

Orthogonal Designs of Kharaghani Type: II

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Abstract

H. Kharaghani, in "Arrays for orthogonal designs", *J. Combin. Designs*, 8 (2000), 166-173, showed how to use amicable sets of matrices to construct orthogonal designs in orders divisible by eight. We show how amicable orthogonal designs can be used to make amicable sets and so obtain infinite families of orthogonal designs in six variables in orders divisible by eight.

Key words and phrases: Orthogonal designs, amicable orthogonal designs, amicable set of matrices, Goethals-Seidel array, Kharaghani array.

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1 Preliminaries

An *orthogonal design* of order n and type (s_1, s_2, \dots, s_k) denoted $OD(n; s_1, s_2, \dots, s_k)$ in variables x_1, x_2, \dots, x_k , is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \dots, \pm x_k\}$ satisfying

$$AA^T = \sum_{i=1}^k (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n .

Let $A = \{A_j : A_j = \{a_{j1}, a_{j2}, \dots, a_{jn}\}, j = 1, \dots, \ell\}$, be a set of ℓ sequences of length n . These sequences are said to be a set of *disjoint sequences* if and only if the set of the corresponding circulant matrices $B_j = \cup_j \{\text{circ}(A_j)\}$, $j = 1, \dots, \ell$ is disjoint, i.e. $B_i * B_j = 0$ for all $i \neq j$, $i, j = 1, 2, \dots, \ell$, where $*$ denotes the Hadamard product. A sequence is said to be *symmetric* if the corresponding circulant matrix is symmetric.

The *non-periodic autocorrelation function* $N_A(s)$ (abbreviated as NPAF) of the above sequences is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \quad s = 0, 1, \dots, n-1. \quad (1)$$

If $A_j(z) = a_{j1} + a_{j2}z + \dots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j , then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^n \sum_{k=1}^n a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}). \quad (2)$$

Given A_ℓ , as above, of length n the *periodic autocorrelation function* $P_A(s)$ (abbreviated as PAF) is defined, reducing $i + s$ modulo n , as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^n a_{ji} a_{j,i+s}, \quad s = 0, 1, \dots, n-1. \quad (3)$$

For the results of this paper generally zero PAF is sufficient. However zero NPAF sequences imply zero PAF sequences exist, the zero NPAF sequence being padded at the end with sufficient zeros to make longer lengths. Hence zero NPAF can give more general results.

Let B_i , $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \dots, \pm x_k\}$ satisfying

$$\sum_{i=1}^4 B_i B_i^T = \sum_{i=1}^k (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2 R & B_3 R & B_4 R \\ -B_2 R & B_1 & B_4^T R & -B_3^T R \\ -B_3 R & -B_4^T R & B_1 & B_2^T R \\ -B_4 R & B_3^T R & -B_2^T R & B_1 \end{pmatrix}$$

where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \dots, s_k)$. See page 107 of [1] for details.

Plotkin [6] showed that, if there is an Hadamard matrix of order $2t$, then there is an $OD(8t; t, t, t, t, t, t, t, t)$. In the same paper he also constructed an $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$. This orthogonal design (OD) has appeared in [1], [7] and in [8]. It is conjectured that there is an $OD(8n; n, n, n, n, n, n, n, n)$ for each odd integer n . Until recently, none except the original for $n = 3$ found by Plotkin, had been constructed in the ensuing twenty eight years. Holzmann and Kharaghani [2] using a new method constructed many new Plotkin OD s of order 24 and two new Plotkin OD s of order 40 and 56. Actually their construction provides many new orthogonal designs in 6, 7 and 8 variables which include the Plotkin OD s of order 40 and 56.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [3] a set $\{A_1, A_2, \dots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^n \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0$$

for some permutation σ of the set $\{1, 2, \dots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^n (A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T) = 0. \quad (4)$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k .

