and a maximum matching M_1 of H. By hypothesis, M^* is not a maximum matching in $G \times H$. Let $W^* = (x_0, y_0)(x_1,y_1)(x_2,y_2)\ldots(x_{2n-1},y_{2n-1})$ be a shortest M^*-augmenting path in $G \times H$, by Theorem A.

Case (i) Either x_0 and x_{2n-1} are M- unsaturated vertices in G or y_0 and y_{2n-1} are M_1- unsaturated vertices in H.

Without loss of generality assume that x_0 and x_{2n-1} are M-unsaturated vertices of G. W^* defines an M- alternating walk $W = x_0\, x_1\, x_2 \ldots\, x_{2n-1}$ of odd length in G (since $(x_i, y_i)(x_{i+1}, y_{i+1})$ is an edge of $G \times H$, $x_ix_{i+1} \in E(G)$, $0 \le i \le 2n-2$). This W must be an M-augmenting walk, since M^* arises out of M and M_1 of G and H, respectively. From W we delete the edges of closed subwalk(s) of even length(s), if any; the resulting walk, say, W_1, is of odd length in which no edge is repeated more than twice, see Figure (6).



A maximum matching M of G is $\{x_1x_2, x_3x_4, x_5x_6, x_7x_8\}$ and a maximum matching M_1 of H is $\{y_0y_1, y_2y_3\}$. Clearly $x_0x_1x_2x_3x_4x_1x_2x_5x_6x_7x_8x_2x_1x_0$ is an alternating walk W defined by the M^*-augmenting path $(x_0,y_0)(x_1,y_1)(x_2,y_0)$ (x_3,y_1) $(x_4,y_0)(x_1,y_2)$ $(x_2,y_3)(x_5,y_2)(x_6,y_3)(x_7,y_2)(x_8,y_3)(x_2,y_1)(x_1,y_0)(x_0,y_1)$ of $G \times H$. Clearly $W_1 = x_0x_1\ x_2x_5x_6x_7x_8\ x_2x_1x_0$ is obtained from W by deleting the edges of the closed subwalk $x_1x_2x_3x_4x_1$.

Figure 6

A similar arugment holds if both y_0 and y_{2n-1} are M_1– unsaturated vertices of H.

Case (ii) Exactly one of the two vertices $\{x_0, x_{2n-1}\}$ is an M– unsaturated vertex in G.

Without out loss of generality, assume that x_0 is an M- unsaturated vertex in G. Then y_{2n-1} must be an M_1– unsaturated vertex in H. For otherwise (x_{2n-1}, y_{2n-1}) would be an M^*– saturated vertex, contradicting the fact that W^* is an M^*– augmenting path. As x_0 is an M– unsaturated vertex in G and y_{2n-1} is an M_1 unsaturated vertex in H, both G and H do not have perfect matchings. ■

Corollary 3.2 Let G be a graph containing a perfect matching and let H be a bipartite graph, then $\tau(G \times H) = \underline{\tau}(G \times H)$.

Proof. Suppose the result is not true, then $\tau(G\times H) > \underline{\tau}(G\times H)$. Then by Theorem 3.1, H should contain an M_1– augmenting walk, where M_1 is a maximum matching in H. As this walk cannot be a path, it should contain an odd cycle, a contradiction to the fact that H is a bipartite graph. ■

Theorem 3.3 If both G and H are bipartite graphs and at least one of them has a perfect matching, then $\alpha(G \times H) = \underline{\alpha}(G \times H)$.

Proof. Without loss of generality assume that G has a perfect matching. As G is a bipartite graph containing a perfect matching, by König's Theorem [3, p. 74].

$$\left.\begin{array}{l}\alpha(G) + \tau(G) = n(G) \text{ and} \\ \alpha(G) = \tau(G)\end{array}\right\} \quad (1)$$

Similary, $\alpha(H) + \tau(H) = n(H)$. (2)

From equations (1) and (2) we have,

$$(\alpha(G) + \tau(G)) . (\alpha(H) + \tau(H)) = n(G) . n(H)$$

$$\alpha(G) . \alpha(H) + \alpha(G) . \tau(H) + \tau(G) . \alpha(H) + \tau(G) . \tau(H) = n(G) . n(H)$$

$$\alpha(H) . (\alpha(G) + \tau(G)) + \alpha(G) . \tau(H) + \tau(G) . \tau(H) = n(G) . n(H)$$

Again by (1), $\alpha(H) . n(G) + 2\ \tau(G) . \tau(H) = n(G) . n(H)$ (3)

As $G \times H$ is a bipartite graph, again by König's Theorem,

$$\alpha(G \times H) + \tau(G \times H) = n(G) . n(H) \quad (4)$$

Using Corollary 3.2, the equation (3) becomes

$$\alpha(H) . n(G) + \tau(G \times H) = n(G) . n(H) \quad (5)$$

From (4) and (5),

$$\alpha(H) . n(G) = \alpha(G \times H)$$

Therefore, by the definition of $\underline{\alpha}(G \times H)$,

$$\alpha(G \times H) = \alpha(H) . n(G) \leq \underline{\alpha}(G \times H) \quad (6)$$

But it is clear that $\alpha(G \times H) \geq \underline{\alpha}(G \times H)$ (7)

From (6) and (7) we have $\alpha(G \times H) = \underline{\alpha}(G \times H)$ ∎

ACKNOWLEDGEMENT

The authors would like to thank Professor P.K. Jha for providing many of his papers in the references. The second author also would like to thank U.G.C., New Delhi, for selecting him in the F.I.P. (award No: FIPTNMD99/SRO/UGC dt. 31.05.1999).

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