Two orthogonal designs $A = OD(m; p_1, p_2, \dots, p_u)$ and $B = OD(m; q_1, q_2, \dots, q_v)$ are said to be *amicable orthogonal designs* of type $AOD(m; p_1, p_2, \dots, p_u; q_1, q_2, \dots, q_v)$, if

$$AB^T = BA^T \quad \text{and} \quad A^T B = B^T A.$$

The maximum number of variables in orthogonal designs and amicable orthogonal designs is intimately related with algebraic forms and has been determined over 20 years ago. An account of this theory is given in [1]. For the purposes of this paper we need to know that the maximum number of variables in orthogonal designs of orders $\equiv 2 \pmod{4}$, $\equiv 4 \pmod{8}$ and $\equiv 8 \pmod{16}$, is 2, 4, and 8 respectively. The maximum number of variables $u + v$ in amicable orthogonal designs of orders $\equiv 2 \pmod{4}$ and $\equiv 4 \pmod{8}$, is 4 and 6 respectively.

A set of matrices $\{A_1, A_2, \dots, A_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \dots, \pm x_k\}$ is said to satisfy the additive property for (s_1, s_2, \dots, s_k) if it satisfies

$$\sum_{i=1}^n A_i A_i^T = \sum_{i=1}^k (s_i x_i^2) I_m. \quad (5)$$

First we need the following array from [3].

Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices of order t , satisfying the additive property for (s_1, s_2, \dots, s_k) . Then the Kharaghani array

$$H = \begin{pmatrix} A_1 & A_2 & A_4 R_n & A_3 R_n & A_6 R_n & A_5 R_n & A_8 R_n & A_7 R_n \\ -A_2 & A_1 & A_3 R_n & -A_4 R_n & A_5 R_n & -A_6 R_n & A_7 R_n & -A_8 R_n \\ -A_4 R_n & -A_3 R_n & A_1 & A_2 & -A_8^T R_n & A_7^T R_n & A_6^T R_n & -A_5^T R_n \\ -A_3 R_n & A_4 R_n & -A_2 & A_1 & A_8^T R_n & A_7^T R_n & -A_6^T R_n & -A_5^T R_n \\ -A_6 R_n & -A_5 R_n & A_8^T R_n & -A_7^T R_n & A_1 & A_2 & -A_4^T R_n & A_3^T R_n \\ -A_5 R_n & A_6 R_n & -A_8^T R_n & -A_7^T R_n & -A_2 & A_1 & A_3^T R_n & A_4^T R_n \\ -A_8 R_n & -A_7 R_n & -A_6^T R_n & A_5^T R_n & A_4^T R_n & -A_3^T R_n & A_1 & A_2 \\ -A_7 R_n & A_8 R_n & A_5^T R_n & A_6^T R_n & -A_3^T R_n & -A_4^T R_n & -A_2 & A_1 \end{pmatrix}$$

is an $OD(8t; s_1, s_2, \dots, s_k)$.

Koukouvinos and Seberry [4] have extended the construction of Holzmam and Kharaghani [2] to find infinite families of Kharaghani type orthogonal designs, and in [5] orthogonal designs $OD(8t; k, k, k, k, k, k)$ in 6 variables for odd t .

2 The construction

First we give the following definition.

Definition 1 Define L -matrices, L_1, L_2, \dots, L_n to be n circulant (or type 1) $(0, \pm 1)$ matrices of order ℓ satisfying

$$(i) \quad L_i * L_j = 0, \quad i \neq j,$$

$$(ii) \quad \sum_{i=1}^n L_i L_i^T = k I_\ell,$$

where $*$ denotes the Hadamard product. We say k is the weight of these L -matrices.

From definition 1 we observe that T -matrices of order t (see Seberry and Yamada [8] for more details) are L -matrices with $\ell = k = t$ and $n = 4$.

Then we have.

Theorem 1 Suppose L_1, L_2, \dots, L_n are n circulant (or type 1) L -matrices of order s and weight k . Some of the L -matrices may be zero. Further suppose $A = (a_{ij})$, $B = (b_{ij})$ are amicable orthogonal designs of type $AOD(n; p_1, p_2, \dots, p_u; q_1, q_2, \dots, q_v)$ on the variables $\{0, \pm x_1, \pm x_2, \dots,$

$\pm x_u\}$, and $\{0, \pm y_1, \pm y_2, \dots, \pm y_v\}$, respectively. Then there exists an amicable set of matrices $\{A_{i=1}^{2n}\}$ which satisfy

$$\sum_{i=1}^{2n} A_i A_i^T = \left(\sum_{i=1}^u p_i x_i^2 + \sum_{i=1}^v q_i y_i^2 \right) \sum_{i=1}^n L_i L_i^T = \left(\sum_{i=1}^u p_i x_i^2 + \sum_{i=1}^v q_i y_i^2 \right) k I_s, \quad (6)$$

and also (4).

Hence these $\{A_i\}_{i=1}^{2n}$ of order s are an amicable set satisfying the additive property for

$$(kp_1, kp_2, \dots, kp_u, kq_1, kq_2, \dots, kq_v).$$

Proof: Use

$$\begin{aligned} A_1 &= a_{11}L_1 + a_{12}L_2 + \dots + a_{1n}L_n, & A_2 &= b_{11}L_1 + b_{12}L_2 + \dots + b_{1n}L_n \\ A_3 &= a_{21}L_1 + a_{22}L_2 + \dots + a_{2n}L_n, & A_4 &= b_{21}L_1 + b_{22}L_2 + \dots + b_{2n}L_n \\ A_5 &= a_{31}L_1 + a_{32}L_2 + \dots + a_{3n}L_n, & A_6 &= b_{31}L_1 + b_{32}L_2 + \dots + b_{3n}L_n \\ &\vdots & &\vdots \\ A_{2n-1} &= a_{n1}L_1 + a_{n2}L_2 + \dots + a_{nn}L_n, & A_{2n} &= b_{n1}L_1 + b_{n2}L_2 + \dots + b_{nn}L_n \end{aligned}$$

First we note that A and B being amicable ensures that the (x, y) entry c_{xy} of $C = AB^T$ is

$$c_{xy} = \sum_{j=1}^n a_{xj} b_{yj} = \sum_{j=1}^n a_{yj} b_{xj} = c_{yx}. \quad (7)$$

We also note that if A and B are amicable then A^T and B^T are also amicable so the (x, y) entry d_{xy} of $D = A^T B$ is

$$d_{xy} = \sum_{j=1}^n a_{jx} b_{jy} = \sum_{j=1}^n a_{jy} b_{jx} = d_{yx}. \quad (8)$$

First let us first multiply out $A_1 A_2^T$, where we will use $(\dots L_\ell L_m^T)_{\ell m}$ to denote the term in $L_\ell L_m^T$. Then

$$A_1 A_2^T = \sum_{j=1}^n a_{1j} b_{1j} L_j L_j^T + \dots + ((a_{1\ell} b_{1m}) L_\ell L_m^T)_{\ell m} + \dots \quad (9)$$

Similarly

$$A_2 A_1^T = \sum_{j=1}^n a_{1j} b_{1j} L_j L_j^T + \dots + ((b_{1\ell} a_{1m}) L_\ell L_m^T)_{\ell m} + \dots \quad (10)$$

Hence $A_1A_2^T - A_2A_1^T$ will have no terms in $L_jL_j^T$, $j = 1, 2, \dots, 2n$. Thus the typical term is given by

$$A_1A_2^T - A_2A_1^T = \dots + ((a_{1\ell}b_{1m} - b_{1\ell}a_{1m})L_\ell L_m^T)_{\ell m} + \dots \quad (11)$$

We now formally multiply out the expression on the left hand side of (4), which gives the following terms in $L_\ell L_m^T$

$$\begin{aligned} \sum_{i=1}^n (A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T) &= \\ &= \dots + \left((\sum_{j=1}^n a_{j\ell}b_{jm} - \sum_{i=1}^n b_{i\ell}a_{im})L_\ell L_m^T \right)_{\ell m} + \dots \\ &= \dots + \left((\sum_{j=1}^n a_{jm}b_{j\ell} - \sum_{i=1}^n a_{im}b_{i\ell})L_\ell L_m^T \right)_{\ell m} + \dots \quad \text{using (8)} \\ &= 0. \end{aligned}$$

This is formally zero and we have (4). These matrices also satisfy (5) and (6) by virtue of A and B being (amicable) orthogonal designs. \square

Remark 1 Although the theorem is true for any pair of amicable orthogonal designs the arrays needed to exploit the full generality of the theorem are only known, at present, to exist for $n = 2$ or 4 .

Recalling that the maximum number of variables in amicable orthogonal designs of orders 2 and 4 are known precisely, and are given in Tables 1 and 2, we have.

Corollary 1 *Suppose there exist $AOD(2; p_1, p_2; q_1, q_2)$. Further suppose there exist two circulant (or type 1) L -matrices of order ℓ and weight k . Then there exists an $OD(4\ell; kp_1, kp_2, kq_1, kq_2)$.*

Proof: We use the L -matrices in the theorem to form an amicable set satisfying the required additive property which is then used in the Goethals-Seidel array to obtain the result. \square

Corollary 2 *Suppose there exist $AOD(4; p_1, p_2, p_3; q_1, q_2, q_3)$. Further suppose there exist four circulant (or type 1) L -matrices of order ℓ and weight k . Then there exists an $OD(8\ell; kp_1, kp_2, kp_3, kq_1, kq_2, kq_3)$.*

Proof: We use the L -matrices in the theorem to form an amicable set satisfying the additive property for $(kp_1, kp_2, kp_3, kq_1, kq_2, kq_3)$. These are then used in the Kharaghani array to obtain the result. \square

Example 1 $n=2$ Let A and B be the $AOD(2; 1, 1; 1, 1)$ given by

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \begin{bmatrix} c & d \\ d & -c \end{bmatrix}.$$

Let L_1 and L_2 be two circulant (or type 1) L -matrices of order ℓ and weight k . Construct

$$\begin{aligned} A_1 &= aL_1 + bL_2, & A_2 &= cL_1 + dL_2 \\ A_3 &= -bL_1 + aL_2, & A_4 &= dL_1 - cL_2. \end{aligned} \quad (12)$$

Then

$$\sum_{i=1}^4 A_i A_i^T = (a^2 + b^2 + c^2 + d^2) \sum_{i=1}^2 L_i L_i^T = k(a^2 + b^2 + c^2 + d^2) I_\ell \quad (13)$$

and

$$A_1 A_2^T - A_2 A_1^T + A_3 A_4^T - A_4 A_3^T = 0. \quad (14)$$

Hence this set of matrices $\{A_1, A_2, \dots, A_4\}$ of order ℓ with entries in $\{0, \pm a, \pm b, \pm c, \pm d\}$ is an *amicable set* satisfying the additive property for $(1, 1, 1, 1)$.

These can be used in a variant of the Goethals-Seidel array

$$G = \begin{pmatrix} A_1 & A_2 & A_3 R & A_4 R \\ -A_2 & A_1 & -A_4 R & A_3 R \\ -A_3 R & A_4 R & A_1 & -A_2 \\ -A_4 R & -A_3 R & A_2 & A_1 \end{pmatrix}$$

where R is the back-diagonal identity matrix, to obtain an $OD(4\ell; k, k, k, k)$.

□

Example 2 $n=4$ Let A and B be the $AOD(4; 1, 1, 1; 1, 1, 1)$ given by

$$\begin{bmatrix} a & b & c & 0 \\ -b & a & 0 & -c \\ -c & 0 & a & b \\ 0 & c & -b & a \end{bmatrix} \quad \begin{bmatrix} d & e & f & 0 \\ e & -d & 0 & -f \\ f & 0 & -d & e \\ 0 & -f & e & d \end{bmatrix}.$$

Let L_1, L_2, \dots, L_4 be four circulant (or type 1) L -matrices of order ℓ and weight k . Construct

$$\begin{aligned} A_1 &= aL_1 + bL_2 + cL_3, & A_2 &= dL_1 + eL_2 + fL_3, \\ A_3 &= -bL_1 + aL_2 - cL_4, & A_4 &= eL_1 - dL_2 - fL_4, \\ A_5 &= -cL_1 + aL_3 + bL_4, & A_6 &= fL_1 - dL_3 + eL_4, \\ A_7 &= cL_2 - bL_3 + aL_4, & A_8 &= -fL_2 + eL_3 + dL_4. \end{aligned} \quad (15)$$

Then

$$\begin{aligned} \sum_{i=1}^8 A_i A_i^T &= (a^2 + b^2 + c^2 + d^2 + e^2 + f^2) \sum_{i=1}^4 L_i L_i^T \\ &= k(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) I_\ell, \end{aligned} \quad (16)$$

and

$$A_1 A_2^T - A_2 A_1^T + A_3 A_4^T - A_4 A_3^T + A_5 A_6^T - A_6 A_5^T + A_7 A_8^T - A_8 A_7^T = 0. \quad (17)$$

Hence this set of matrices $\{A_1, A_2, \dots, A_8\}$ of order ℓ with entries in $\{0, \pm a, \pm b, \pm c, \pm d, \pm e, \pm f\}$ is an *amicable set* satisfying the additive property for $(1, 1, 1, 1, 1, 1)$.

These can be used in the Kharaghani array to obtain an $OD(8\ell; k, k, k, k, k, k, k)$. \square

Example 3 $n=4$ Let A and B be the $AOD(4; 1, 1, 2; 1, 1, 2)$ given by

$$\begin{bmatrix} a & b & c & c \\ -b & a & c & -c \\ c & c & -a & -b \\ c & -c & b & -a \end{bmatrix} \quad \begin{bmatrix} d & e & f & f \\ e & -d & f & -f \\ -f & -f & e & d \\ -f & f & d & -e \end{bmatrix}.$$

Let L_1, L_2, \dots, L_4 four circulant (or type 1) L -matrices of order ℓ and weight k . Construct

$$\begin{aligned} A_1 &= aL_1 & +bL_2 & +cL_3 & +cL_4 & A_2 &= dL_1 & +eL_2 & +fL_3 & +fL_4, \\ A_3 &= -bL_1 & +aL_2 & +cL_3 & -cL_4, & A_4 &= eL_1 & -dL_2 & +fL_3 & -fL_4, \\ A_5 &= cL_1 & +cL_2 & -aL_3 & -bL_4, & A_6 &= -fL_1 & -fL_2 & +eL_3 & +dL_4, \\ A_7 &= cL_1 & -cL_2 & +bL_3 & -aL_4, & A_8 &= -fL_1 & +fL_2 & +dL_3 & -eL_4. \end{aligned} \quad (18)$$

Then

$$\begin{aligned} \sum_{i=1}^8 A_i A_i^T &= (a^2 + b^2 + 2c^2 + d^2 + e^2 + 2f^2) \sum_{i=1}^4 L_i L_i^T \\ &= k(a^2 + b^2 + 2c^2 + d^2 + e^2 + 2f^2) I_\ell, \end{aligned} \quad (19)$$

and

$$A_1 A_2^T - A_2 A_1^T + A_3 A_4^T - A_4 A_3^T + A_5 A_6^T - A_6 A_5^T + A_7 A_8^T - A_8 A_7^T = 0. \quad (20)$$

Hence this set of matrices $\{A_1, A_2, \dots, A_8\}$ of order ℓ with entries in $\{0, \pm a, \pm b, \pm c, \pm d, \pm e, \pm f\}$ is an *amicable set* satisfying the additive property for $(1, 1, 2, 1, 1, 2)$.

These can be used in the Kharaghani array to obtain an $OD(8\ell; k, k, k, k, k, 2k, 2k)$. \square

Table 1: Amicable Orthogonal Designs in Order 2

X\Y	(1)	(2)	(1,1)
(1)	*	*	*
(2)		*	*
(1,1)			*

Table 2: Amicable Orthogonal Designs in Order 4

X\Y	(1)	(2)	(3)	(4)	(1,1)	(1,2)	(1,3)	(2,2)	(1,1,1)	(1,1,2)
(1)	*	*	*	*	*	*	*	*	*	*
(2)		*	*	*	*	*	*	*	*	*
(3)			*	*	*	*	*	*	*	*
(4)				*	*	*	*	*	*	*
(1,1)					*	*	*	*	*	*
(1,2)						*	*	*	*	*
(1,3)							*	*		*
(2,2)								*		*
(1,1,1)									*	
(1,1,2)										*

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